J. Austral. Math. Soc. (Series A) 31 (1981), 202–206

ESTIMATING $|\alpha - p/q|$

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(Received 14 August 1980)

Communicated by A. J. van der Poorten

Abstract

An estimate for $q^2|\alpha - p/q|$ is obtained by considering the relation between the continued fractions for α and p/q. This leads to an extension of the standard result "$q^2|\alpha - p/q| < 1$ implies that for some $n, p/q = (ip_n + p_{n-1})/(iq_n + q_{n-1})$ where $i = 0, 1$ or $a_{n+1} - 1$".

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 10 F 05.

1. Introduction

We shall take α to be a positive real irrational number and p/q to be a rational approximation to α, written in reduced form (so that p and q are relatively prime positive integers). We write both α and p/q as simple continued fractions

$$\alpha = [a_0, a_1, \ldots, a_n, a_{n+1}, \ldots],$$
$$p/q = [a_0, a_1, \ldots, a_n, b_1, \ldots, b_r],$$

where, without loss of generality, $b_r \geqslant 2$. We assume p/q is not a convergent p_m/q_m to α, so we may require $b_1 \neq a_{n+1}$ and $r \geqslant 1$.

There are a number of results on $|\alpha - p/q|$. For example there are two classical results (see Lang (1966)):

CR1. If $q^2|\alpha - p/q| < \frac{1}{2}$ then p/q is a convergent to α.

CR2. If $q^2|\alpha - p/q| < 1$ then, for some $n, p/q = (ip_n + p_{n-1})/(iq_n + q_{n-1})$ where $i = 0, 1$ or $a_{n+1} - 1$.

More recently I had need to know what could be said if $q^2|\alpha - p/q| < 2$. In Worley (1977) it was shown that in this case $p/q = (ip_n + p_{n-1})/(iq_n + q_{n-1})$ where either i is an integer, $0 \leqslant i \leqslant a_{n+1} - 1$, or i is a rational number of the form $j + \frac{1}{2}$ where $0 \leqslant j \leqslant a_{n+1} - 1$ and j is integral.

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The purpose of this paper is to strengthen and generalize this last result. To do so we make use of the following results.

LEMMA 1 (Niven and Zuckerman (1972), Theorem 7.3). *For any positive real number* x,

$$[a_0, a_1, \dots, a_n, x] = (xp_n + p_{n-1})/(xq_n + q_{n-1}).$$

LEMMA 2 (Niven and Zuckerman (1972), Theorem 7.5).

$$|p_n q_{n-1} - p_{n-1} q_n| = 1.$$

LEMMA 3 (Worley (1973), Lemma 1). *Let*

$$P_m/Q_m = [0, a_{m+2}, a_{m+3}, \dots, a_{m+k}],$$

where the continued fraction is to be interpreted as $0/1$ *if* $k = 1$. *Then*

$$q_{m+k} = q_m Q_m([a_{m+1}, \dots, a_{m+k}] + q_{n-1}/q_n).$$

In the notation introduced earlier, Lemma 3 shows that

$$q = q_n d([b_1, \dots, b_r] + q_{n-1}/q_n),$$

where d is the denominator of $[b_1, \dots, b_r]$.

By combining the previous three lemmas we obtain

LEMMA 4. *If* p/q *is not a convergent to* α, *then*

$$q^2|\alpha - p/q| > \begin{cases} d^2(\beta - \beta^2/\gamma) & \text{if } \beta < \gamma, \\ d^2(\beta - \gamma)(\beta + 1)/(\gamma + 1) & \text{if } \beta > \gamma, \end{cases}$$

where $\beta = [b_1, \dots, b_r]$ *and* $\gamma = [a_{n+1}, a_{n+2}, \dots]$ *using the notation introduced earlier.*

PROOF. By Lemmas 1 and 2

$$|\alpha - p/q| = \left| \frac{\gamma p_n + p_{n-1}}{\gamma q_n + q_{n-1}} - \frac{\beta p_n + p_{n-1}}{\beta q_n + q_{n-1}} \right|$$
$$= \frac{|\gamma - \beta|}{(\gamma q_n + q_{n-1})(\beta q_n + q_{n-1})}.$$

Using the comment after the statement of Lemma 3 this gives

$$q^2|\alpha - p/q| = \frac{d^2|\gamma - \beta|(\beta + q_{n-1}/q_n)}{(\gamma + q_{n-1}/q_n)}.$$

Since $(\beta + x)/(\gamma + x)$ is, for $0 \leqslant x < 1$, an increasing function of x if $\beta < \gamma$ and a decreasing function if $\beta > \gamma$ the lemma follows.

