



RESEARCH ARTICLE

# Abelian absolute Galois groups

## In Erinnerung an Wulf-Dieter Geyer (1939–2019)

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### Abstract

Generalizing a result of Wulf-Dieter Geyer in his thesis, we prove that if  $K$  is a finitely generated extension of transcendence degree  $r$  of a global field and  $A$  is a closed abelian subgroup of  $\text{Gal}(K)$ , then  $\text{rank}(A) \leq r + 1$ . Moreover, if  $\text{char}(K) = 0$ , then  $\hat{\mathbb{Z}}^{r+1}$  is isomorphic to a closed subgroup of  $\text{Gal}(K)$ .

### 1. Introduction

A consequence of class field theory appearing in [9, p. 302, Thm. 8.8(b)(iii)] says that the cohomological dimension of every number field  $K$  which is not embeddable in  $\mathbb{R}$  is 2. On the other hand,  $\text{cd}(\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}) = 2$  [9, p. 217, Cor. 3.2 and p. 221, Prop. 4.4] and the group  $\hat{\mathbb{Z}}$  occurs as a closed subgroup of  $\text{Gal}(\mathbb{Q})$  in many ways [3, p. 379, Thm. 18.5.6]. One may therefore wonder whether  $\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$  is isomorphic to a closed subgroup of  $\text{Gal}(\mathbb{Q})$ .

A somewhat surprising result of Geyer’s thesis says that this is not the case. Indeed, every closed abelian subgroup of  $\text{Gal}(\mathbb{Q})$  is procyclic [4, p. 357, Satz 2.3] (see also [9, p. 306, Thm. 9.1]).

We generalize this result for every finitely generated extension  $K$  of transcendence degree  $r$  of a global field. We prove that if a profinite group  $A$  is isomorphic to a closed abelian subgroup of  $\text{Gal}(K)$ , then  $\text{rank}(A) \leq r + 1$ . In particular,  $\hat{\mathbb{Z}}^{r+2}$  is not a subgroup of  $\text{Gal}(K)$  (Proposition 4.3).

In the rest of this note, we abuse our language and write “ $A$  is a closed subgroup of  $\text{Gal}(K)$ ” rather than “ $A$  is isomorphic to a closed subgroup of  $\text{Gal}(K)$ .”

It turns out that the latter inequality is sharp. Indeed, if  $\text{char}(K) = 0$ , then  $\hat{\mathbb{Z}}^{r+1}$  is a closed subgroup of  $\text{Gal}(K)$ , while if  $\text{char}(K) = p > 0$ , then  $\hat{\mathbb{Z}}$  is a closed subgroup of  $\text{Gal}(K)$ ,  $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$  is a closed subgroup of  $\text{Gal}(K)$  if  $r \geq 0$  (Theorem 5.7), but  $\hat{\mathbb{Z}}^{r+1}$  is not a closed subgroup of  $\text{Gal}(K)$  if  $r \geq 1$  (Remark 5.8). Here  $l$  ranges over the prime numbers. The exclusion of the factor  $\mathbb{Z}_p$  in the case when  $p > 0$  and  $r \geq 1$  follows from the rule  $\text{cd}_p(\text{Gal}(F)) \leq 1$  for each field  $F$  of characteristic  $p$  [9, p. 256, Thm. 3.3].

### 2. Preliminaries

One of the basic tools needed in the proof of the generalization of Geyer’s result is a special case of the renowned Pontryagin–van Kampen theorem. Here, and in the rest of this note,  $l$  stands for a prime number,  $\mathbb{Z}_l$  is the ring of  $l$ -adic numbers, viewed as a profinite abelian group or as a principal ideal domain. We also write  $\hat{\mathbb{Z}} := \prod_l \mathbb{Z}_l$  for the Prüfer group [3, p. 12]. Thus,  $\mathbb{Z}_l$  is the free pro- $l$  cyclic group and  $\hat{\mathbb{Z}}$  is the free procyclic group.

**Proposition 2.1** ([10, p. 129, Thm. 4.3.3]). *Let  $A$  be a torsion-free abelian profinite group. Then  $A \cong \prod_l \mathbb{Z}_l^{r_l}$ , where  $r_l$  is a cardinal number for each  $l$ .*

The proof of Proposition 2.1 uses a special case of the Pontryagin–van Kampen duality theorem saying that every locally compact abelian topological group  $A$  is canonically isomorphic to its double dual group  $A^{**}$ , where  $A^* = \text{Hom}(A, \mathbb{R}/\mathbb{Z})$ . The proof of that special case needed in our proposition, dealing only with abelian profinite groups, appears in [10, Section 2.9]. It is much simpler than the proof of the general theorem [5, p. 376, Thm. 24.2].

We denote the algebraic closure of a field  $K$  by  $\tilde{K}$  and its separable algebraic closure by  $K_{\text{sep}}$ . We write  $\text{Gal}(K)$  for the absolute Galois group  $\text{Gal}(K_{\text{sep}}/K)$  of  $K$ . If  $A$  is a closed subgroup of  $\text{Gal}(K)$ , then  $K_{\text{sep}}(A)$  denotes the fixed field of  $A$  in  $K_{\text{sep}}$ .

**Lemma 2.2.** *Let  $K$  be a field and  $A$  a nontrivial finite subgroup of  $\text{Gal}(K)$ . Then,  $A \cong \mathbb{Z}/2\mathbb{Z}$ ,  $\text{char}(K) = 0$ , and the fixed field  $\tilde{K}(A)$  of  $A$  in  $\tilde{K}$  is real closed. In addition,  $A$  is the centralizer of itself in  $\text{Gal}(K)$ .*

*Proof.* Let  $R = K_{\text{sep}}(A)$ . Then, a theorem of Artin says that  $\text{char}(K) = 0$ ,  $K_{\text{sep}} = \tilde{K}$ , and  $\tilde{K} = R(\sqrt{-1})$  [7, p. 299, Cor. 9.3]. Let  $\tau$  be the unique element of order 2 of  $\text{Gal}(R)$  defined by  $\tau(\sqrt{-1}) = -\sqrt{-1}$ .

By [7, p. 452, Prop. 2.4],  $R$  is real closed. Let  $<$  be the ordering of  $K$  induced by the unique ordering of  $R$ . If  $R'$  is a real closed field extension of  $K$  in  $\tilde{K}$  whose ordering extends  $<$ , then by [7, p. 455, Thm. 2.9], there exists a unique  $K$ -isomorphism  $R \rightarrow R'$ .

