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BOUNDS ON THE COARSENESS OF THE n-CUBE

BY

JEHUDA HARTMAN

ABSTRACT. The coarseness, $c(G)$, of a graph G is the maximum number of edge disjoint nonplanar subgraphs contained in G. For the n-dimensional cube Q^n we obtain the inequalities

$$\left[\frac{n\cdot 2^{n-2}}{7}\right]\geq c(Q^n)\geq\left[\frac{n}{4}\right]2^{n-3}.$$

Introduction. A graph is said to be *planar* if it can be drawn in the plane (or on a sphere) so that no two of its edges intersect. The *coarseness*, $c(G)$ of a graph G, a concept introduced first by P. Erdös, is the maximum number of edge disjoint non-planar subgraphs contained in G. The coarseness of the complete graphs K_n and the complete bipartite graphs $K_{m,n}$ has been evaluated in [1]–[4], where exact values of $c(G)$ are given in nearly all cases. In the present article we obtain upper and lower bounds for the coarseness of the n-dimensional cube.

Some definitions. We adopt the terminology and notation of F. Harary [5]. All graphs considered are finite, undirected and without loops or multiple edges.

An edge $x=uv$ of a graph G is called *subdivided* if it is replaced by a vertex w, called a *refinement vertex*, and by new edges uw and wv. A graph G' is a *subdivision* of G, if it is obtained from G by a subdivision of an edge of G. A *refinement* $\hat{G}$ of G is a graph obtained from G by a finite sequence of subdivisions. Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by a sequence of subdivisions of edges. The *n-cube* Q^n is defined inductively as a Cartesian product, where $Q^1=K_2$ and $Q^n=K_2\times Q^{n-1}$. A graph isomorphic to a subgraph of Q^n is called *cubical*. A refinement $\hat{G}$ of G which is cubical is called a *cubical* refinement of G. Since a graph G is planar if and only if $c(G)=0$, it follows that $c(Q^n)=0$ for $n=1,2,3$.

Main results. Upper and lower bounds are established for $c(Q^n)$ in Theorem 1 with the aid of the Lemmas. The number of vertices and edges of a graph G will be denoted by $v(G)$ and $e(G)$ respectively. In particular, $v(Q^n)=2^n$ and $e(Q^n)=n\cdot 2^{n-1}$, and each vertex of Q^n has degree n.

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LEMMA 1. *Let G_i $(i=1,\ldots,n)$ be finite graphs and $G=G_1\times G_2\times\cdots\times G_n$.* Then

$$\frac{c(G)}{v(G)}\geq\sum_{i=1}^{n}\frac{c(G_i)}{v(G_i)}. \tag{1}$$

Proof. First consider $n=2$, and let V_i, E_i be respectively the sets of vertices and edges of G_i, $(i=1,2)$. For $a\in V_1$ let G_{a_2} be the maximal subgraph of $G_1\times G_2$ with vertex set that of $\{a\}\times V_2$. Then G_{a_2} is isomorphic to G_2 and $G_1\times G_2$ contains $v(G_1)$ edge disjoint isomorphs of G_2. Similarly, there are $v(G_2)$ edge disjoint isomorphs of G_1 in $G_1\times G_2$. Hence

$$c(G_1\times G_2)\geq v(G_1)c(G_2)+v(G_2)c(G_1). \tag{2}$$

The following inequality, equivalent to (1) since $v(G)=\prod_{i=1}^{n}v(G_i)$, can be proved by induction on n.

$$c(G)\geq\sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\neq 1}}^{n}v(G_j)\right)c(G_i). \tag{3}$$

For $n=2$ we have (2), anchoring the induction. Applying (2),

$$c(G_1\times G_2\times\cdots\times G_n\times G_{n+1})\geq v(G_{n+1})c(G_1\times G_2\times\cdots\times G_n)+\prod_{i=1}^{n}\cdots.$$

Using the induction hypothesis, inequality (3) follows.

In particular, if the graphs G_i are all isomorphic copies of a graph H, and if H^n designates the Cartesian product of n copies of H, then

$$c(H^n)\geq n\cdot v^{n-1}(H)\cdot c(H). \tag{4}$$

Note that in some cases equality holds in (1).

By Kuratowski's theorem [9], every nonplanar graph contains a subgraph homeomorphic to K_5 or to $K_{3,3}$. If $c(G)=k$, then G contains k edge disjoint subgraphs, each homeomorphic to K_5 or to $K_{3,3}$, but does not contain $k+1$ such subgraphs. To establish an upper bound for $c(Q^n)$, we calculate the minimum number of edges of cubical refinements of K_5 and $K_{3,3}$. As noted in [1], $c(G)\leq[e(G)/9]$ for every nonplanar graph G. This bound will be improved, using Lemmas 2 and 3.

LEMMA 2. *Every cubical refinement of K_5 has at least 16 edges. Moreover, if a cubical refinement of K_5 has exactly 16 edges, it is isomorphic to the graph shown in Figure 1.*

The proof appears in [6], [8], and is omitted.

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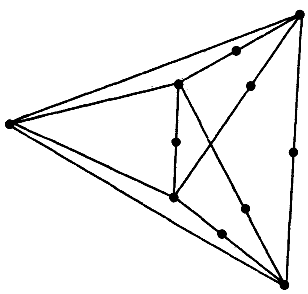

Fig. 1

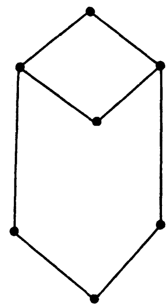

Fig. 2

LEMMA 3. *Every cubical refinement of* $K_{3,3}$ *has at least* 14 *edges.*

Proof. The graph $K_{2,3}$ is not cubical, but the refinement of $K_{2,3}$ depicted in Figure 2 is cubical. It is also clear that any cubical refinement of $K_{2,3}$ must contain at least 8 edges. Furthermore, any cubical refinement of $K_{2,3}$ with exactly 8 edges is isomorphic to the graph shown in Figure 2. It should be noted that there are refinements of $K_{2,3}$ having 8 edges which are not cubical.