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2. The main results

By close analysis of the bounds given in Lemma 4, we obtain the following theorem, which includes the classical results CR1 and CR2 as the special cases $k = \frac{1}{2}$ and $k = 1$.

THEOREM 1. *If α is irrational, $k \geqslant \frac{1}{2}$, and p/q is a rational approximation to α (in reduced form) for which*

$$q^2|\alpha - p/q| < k$$

then either p/q is a convergent p_n/q_n to α or p/q has one of the following forms:

$$\text{(i)} \quad \frac{p}{q} = \frac{ap_n + bp_{n-1}}{aq_n + bq_{n-1}}, \qquad ab < 2k,$$

$$\text{(ii)} \quad \frac{p}{q} = \frac{ap_n - bp_{n-1}}{aq_n + bq_{n-1}}, \qquad ab < 2k,$$

$$\text{(iii)} \quad \frac{p}{q} = \frac{ap_{n+1} + bp_{n-1}}{aq_{n+1} + bq_{n-1}}, \quad ab < k, a_{n+1} = 1,$$

where a and b are positive integers.

To obtain the symmetry of this result, there has been a slight sacrifice of detail. Some applications may require the slightly stronger version:

THEOREM 2. *If α is irrational, $k \geqslant \frac{1}{2}$, and p/q is a rational approximation to α (in reduced form) for which*

$$q^2|\alpha - p/q| < k$$

then either p/q is a convergent p_n/q_n to α or p/q has one of the following forms:

$$\text{(i)}' \quad p/q = \frac{ap_n + bp_{n-1}}{aq_n + bq_{n-1}} \quad \begin{array}{lll} a \geqslant b & \text{and} & ab < 2k, \quad \text{or} \\ a \leqslant b & \text{and} & ab < k + a^2/a_{n+1}, \end{array}$$

$$\text{(ii)}' \quad p/q = \frac{ap_n - bp_{n-1}}{aq_n - bq_{n-1}} \quad \begin{array}{lll} a \leqslant b & \text{and} & ab < 2k, \quad \text{or} \\ a \geqslant b & \text{and} & ab(1 - b/2a) < k, \end{array}$$

where a and b are positive integers.

Theorem 1 follows from Theorem 2 on writing $p_{n+1} = p_n + p_{n-1}$, $q_{n+1} = q_n + q_{n-1}$ if p/q has the form (i)$'$ where $a < b$ and $a_{n+1} = 1$. In all other cases of (i)$'$ and (ii)$'$ we have $ab < 2k$.

PROOF OF THEOREM 2. We suppose $q^2|\alpha - p/q| < k$ and p/q is not a convergent to α. We write α and p/q in the form described in Section 1 and

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compare k with the bounds given by Lemma 4. We also make use of the fact that the graph of $f_\gamma(x) = x - x^2/\gamma$ is a parabola with maximum at $x = \frac{1}{2}\gamma$ and axis of symmetry the line $x = \frac{1}{2}\gamma$. From this it follows that if $m \leqslant \min(\beta, \gamma - \beta)$ then

$$\beta - \beta^2/\gamma \geqslant m(1 - m/\gamma). \tag{1}$$

It is convenient to break the proof into separate cases.

Case 1. $r = 1$, $\beta = b_1 < \gamma$. In this case we have $d = 1$, and setting $m = \min(b_1, a_{n+1} - b_1)$ we obtain from (1) and Lemma 4 that

$$k > q^2|\alpha - p/q| > \beta - \beta^2/\gamma \geqslant m(1 - m/\gamma).$$

Since $\gamma > \alpha_{n+1} \geqslant 2m$ we conclude that $m < 2k$. On observing that if $\beta = b_1 = m$ then

$$p/q = (mp_n + p_{n-1})/(mq_n + q_{n-1})$$

and if $\beta = b_1 = a_{n+1} - m$ then

$$p/q = (p_{n+1} - mp_n)/(q_{n+1} - mq_n).$$

The proof of Theorem 2 is complete in this case.