Let  $\sigma$  be an element of the centralizer  $C_{\text{Gal}(K)}(A)$  of  $A$  in  $\text{Gal}(K)$ . Then,  $\sigma R$  is a real closure of  $(K, <)$  and  $\text{Gal}(\sigma R) \cong \mathbb{Z}/2\mathbb{Z}$ . Also,  $\tau(\sigma R) = \tau\sigma R = \sigma\tau R = \sigma R$ . By the preceding paragraph applied to  $\sigma R$  rather than to  $R$ , the restriction of  $\tau$  to  $\sigma R$  is the identity map. In other words,  $\tau \in \text{Gal}(\sigma R)$ . Since  $\text{ord}(\tau) = 2$ , the element  $\tau$  generates  $\text{Gal}(\sigma R)$ , so  $R = \sigma R$ . The uniqueness of the  $K$ -isomorphism of  $R$  into  $R$  implies that  $\sigma \in \text{Gal}(R) = A$ , as desired. □

**Corollary 2.3.** *Let  $K$  be a field and  $A$  a closed abelian subgroup of  $\text{Gal}(K)$ . Then,  $A \cong \mathbb{Z}/2\mathbb{Z}$  or  $A \cong \prod_l \mathbb{Z}_l^{r_l}$ , where  $l$  ranges over all prime numbers and  $r_l$  is a cardinal number.*

*Proof.* If  $A$  has a non-unit element  $\alpha$  of a finite order, then by Lemma 2.2,  $\langle \alpha \rangle \cong \mathbb{Z}/2\mathbb{Z}$  and  $\langle \alpha \rangle$  is its own centralizer in  $\text{Gal}(K)$ . Since  $A$  is abelian,  $A$  is contained in that centralizer. Therefore,  $A = \langle \alpha \rangle$ .

Otherwise,  $A$  is torsion-free. Hence, by Proposition 2.1,  $A$  has the desired structure. □

Given a profinite group  $G$  and a prime number  $l$ , we write  $\text{cd}_l(G)$  for the  **$l$ th cohomology dimension of  $G$**  [9, p. 196, Def. 1.1]. Also, we write  $\zeta_n$  for a primitive root of unity of order  $n$ .

**Lemma 2.4.** *The following statements hold for prime numbers  $p, l$ , and a finite extension  $E$  of  $\mathbb{Q}_p$ :*

- (a)  $E$  contains only finitely many roots of unity.
- (b)  $l^\infty | [E(\zeta_{l^j})_{j \geq 1} : E]$ .
- (c)  $\text{cd}_l(\text{Gal}(E(\zeta_{l^j})_{j \geq 1})) \leq 1$ .

*Proof of (a).* Let  $O$  be the ring of integers of  $E$ ,  $\bar{E}$  the residue field of  $E$ ,  $\pi$  a prime element of  $O$ ,  $U$  the group of invertible elements of  $O$ , and  $U^{(1)} = 1 + \pi O$  the subgroup of 1-units of  $O$ . Reduction modulo  $\pi O$  yields the following short exact sequence

$$\mathbf{1} \longrightarrow U^{(1)} \longrightarrow U \longrightarrow \bar{E}^\times \longrightarrow \mathbf{1},$$

where  $\mathbf{1}$  is the trivial group. By [11, p. 213, Chap. XIV, Prop. 10],  $U^{(1)}$  is isomorphic to a direct product of a finite abelian group with a free abelian group. Since  $\bar{E}^\times$  is also finite, the torsion group of  $U$  is finite. That group is the group of roots of unity in  $E$ . □

*Proof of (b).* By (a),  $E$  has only finitely many roots of unity of order  $l^j$  with  $j \geq 1$ . Thus, there exists a non-negative integer  $j$  with  $\zeta_{l^j} \in E$  and  $\zeta_{l^{j+1}} \notin E$ . By [7, p. 297, Thm. 9.1],  $[E(\zeta_{l^{j+1}}) : E(\zeta_{l^j})] = l$ . Apply the

same argument to the field  $E_1 := E(\zeta_{j_1})$  to find an integer  $j_2 > j_1 := j$  such that  $\zeta_{j_2} \in E_1$  and  $\zeta_{j_2+1} \notin E_1$ , so  $[E_2 : E_1] = l$  with  $E_2 := E(\zeta_{j_1+1}, \zeta_{j_2+1})$ . Continue to find a sequence  $j_1 < j_2 < j_3 < \dots$  and fields  $E \subset E_1 \subset E_2 \subset E_3 \subset \dots$  such that  $\zeta_{j_{n+1}} \in E_n := E(\zeta_{j_i+1})_{i=1}^n$  and  $\zeta_{j_{n+1}+1} \notin E_n$ , so  $[E_{n+1} : E_n] = l$ , for each  $n \geq 1$ . Hence,  $l^\infty | [E(\zeta_{j_i})_{i \geq 1} : E]$ . □

*Proof of (c).* The claim follows from (b) and [9, p. 291, Cor. 7.4(i),(ii)]. □

Note that the citation in the proof of (c) relies on local class field theory.

### 3. Geyer’s theorem

We generalize Geyer’s theorem which asserts that every closed abelian subgroup of  $\text{Gal}(\mathbb{Q})$  is procyclic [4, p. 357, Satz 2.3].

**Lemma 3.1.** *Let  $F$  be a field of positive characteristic  $p$ . Then, no pro- $p$  closed subgroup of  $\text{Gal}(F)$  is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$ .*

*Proof.* Let  $G$  be a closed pro- $p$  subgroup of  $\text{Gal}(F)$ . By [9, p. 256, Thm. 3.3],  $\text{cd}(G) \leq 1$ . On the other hand,  $\mathbb{Z}_p$  is a free pro- $p$  group of rank 1. Hence, by [9, p. 217, Cor. 3.2],  $\text{cd}(\mathbb{Z}_p) = 1$ . It follows from [9, p. 221, Prop. 4.4] that  $\text{cd}(\mathbb{Z}_p \times \mathbb{Z}_p) = \text{cd}(\mathbb{Z}_p) + \text{cd}(\mathbb{Z}_p) = 2$ . Therefore,  $G \not\cong \mathbb{Z}_p \times \mathbb{Z}_p$ , as claimed. □

**Lemma 3.2.** *Let  $K$  be a global field,  $l \neq \text{char}(K)$  a prime number, and  $M$  a separable algebraic extension of  $K$ . Suppose that  $M$  contains all of the roots of unity of order  $l^i$  for  $i = 1, 2, 3, \dots$ . Then,  $\text{cd}_l(\text{Gal}(M)) \leq 1$ . In particular,  $\text{Gal}(M) \not\cong \mathbb{Z}_l \times \mathbb{Z}_l$ .*

*Proof.* We distinguish between two cases:

Case A:  $K$  is a number field. We assume without loss that  $K = \mathbb{Q}$ . By assumption,  $\zeta_{l^2} \in M \setminus \mathbb{R}$ . Thus,  $M$  cannot be embedded into  $\mathbb{R}$ , that is  $M$  is **totally imaginary**. Hence by [9, p. 302, Thm. 8.8(a)],  $\text{cd}_l(\text{Gal}(M)) \neq \infty$ .