Denote the sets of vertices of $K_{3,3}$ by $\{x_i\}$, $\{y_i\}$, $i=1,2,3$. Let $\hat{K}_{3,3}$ be a cubical refinement of $K_{3,3}$. The graph obtained from $\hat{K}_{3,3}$ by elimination of the vertex x_i and all the refinement vertices on the edges of $K_{3,3}$ incident with x_i is denoted by $\hat{K}_{3,3}-x_i$ and is a cubical refinement of $K_{2,3}$. Therefore

$$2e(\hat{K}_{3,3})=\sum_{i=1}^{3} e(\hat{K}_{3,3}-x_i)\geq 3.8, \tag{5}$$

so every cubical refinement of $K_{3,3}$ has at least 12 edges.

If $e(\hat{K}_{3,3})=12$, then $\hat{K}_{3,3}-x_1$ is cubical with 8 edges. Consequently, there must be a vertex y_k in $\hat{K}_{3,3}-x_1$ with at least two refinement vertices on edges of $K_{3,3}-x_1$ incident with y_k, and in that case $e(\hat{K}_{3,3}-y_k)\leq 7$, a contradiction. Hence, $e(\hat{K}_{3,3})\geq 13$. If equality holds, then there must be a y_j such that $e(\hat{K}_{3,3}-y_j)=8$, otherwise $2\cdot 13=\sum_{i=1}^{3} e(\hat{K}_{3,3}-y_i)\geq 3.9$, a contradiction. Let the notation be chosen so that $e(\hat{K}_{3,3}-y_1)=8$. Then $\hat{K}_{3,3}-y_1$ is isomorphic to the graph of Figure 2. Furthermore, we may assume that the other two refinement vertices of $\hat{K}_{3,3}$ are on the edges x_1y_1 or x_2y_1. Since Q^n is bipartite, the subgraph $\hat{K}_{3,3}$ is also bipartite and must be isomorphic to one of the graphs of Figure 3.

However, if the graph in Figure 3(a) is cubical, then by simple distance considerations at least one of the edges x_1y_1, x_3y_3 must contain four or more refinement vertices. Similarly, if the graph in Figure 3(b) is cubical, then the edge x_2y_2 must contain at least one refinement vertex. Thus neither graph is cubical and $e(\hat{K}_{3,3})\geq 14$.

It can also be shown that any cubical refinement of $K_{3,3}$ having 14 edges is isomorphic to the graph in Figure 4.

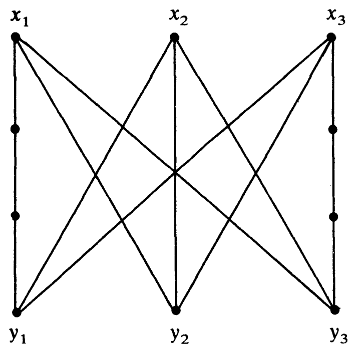

Fig. 3(a)

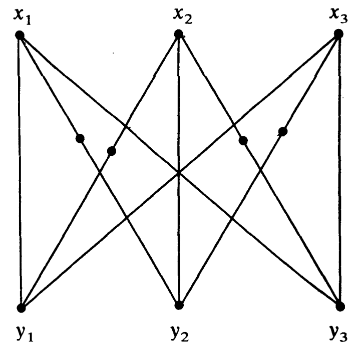

Fig. 3(b)

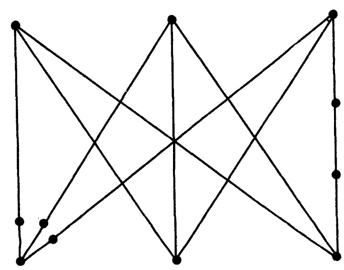
Fig. 4

Lemmas 2 and 3 show that every cubical refinement of Q^n has at least 14 edges, so that

(6) $$c(Q^n) \leq \left[\frac{n \cdot 2^{n-2}}{7}\right].$$

From Lemma 1 and the fact that Q^4 is the edge-disjoint union of two isomorphic cubical refinements of K_5, one of which is shown in Figure 1, we conclude that $c(Q^4)=2$. Writing $n=4k+r$, $r=0, 1, 2, 3$, then $Q^n=(Q^4)^k \times K_2^r$, and by (2) with $G_1=Q^{4k}$ and $G_2=K_2^r$, $c(Q^n) \geq 2^r c(Q^4)^k$. Using (4) with $H=Q^4$ and $n=k$, $c(Q^n) \geq 2^r k v^{k-1}(Q^4) c(Q^4)$, from which, for all positive integers n,

(7) $$c(Q^n) \geq \left[\frac{n}{4}\right] 2^{n-3}.$$

Inequalities (6) and (7) imply our main result, Theorem 1.

THEOREM 1. *For any positive integer n,*

$$\left[\frac{n \cdot 2^{n-2}}{7}\right] \geq c(Q^n) \geq \left[\frac{n}{4}\right] \cdot 2^{n-3}.$$

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Department of Mathematics
The University of Alberta
Edmonton, Alberta, Canada
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