Case 2. $r > 1$, $\beta < \gamma$. To use (1) we write β as $m + \rho/d$ if $\beta \leqslant \frac{1}{2}a_{n+1}$ and as $a_{n+1} - m - \rho/d$ if $\beta > \frac{1}{2}a_{n+1}$, where we take $1 \leqslant \rho \leqslant d - 1$. Using (1) and Lemma 4 we obtain

$$\begin{aligned} k > q^2|\alpha - p/q| &> d^2(\beta - \beta^2/\gamma) \\ &\geqslant d^2(m + \rho/d)(1 - (m + \rho/d)/\gamma) \\ &= d(dm + \rho)(1 - (m + \rho/d)/\gamma). \end{aligned}$$

If $m \geqslant 1$ we observe that, because $\gamma > a_{n+1} > 2(m + \rho/d)$, it follows that $d(dm + \rho) < 2k$. However if $m = 0$ we use the fact that $\gamma > a_{n+1} > b_1 \geqslant 1$, so $\gamma > 2$. This yields $d\rho(1 - \rho/2d) < k$. Theorem 2 now follows in this case, on observing that if $\beta = m + \rho/d$ then

$$p/q = (ap_n + bp_{n-1})/(aq_n + bq_{n-1}),$$

where $a = dm + \rho$ and $b = d$, and if $\beta = a_{n+1} - m - \rho/d$ then

$$p/q = (ap_{n+1} - bp_n)/(aq_{n+1} - bq_n),$$

where $a = d$ and $b = dm + \rho$.

Case 3. $r = 1$, $\beta = b_1 > \gamma$. In this case $d = 1$ and

$$k > q^2|\alpha - p/q| > (\beta - \gamma)(\beta + 1)/(\gamma + 1) > \beta - \gamma.$$

We observe that $\gamma < a_{n+1} + 1/a_{n+2}$, so we have $b_1 - a_{n+1} - 1/a_{n+2} < k$. Writing $b_1 = a_{n+1} + b$ we have

$$p/q = (p_{n+1} + bp_n)/(q_{n+1} + bq_n),$$

where $b < k + 1/a_{n+2}$. Thus p/q has the form specified in (i).

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Case 4. $r > 1$, $\beta > \gamma$. In this case we write $\beta = b_1 + \rho/d$ where $1 < \rho < d - 1$. We have

$$\begin{aligned} k > q^2|\alpha - p/q| &> d^2(\beta - \alpha)(\beta + 1)/(\gamma + 1) \\ &> d^2(\beta - \gamma) \\ &> d(db_1 + \rho - da_{n+1} - d/a_{n+2}). \end{aligned}$$

Writing $m = b_1 - a_{n+1}$ this gives $d(dm + \rho) < k + d^2/a_{n+2}$. Since

$$p/q = (dp_{n+1} + (dm + \rho)p_n)/(dq_{n+1} + (dm + \rho)q_n)$$

the proof of Theorem 2 is now complete.

Since the particular case $k = 2$ arose earlier, and the result has a nice form in this case, we conclude with it.

COROLLARY. *If $q^2|\alpha - p/q| < 2$ then either p/q is a convergent p_n/q_n to α or p/q has one of the following two forms*

(i)″ $p/q = (ap_n + p_{n-1})/(aq_n + q_{n-1})$

$a = 1, 2, 3, a_{n+1} - 3, a_{n+1} - 2, a_{n+1} - 1,$

(ii)″ $p/q = (ap_n + 2p_{n-1})/(aq_n + 2q_{n-1})$ $a = 1$ *or* $2a_{n+1} - 1$.

PROOF. The possibilities given by Theorem 2 for form (i)′ are $(a, b) = (1, 1)$, $(2, 1)$, $(3, 1)$, $(1, 2)$ which correspond to the first three possibilities given for (i)″ and the first for (ii)″. The possibilities given by Theorem 2 for form (ii)′ are $(a, b) = (1, 1), (1, 2), (1, 3), (2, 1)$ and correspond to the other possibilities given for (i)″ and (ii)″.

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https://doi.org/10.1017/S1446788700033486 Published online by Cambridge University Press