Now we consider a prime number  $p$ , a valuation  $v$  of  $M$  lying over  $p$ , and the completion  $\hat{M}_v$  of  $M$  at  $v$ . Then,  $\zeta_{l^i} \in M \subseteq \hat{M}_v$  for each  $i$ . Hence, by Lemma 2.4(b),  $l^\infty | [\hat{M}_v : \mathbb{Q}_p]$ . Therefore, by [9, p. 302, Thm. 8.8(b)],  $\text{cd}_l(\text{Gal}(M)) \leq 1$ .

Finally, by [9, p. 217, Cor. 3.2 and p. 221, Prop. 4.4] and [9, p. 217, Cor. 3.2],

$$\text{cd}_l(\mathbb{Z}_l \times \mathbb{Z}_l) = \text{cd}_l(\mathbb{Z}_l) + \text{cd}_l(\mathbb{Z}_l) = 1 + 1 = 2.$$

Hence,  $\text{Gal}(M) \not\cong \mathbb{Z}_l \times \mathbb{Z}_l$ , as claimed.

Case B:  $K$  is a finite separable extension of  $\mathbb{F}_p(t)$  with  $t$  transcendental over  $\mathbb{F}_p$ . We assume without loss that  $K = \mathbb{F}_p(t)$ . By assumption,  $M$  contains the field  $L := \mathbb{F}_p(\zeta_{l^i})_{i \geq 1}$ , so  $L(t) \subseteq M$ . Since there are infinitely many roots of unity  $\zeta_{l^i}$  in  $\tilde{\mathbb{F}}_p$  and only finitely many of them belong to each finite field,  $L$  is an infinite field. In addition, for each  $i \geq 1$  the extension  $\mathbb{F}_p(\zeta_{l^{i+1}})/\mathbb{F}_p(\zeta_{l^i})$  is cyclic of degree  $l$  or trivial. Hence,  $\text{Gal}(L/\mathbb{F}_p(\zeta_{l^i})) \cong \mathbb{Z}_l$ . Therefore,  $L$  is contained in the maximal extension  $L'$  of  $\mathbb{F}_p(\zeta_{l^i})$  of an  $l^i$ th power degree. Since  $\text{Gal}(L'/\mathbb{F}_p(\zeta_{l^i})) \cong \mathbb{Z}_l$ , the restriction map  $\text{Gal}(L'/\mathbb{F}_p(\zeta_{l^i})) \rightarrow \text{Gal}(L/\mathbb{F}_p(\zeta_{l^i}))$  is surjective, and  $\mathbb{Z}_l$  is generated by one element, that map is an isomorphism [3, p. 331, Cor. 16.10.8]. It follows that  $L = L'$ . Therefore,  $l$  does not divide the order of  $\text{Gal}(L)$ .

By [9, p. 208, Cor. 2.3],  $\text{cd}_l(\text{Gal}(L)) = 0$ . Hence, by [9, p. 272, Prop. 5.2],  $\text{cd}_l(\text{Gal}(L(t))) = 1$ . Since  $\text{Gal}(M) \leq \text{Gal}(L(t))$ , we have by [9, p. 204, Prop. 2.1(a)], that  $\text{cd}_l(\text{Gal}(M)) \leq 1$ . As in Case A, this inequality implies that  $\text{Gal}(M) \not\cong \mathbb{Z}_l \times \mathbb{Z}_l$ , as claimed. □

Here is the promised result of Geyer.

**Theorem 3.3.** *Let  $K$  be a global field and  $A$  a closed abelian subgroup of  $\text{Gal}(K)$ . Then,  $A$  is procyclic.*

*Proof.* We start the proof with the special case where the torsion group  $A_{\text{tor}}$  of  $A$  is nontrivial. In this case, there exists a non-unit  $\tau \in A$  of finite order. By Lemma 2.2,  $\text{char}(K) = 0$  and  $A \cong \mathbb{Z}/2\mathbb{Z}$ . In particular,  $A$  is procyclic.

We may therefore assume that  $A$  is a nontrivial torsion-free abelian profinite group. By Proposition 2.1,  $A \cong \prod_l \mathbb{Z}_l^{r_l}$ , where  $l$  ranges over all prime numbers and for each  $l$ ,  $r_l$  is a cardinal number, so we may assume that  $A \cong \mathbb{Z}_l^r$  for a prime number  $l$  and a positive cardinal number  $r$  and prove that  $A \cong \mathbb{Z}_l$ .

Otherwise,  $A$  contains a closed subgroup which is isomorphic to  $\mathbb{Z}_l \times \mathbb{Z}_l$ . Thus, we may assume that  $A \cong \mathbb{Z}_l \times \mathbb{Z}_l$  and prove that this assumption leads to a contradiction.

To this end, we denote the fixed field of  $A$  in  $K_{\text{sep}}$  by  $M$  and identify  $\text{Gal}(M)$  with  $A$ . By Lemma 3.1,  $l \neq \text{char}(K)$ .

Claim:  $M$  contains a root of unity  $\zeta_l$  of order  $l$ . Indeed, if  $l = 2$ , then  $\zeta_l = -1 \in M$ . Otherwise,  $l > 2$  and if  $\zeta_l \notin M$ , then  $[M(\zeta_l) : M]$  is a divisor of  $l - 1$  which is greater than 1. On the other hand,  $[M(\zeta_l) : M]$  divides the (profinite) order of  $A$  which is  $l^\infty$ , a contradiction.

Since  $\text{Gal}(M) \cong \mathbb{Z}_l \times \mathbb{Z}_l$ , Lemma 3.2 implies that not all roots of unity of order  $l^i$  with  $i \geq 1$  belong to  $M$ . Let  $n$  be the smallest positive integer such that  $M$  contains a root of unity of order  $l^{n-1}$  but does not contain a root of unity of order  $l^n$ . Choose a root of unity  $\zeta_{l^n}$  and set  $M_1 = M(\zeta_{l^n})$ . Then,  $\zeta_{l^n}^l \in M$  but  $\zeta_{l^n} \notin M$ . Hence,  $[M_1 : M] | l$  and  $[M_1 : M] \neq 1$  (by the Claim and [7, p. 289, Thm. 6.2(ii)]), so  $[M_1 : M] = l$ .

Let  $U$  be the open subgroup of  $\mathbb{Z}_l$  of index  $l$ . Then, the index of each of the subgroups  $\mathbb{Z}_l \times U$  and  $U \times \mathbb{Z}_l$  of  $\text{Gal}(M)$  is  $l$ . We choose one of them which is different from  $\text{Gal}(M_1)$  and denote its fixed field in  $K_{\text{sep}}$  by  $M_2$ . Then,  $M_2$  is a cyclic extension of  $M$  of degree  $l$  and  $M_1 \neq M_2$ .

Since  $\zeta_l \in M$ , [7, p. 289, Thm. 6.2(i)] implies the existence of  $a, x \in K_{\text{sep}}$  with  $M_2 = M(x)$  and  $a := x^l \in M$ . Choose  $b \in K_{\text{sep}}$  with  $b^{l^{n-1}} = x$ , so  $b^l = a$ . In particular,  $M_2 = M(b^{l^{n-1}}) \subseteq M(b)$  and  $[M(b) : M_2] \leq l^{n-1}$ . It follows from the preceding paragraph that

$$[M(b) : M] \leq l^n. \tag{3.1}$$

Next choose  $\sigma \in A$  such that  $\sigma|_{M_1} = \text{id}$  and  $\sigma|_{M_2} \neq \text{id}$ . In particular,  $\sigma x \neq x$ , so  $\zeta := (\sigma b)b^{-1}$  satisfies

$$\zeta^l = \sigma b^l \cdot b^{-l} = \sigma a \cdot a^{-1} = aa^{-1} = 1 \text{ and } \zeta^{l^{n-1}} = \sigma b^{l^{n-1}} \cdot b^{-l^{n-1}} = \sigma x \cdot x^{-1} \neq 1,$$

thus  $\zeta$  is a primitive root of 1 of order  $l^n$ .

The definition of  $M_1$  implies that  $M_1 = M(\zeta)$ . But  $M(b)$  is a Galois extension of  $M$  (because  $\text{Gal}(M)$  is abelian). Hence,  $\zeta = (\sigma b)b^{-1} \in M(b)$ , so  $M_1 \subseteq M(b)$ . Since  $[M_1 : M] = l$ , we have by (3.1) that  $[M(b) : M_1] \leq l^{n-1}$ . Since  $\sigma$  is the identity on  $M_1$ , the latter inequality implies that  $\text{ord}(\sigma|_{M(b)}) \leq l^{n-1}$ .

On the other hand, the relation  $\sigma b = b\zeta$  implies by induction on  $i$  that  $\sigma^i b = b\zeta^i \neq b$  for each  $1 \leq i \leq l^{n-1}$ . Hence,  $\text{ord}(\sigma|_{M(b)}) > l^{n-1}$ . This contradicts the conclusion of the preceding paragraph, as required. □

### 4. Generalization of Geyer’s theorem

The central part of the proof of Geyer’s theorem says that for each prime number  $l$ , the largest positive integer  $n$  for which  $\mathbb{Z}_l^n$  is a closed subgroup of  $\text{Gal}(\mathbb{Q})$  or of  $\text{Gal}(\mathbb{F}_p(t))$  is 1. The next lemma will allow us to generalize that statement to each finitely generated extension of a global field.

**Remark 4.1.** *Let  $A$  be a finitely generated torsion-free abelian pro- $l$  group for a prime number  $l$ . [3, p. 519, Prop. 22.7.12(a)] allows us to also consider  $A$  as a finitely generated  $\mathbb{Z}_l$ -module. Since  $\mathbb{Z}_l$  is a principal ideal domain, [7, p. 147, Thm. 7.3] implies that  $A = \mathbb{Z}_l^n$  is a finitely generated free  $\mathbb{Z}_l$ -module of rank  $n$  for some non-negative integer  $n$ . Since  $\mathbb{Z}_l$  is generated, as a profinite group, by one element,  $n$  is also the rank,  $\text{rank}(A)$ , of  $A$  as a profinite group. In other words,  $\text{rank}(A) = \text{rank}_{\mathbb{Z}_l}(A)$ .*

**Lemma 4.2.** *Let  $K$  be a field,  $t$  an indeterminate, and  $l$  a prime number. Suppose that  $n$  is the largest positive integer for which  $\mathbb{Z}_l^n$  is a closed subgroup of  $\text{Gal}(K)$ . Then, the largest positive integer  $m$  for which  $\mathbb{Z}_l^m$  is a closed subgroup of  $\text{Gal}(K(t))$  does not exceed  $n + 1$ .*

*Proof.* Suppose that  $A := \mathbb{Z}_l^{n'}$  is a closed subgroup of  $\text{Gal}(K(t))$  for some positive integer  $n'$ . Let  $\varphi : \text{Gal}(K(t)) \rightarrow \text{Gal}(K)$  be the restriction map. Then,  $\text{Ker}(\varphi) = \text{Gal}(K_{\text{sep}}(t))$ . Setting  $\bar{A} = \varphi(A)$  and  $A_0 = \text{Ker}(\varphi) \cap A$ , we get the following commutative diagram of profinite groups:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \text{Gal}(K_{\text{sep}}(t)) & \longrightarrow & \text{Gal}(K(t)) & \xrightarrow{\varphi} & \text{Gal}(K) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 \mathbf{0} & \longrightarrow & A_0 & \longrightarrow & A & \longrightarrow & \bar{A} & \longrightarrow & \mathbf{0},
 \end{array}$$

where  $\mathbf{0}$  stands for the trivial group of an additive abelian group. Since  $\mathbb{Z}_l$  is a principal ideal domain and  $A$  is a free  $\mathbb{Z}_l$ -module of rank  $n'$ ,  $A_0$  is a free  $\mathbb{Z}_l$ -module, by [7, p. 146, Thm. 7.1]. Also, by [7, p. 148, Lemma 7.4],  $\bar{A}$  is a free  $\mathbb{Z}_l$ -module and  $n' = \text{rank}(A_0) + \text{rank}(\bar{A})$ .

By [9, p. 272, Prop. 5.2],  $\text{Gal}(K_{\text{sep}}(t))$  is a projective group, so also  $A_0$  is a projective group. In other words,  $\text{rank}(A_0) \leq 1$ . Also, by Corollary 2.3 and the assumption of the lemma,  $\bar{A} = \mathbb{Z}_l^m$  with  $m \leq n$  or  $l = 2$  and  $\bar{A} \cong \mathbb{Z}/2\mathbb{Z}$ . In each case,  $\text{rank}(\bar{A}) \leq n$ , hence  $\text{rank}(A) = \text{rank}(\bar{A}) + \text{rank}(A_0) \leq n + 1$ , as claimed. □

**Proposition 4.3.** *Let  $K$  be a finitely generated extension with transcendence degree  $r$  of a global field  $K_0$  and let  $A$  be a closed abelian subgroup of  $\text{Gal}(K)$ . Then,  $A \cong \mathbb{Z}/2\mathbb{Z}$  or  $A \cong \prod_l \mathbb{Z}_l^{r_l}$ , where  $l$  ranges over all prime numbers and  $r_l \leq r + 1$  for each prime number  $l$ .*

*Proof.* By Corollary 2.3,  $A \cong \mathbb{Z}/2\mathbb{Z}$  or  $A \cong \prod_l \mathbb{Z}_l^{r_l}$ , with cardinal numbers  $r_l$ . Assume the latter case. If  $K$  is a global field, then  $r = 0$ . Hence, by Theorem 3.3,  $r_l \leq 0 + 1$  for each  $l$ .

Otherwise,  $r \geq 1$  and  $K$  is a finitely generated extension of transcendence degree 1 of a finitely generated extension  $K'_0$  of transcendence degree  $r - 1$  of  $K_0$ . By induction, for each prime number  $l$ ,  $r$  is the largest positive integer such that  $\mathbb{Z}_l^r$  is a closed subgroup of  $\text{Gal}(K'_0)$ . Hence, by Lemma 4.2,  $r + 1$  is the largest positive number for which  $\mathbb{Z}_l^{r+1}$  is a closed subgroup of  $\text{Gal}(K)$ . In particular,  $r_l \leq r + 1$ , as claimed. □

### 5. Realizing $\hat{\mathbb{Z}}^{r+1}$ as a closed subgroup of $\text{Gal}(K)$

Let  $K$  be a finitely generated extension of  $\mathbb{Q}$  of transcendence degree  $r$ . We complete Proposition 4.3 in this section by proving that  $\hat{\mathbb{Z}}^{r+1}$  is a closed subgroup of  $\text{Gal}(K)$ . An analogous result holds for a finitely generated extension  $K$  of transcendence degree  $r$  of  $\mathbb{F}_p(t)$ , in which case  $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$  replaces  $\hat{\mathbb{Z}}^{r+1}$ .

**Remark 5.1** (Valued fields). *We denote the residue field of a valued field  $(F, v)$  by  $\bar{F}_v$ , and its value group by  $v(F^\times)$ . In addition, we extend  $v$  to a valuation of  $F_{\text{sep}}$  that we also denote by  $v$ , consider its valuation ring  $O_{v,\text{sep}}$ , and let  $D_{v,\text{sep}} = \{\sigma \in \text{Gal}(F) \mid \sigma O_{v,\text{sep}} = O_{v,\text{sep}}\}$  be the corresponding **decomposition group**. Then, we let  $F_v$  be the fixed field of  $D_{v,\text{sep}}$  in  $F_{\text{sep}}$ . Abusing our notation, we also let  $v$  be the restriction of  $v$  to  $F_v$ . Then,  $(F_v, v)$  is the **Henselization** of  $(F, v)$ .*

One knows that  $(F_v, v)$  has the same residue field and value group as those of  $(F, v)$  [2, p. 138, Prop. 15.3.7]. Moreover, the valued fields  $(F_{\text{sep}}, v)$  and  $(F_v, v)$  depend on the extension of  $v$  to  $F_{\text{sep}}$  up to isomorphism [2, p. 138, Cor. 15.3.6].

If  $v$  is a rank-1 valuation, then so is its extension to  $F_v$ . In this case, the completion  $(\hat{F}_v, v)$  of  $(F, v)$  is also discrete with the same value group and residue field as those of  $(F, v)$ . Moreover,  $(\hat{F}_v, v)$  is also the completion of  $(F_v, v)$ . By Hensel's lemma,  $(\hat{F}_v, v)$  is also Henselian [2, p. 167, Cor. 18.3.2]. We embed  $F_{\text{sep}}$  into  $\hat{F}_v$  and observe that  $F_{\text{sep}} \cap \hat{F}_v = F_v$  (since  $(F_{\text{sep}} \cap \hat{F}_v, v)$  is an immediate separable algebraic

extension of  $(F_v, v)$  and  $F_{\text{sep}}\hat{F}_v = \hat{F}_{v,\text{sep}}$  (by the Krasner-Ostrowski lemma [2, p. 172, Cor. 18.5.3]). Thus, restriction gives an isomorphism  $\text{Gal}(\hat{F}_v) \cong \text{Gal}(F_v)$  of the corresponding absolute Galois groups.

We denote the maximal unramified extension of  $F_v$  (resp.  $\hat{F}_v$ ) by  $F_{v,\text{ur}}$  (resp.  $\hat{F}_{v,\text{ur}}$ ) and the maximal tamely ramified extension by  $F_{v,\text{tr}}$  (resp.  $\hat{F}_{v,\text{tr}}$ ). These fields are Galois extensions of  $F_v$  (resp.  $\hat{F}_v$ ). As in [2, p. 133, p. 141, and p. 145], we set  $Z(v) = \text{Gal}(F_v)$  for the **decomposition group**,  $T(v) = \text{Gal}(F_{v,\text{ur}})$  for the **inertia group**, and  $V(v) = \text{Gal}(F_{v,\text{tr}})$  for the **ramification group** of  $(F, v)$ . The letters  $Z$ ,  $T$ , and  $V$  are borrowed from the German translations Zerlegungsgruppe, Trägheitsgruppe, and Verzweigungsgruppe of the English expressions decomposition group, inertia group, and ramification group,

$$\begin{array}{ccccccc}
 & & & & Z(v) & & \\
 & & & & \curvearrowright & & \\
 & & & & T(v) & & \\
 & & & & \curvearrowright & & \\
 F & \text{---} & F_v & \text{---} & F_{v,\text{ur}} & \text{---} & F_{v,\text{tr}} & \text{---} & F_{\text{sep}} \\
 & & & & \text{---} & & \text{---} & & \\
 & & & & V(v) & & & & 
 \end{array} \tag{5.1}$$

Each of the fields  $F_{v,\text{ur}}$ ,  $F_{v,\text{tr}}$ , and  $F_{\text{sep}}$  is a Galois extension of  $F_v$ . By [2, p. 199, Thm. 22.1.1] and [6, Thm. 2.2] (resp. [2, p. 203, Thm. 22.2.1]) both restriction maps

$$\text{Gal}(F_{v,\text{tr}}/F_v) \rightarrow \text{Gal}(F_{v,\text{ur}}/F_v) \quad \text{and} \quad \text{Gal}(F_v) \rightarrow \text{Gal}(F_{v,\text{tr}}/F_v)$$

split. In particular, each closed subgroup of  $\text{Gal}(F_{v,\text{ur}}/F_v)$ ; hence, each closed subgroup of  $\text{Gal}(\hat{F}_v)$  is isomorphic to a closed subgroup of  $\text{Gal}(F_{v,\text{tr}}/F_v)$ . Also, each closed subgroup of  $\text{Gal}(F_{v,\text{tr}}/F_v)$  is isomorphic to a closed subgroup of  $\text{Gal}(F_v)$ .

Note that  $E$  in Theorem 22.1.1 of [2] is  $F_{\text{sep}}$ , in our notation, so it satisfies the condition  $E = E^l$  for all prime numbers  $l \neq \text{char}(\hat{F}_v)$  needed in that theorem.

**Notation 5.2.** We denote the group of roots of unity in a field  $F$  by  $\mu(F)$ . If  $\text{char}(F) = p > 0$  and  $F$  is separably closed, then  $\mu(F) = \tilde{\mathbb{F}}_p^\times$ . If  $\text{char}(F) = 0$  and  $F$  is algebraically closed, then  $\mu(F) = \mu(\mathbb{Q})$  and  $\mathbb{Q}_{\text{ab}} := \mathbb{Q}(\mu(\mathbb{Q}))$  is the maximal abelian extension of  $\mathbb{Q}$  (by the theorem of Kronecker–Weber [8, p. 324, Thm. 110]).

**Remark 5.3.** Given a field  $K$ , the **field of formal power series**  $K((t))$  in the variable  $t$  with coefficients in  $K$ , also called the **field of Laurent series over  $K$** , is the field of all formal power series  $\sum_{i=m}^\infty a_i t^i$  with  $m \in \mathbb{Z}$  and  $a_i \in K$  for all  $i \geq m$ . If  $l < m$ , then  $\sum_{i=m}^\infty a_i t^i$  is identified with  $\sum_{i=l}^\infty a_i t^i$  with  $a_i = 0$  for each  $l \leq i < m$ . Summation and multiplication in  $K((t))$  are defined by the following rules:

$$\begin{aligned}
 \sum_{i=m}^\infty a_i t^i + \sum_{i=m'}^\infty a'_i t^i &= \sum_{i=\min(m,m')}^\infty (a_i + a'_i) t^i, \\
 \left( \sum_{i=m}^\infty a_i t^i \right) \left( \sum_{j=m'}^\infty a'_j t^j \right) &= \sum_{k=m+m'}^\infty \left( \sum_{i+j=k} a_i a'_j \right) t^k.
 \end{aligned}$$

Let  $v$  be the unique discrete valuation of  $K((t))$  with  $v(a) = 0$  for each  $a \in K$  and  $v(t) = 1$ . Then,  $(K((t)), v)$  is the completion of  $(K(t), v)$ , where  $v(\sum_{i=m}^\infty a_i t^i) = m$  whenever  $a_m \neq 0$ . By [2, p. 167, Cor. 18.3.2],  $K((t))$  is Henselian with respect to  $v$ .

By [1, p. 28, Cor. 2] (or [2, p. 141, Thm. 16.1.1]),

$$\text{Gal}(K((t))_{\text{ur}}/K((t))) \cong \text{Gal}(K).$$

Replacing  $K$  by  $K_{\text{sep}}$ , we have that  $K_{\text{sep}}((t))_{\text{ur}} = K_{\text{sep}}((t))$ . Since the roots of unity of order  $n$  with  $\text{char}(K) \nmid n$  are in  $K_{\text{sep}}$ , we have that  $K_{\text{sep}}((t))$  has a cyclic extension of degree  $n$  in  $K_{\text{sep}}((t))_{\text{tr}}$ . Indeed, that extension is  $K_{\text{sep}}((t^{1/n}))$ .

Going to the limit of these extensions, we obtain with  $p := \text{char}(K)$  that  $K_{\text{sep}}((t))_{\text{tr}} = \bigcup_{p \nmid n} K_{\text{sep}}((t^{1/n}))$  and  $\text{Gal}(K_{\text{sep}}((t))_{\text{tr}}/K_{\text{sep}}((t))) \cong \prod_{l \neq p} \mathbb{Z}_l$ .

Moreover, if  $\text{char}(K) = 0$ , then the ramification group  $\text{Gal}(\tilde{K}((t))_{\text{tr}})$  of  $\tilde{K}((t))$  is trivial [2, p. 145, Thm. 16.2.3], so  $\tilde{K}((t))_{\text{tr}} = \tilde{K}((t))$ . Thus, by the preceding paragraph, in this case,  $\text{Gal}(\tilde{K}((t))) \cong \hat{\mathbb{Z}}$ .

**Lemma 5.4.** *Let  $K_0$  be a field of characteristic  $p$ ,  $t$  an indeterminate, and  $r$  a positive integer. Suppose that  $\mu(K_{0,\text{sep}}) \subseteq K_0$  and  $\prod_{l \neq p} \mathbb{Z}_l^r$  is a closed subgroup of  $\text{Gal}(K_0)$ . Then,  $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$  is a closed subgroup of  $\text{Gal}(K_0(t))$ .*

*Proof.* By assumption, the field  $K_0$  has a separable algebraic extension  $K$  with  $\text{Gal}(K) \cong \prod_{l \neq p} \mathbb{Z}_l^r$ . Let  $v$  be the discrete  $K$ -valuation of  $K(t)$  with  $v(t) = 1$  and choose a Henselization  $M := K(t)_v$  of  $K(t)$  with respect to  $v$ . Then,

$$\bar{M} := \overline{K(t)}_v = K \tag{5.2}$$

is the residue field of both  $K(t)$  and  $M$  with respect to  $v$ .

Claim:  $M$  is linearly disjoint from  $\tilde{K}$  over  $K$ . Indeed, let  $\tilde{K}_1, \dots, \tilde{K}_n$  be linearly independent elements of  $\tilde{K}$  over  $K$ . Assume toward contradiction that there exist  $m_1, \dots, m_n \in M$  not all zero with  $\sum_{i=1}^n m_i \tilde{K}_i = 0$ . Dividing  $m_1, \dots, m_n$  by the element with the least  $v$ -value, we may assume that the  $v$ -residues  $\bar{m}_1, \dots, \bar{m}_n$  are elements of  $K$  and one of them is non-zero. Thus,  $\sum_{i=1}^n \bar{m}_i \tilde{K}_i = 0$ , contradicting the assumption on  $\tilde{K}_1, \dots, \tilde{K}_n$ . This proves our claim.

By [2, p. 200, Cor. 22.1.2],

$$Z(v)/V(v) \cong \chi(v) \rtimes \text{Gal}(\bar{M}) \stackrel{(5.2)}{=} \chi(v) \rtimes \text{Gal}(K), \tag{5.3}$$

where  $Z(v) = \text{Gal}(M)$  and  $V(v)$  are respectively the corresponding decomposition and the ramification groups of  $M$  and

$$\chi(v) = \text{Hom}(v(M_{\text{sep}}^\times)/v(M^\times), \mu(K_{0,\text{sep}})). \tag{5.4}$$

See [2, last line of page 144] with  $\bar{\mu}$  in that line being  $\mu(K_{0,\text{sep}})$ , as introduced in the first paragraph of [2, p. 143, Sec. 16.2].

The action of  $\text{Gal}(K)$  on  $\chi(v)$  is given for each  $\tau \in \text{Gal}(K)$ , each homomorphism  $h : v(M_{\text{sep}}^\times)/v(M^\times) \rightarrow \mu(K_{0,\text{sep}})$ , and every  $\gamma \in v(M_{\text{sep}}^\times)$ , by

$$\tau(h)(\gamma + v(M^\times)) = \tau(h(\gamma + v(M^\times))) = h(\gamma + v(M^\times)),$$

where the latter equality holds because  $\mu(K_{0,\text{sep}}) \subseteq K_0 \subseteq K$ . In other words, that action is trivial. It follows that

$$\text{Gal}(M_{\text{tr}}/M) \stackrel{(5.1)}{\cong} Z(v)/V(v) \stackrel{(5.3)}{\cong} \chi(v) \times \text{Gal}(K). \tag{5.5}$$

By [2, p. 147, Cor. 16.2.7], there is a short exact sequence

$$\mathbf{1} \longrightarrow V(v) \longrightarrow T(v) \longrightarrow \chi(v) \longrightarrow \mathbf{1}.$$

Hence,  $\chi(v) \cong T(v)/V(v)$ .

By our choice of  $v$ , the completion of  $K(t)$  with respect to  $v$  (which is also the completion of the Henselian field  $M$ ) is the field  $K((t))$  of formal power series in  $t$  with coefficients in  $K$  [2, p. 83, Example 9.2.2]. The maximal unramified extension of  $K((t))$  is  $K_{\text{sep}}((t))$  and by Remark 5.3,  $\chi(v) \cong T(v)/V(v) \cong \text{Gal}(M_{\text{tr}}/M_{\text{ur}}) \cong \prod_{l \neq p} \mathbb{Z}_l$ .

By the definition of  $K$ ,  $\text{Gal}(K) \cong \prod_{l \neq p} \mathbb{Z}_l^r$ . Hence, by the preceding paragraph,

$$\text{Gal}(M_{\text{tr}}/M) \stackrel{(5.5)}{\cong} \chi(v) \times \text{Gal}(K) \cong \prod_{l \neq p} \mathbb{Z}_l \times \prod_{l \neq p} \mathbb{Z}_l^r = \prod_{l \neq p} \mathbb{Z}_l^{r+1}. \tag{5.5}$$

Since by [6, Thm. 2.2], the epimorphism  $\text{Gal}(M) \rightarrow \text{Gal}(M_{tr}/M)$  splits,  $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$  is a closed subgroup of  $\text{Gal}(M)$ . Since  $M$  is a separable algebraic extension of  $K_0(t)$  [2, p. 137, Thm. 15.3.5],  $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$  is also a closed subgroup of  $\text{Gal}(K_0(t))$ , as claimed.  $\square$

**Remark 5.5.** Note that the references that support both (5.3) and (5.4) hold also in the case where  $\text{char}(K_0) = 0$ .

The following result will be needed in Theorem 5.7.

**Lemma 5.6.** Let  $L$  be a set of prime numbers and  $H$  an open subgroup of  $\prod_{l \in L} \mathbb{Z}_l$ . Then,  $H \cong \prod_{l \in L} \mathbb{Z}_l$ .

*Proof.* We set  $Z := \prod_{l \in L} \mathbb{Z}_l$  and consider all the groups appearing in this proof as additive groups. Since  $H$  is open in  $Z$ , its index  $n := (Z : H)$  is a positive integer. Since  $Z$  is abelian,  $H$  is normal in  $Z$ , so  $nZ \leq H$ .

By [3, p. 13, Lemma 1.4.2(e)],  $n\mathbb{Z}_l \cong \mathbb{Z}_l$  for each  $l \in L$ . Hence,  $nZ = \prod_{l \in L} n\mathbb{Z}_l \cong \prod_{l \in L} \mathbb{Z}_l = Z$ .

Let  $n = \prod_{l \in L'} l^{i(l)}$  be the decomposition of  $n$  into a product of prime powers. If  $l$  and  $l'$  are distinct prime numbers, then  $l'$  is a unit of the ring  $\mathbb{Z}_l$ , so  $l'\mathbb{Z}_l = \mathbb{Z}_l$ . Hence,  $nZ = \prod_{l \in L \cap L'} l^{i(l)}\mathbb{Z}_l \times \prod_{l \in L \setminus L'} \mathbb{Z}_l$ . Therefore,  $(Z : nZ) = \prod_{l \in L \cap L'} (\mathbb{Z}_l : l^{i(l)}\mathbb{Z}_l) = \prod_{l \in L \cap L'} l^{i(l)} \leq n = (Z : H)$ . Combining this result with the result of the first paragraph of the proof, we have  $H = nZ$ . Therefore, by the second paragraph of the proof,  $H \cong Z$ , as claimed.  $\square$

This brings us to the main result of the current section.

**Theorem 5.7.** Let  $F$  be a finitely generated extension of transcendence degree  $r \geq 0$  of a global field  $F_0$  of characteristic  $p$  and let  $F' = F(\mu(F_{0,\text{sep}}))$ . Then,  $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$  is a closed subgroup of  $\text{Gal}(F')$ , hence also of  $\text{Gal}(F)$ .

*Proof.* In the case where  $r = 0$ ,  $F$  itself is a global field, hence Hilbertian [3, p. 242, Thm. 13.4.2]. Since  $F'$  is an abelian extension of  $F$ , a theorem of Kuyk asserts that  $F'$  is also Hilbertian [3, p. 333, Thm. 16.11.3]. Since  $F$  is countable, so is  $F'$ . By [3, p. 379, Thm. 18.5.6], for almost all  $\sigma \in \text{Gal}(F')$  (in the sense of the Haar measure of  $\text{Gal}(F')$ ) the closed subgroup  $\langle \sigma \rangle$  of  $\text{Gal}(F')$  generated by  $\sigma$  is isomorphic to  $\hat{\mathbb{Z}}$ . Since  $\prod_{l \neq p} \mathbb{Z}_l$  is a closed subgroup of  $\prod_l \mathbb{Z}_l$  and  $\prod_l \mathbb{Z}_l \cong \hat{\mathbb{Z}}$  [3, p. 15, Lemma 1.4.5],  $\prod_{l \neq p} \mathbb{Z}_l$  is a closed subgroup of  $\text{Gal}(F')$ .

Alternatively, by a theorem of Whaples, for each  $l \neq p$  the field  $F'$  has a Galois extension  $F'_l$  with  $\text{Gal}(F'_l/F') \cong \mathbb{Z}_l$  [3, p. 314, Cor. 16.6.7]. Then,  $F'' := \prod_{l \neq p} F'_l$  is a Galois extension of  $F'$  with  $\text{Gal}(F''/F') \cong \prod_{l \neq p} \mathbb{Z}_l$ . Since  $\prod_{l \neq p} \mathbb{Z}_l$  is projective [3, p. 507, Cor. 22.4.6], the restriction map  $\text{Gal}(F') \rightarrow \text{Gal}(F''/F')$  splits [3, p. 506, Remark 22.4.2]. Hence, again,  $\prod_{l \neq p} \mathbb{Z}_l$  is a closed subgroup of  $\text{Gal}(F')$ .

Next assume by induction that  $r \geq 1$  and the theorem holds for  $r - 1$ . Choose a finitely generated extension  $F_{r-1}$  of transcendence degree  $r - 1$  of  $F_0$  in  $F$  and let  $F'_{r-1} = F_{r-1}(\mu(F_{0,\text{sep}}))$ . Since  $F$  is finitely generated over  $F_0$  of transcendence degree  $r$ , we may choose  $t$  in  $F$  which is transcendental over  $F_{r-1}$  and  $[F : F_{r-1}(t)] < \infty$ . Then,  $F' = F'_{r-1}F$  is a finite extension of  $F'_{r-1}(t)$ . Let  $L$  be the maximal separable extension of  $F'_{r-1}(t)$  in  $F'$ , so  $F'/L$  is a purely inseparable extension of  $L$ . Then,  $L$  is a finite separable extension of  $F'_{r-1}(t)$ .

$$\begin{array}{ccccccc}
 F_{r-1}(\mu(F_{0,\text{sep}})) & = & F'_{r-1} & \text{---} & F'_{r-1}(t) & \text{---} & L & \text{---} & F' = F'_{r-1}F \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & F_{r-1} & \text{---} & F_{r-1}(t) & \text{---} & & \text{---} & F
 \end{array}$$



Hence,

$$\text{Gal}(L) \text{ is an open subgroup of } \text{Gal}(F'_{r-1}(t)). \quad (5.6)$$

By the induction hypothesis,  $\prod_{l \neq p} \mathbb{Z}'_l$  is a closed subgroup of  $\text{Gal}(F'_{r-1})$ . Therefore, by (5.6), Lemma 5.4, and Lemma 5.6,  $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$  is a closed subgroup of  $\text{Gal}(L)$ . Since  $F'/L$  is a purely inseparable extension (in particular  $F' = L$  if  $\text{char}(F_0) = 0$ ),  $\prod_{l \neq p} \mathbb{Z}_l^{r+1}$  is a closed subgroup of  $\text{Gal}(F')$ , hence also of  $\text{Gal}(F)$ , as claimed.  $\square$

**Remark 5.8.** Let  $F$  be a field as in Theorem 5.7. If  $p = 0$ , then  $\hat{\mathbb{Z}}^{r+1} = \prod_{l \neq p} \mathbb{Z}_l^{r+1}$ . Hence, by that theorem,  $\hat{\mathbb{Z}}^{r+1}$  is isomorphic to a closed subgroup of  $\text{Gal}(F)$ .

If  $p \neq 0$  but  $r = 0$ , then  $F = F_0$  is a countable Hilbertian field and again, by [3, p. 379, Thm. 18.5.6], for almost all  $\sigma \in \text{Gal}(F)$  we have  $\langle \sigma \rangle \cong \hat{\mathbb{Z}}$ .

However, by [9, p. 256, Thm. 3.3],  $\text{cd}_p(\text{Gal}(F)) \leq 1$ . On the other hand, by [9, p. 221, Prop. 4.4],  $\text{cd}_p(\mathbb{Z}_p^{r+1}) = r + 1 \geq 2$  if  $r \geq 1$ . Hence,  $\mathbb{Z}_p^{r+1}$  is isomorphic to no closed subgroup of  $\text{Gal}(F)$ . Therefore,  $\hat{\mathbb{Z}}^{r+1}$  is isomorphic to no closed subgroup of  $\text{Gal}(F)$ .

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