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THE POLYTABLOID BASIS EXPANDS POSITIVELY INTO THE WEB BASIS

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Abstract

We show that the transition matrix from the polytabloid basis to the web basis of the irreducible $\mathfrak{S}_{2n}$-representation of shape (n, n) has nonnegative integer entries. This proves a conjecture of Russell and Tymoczko [*Int. Math. Res. Not.*, **2019**(5) (2019), 1479–1502].

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1. Main

The purpose of this note is to establish a positivity relation between the polytabloid and web bases of the Catalan-dimensional irreducible $\mathfrak{S}_{2n}$-representation of shape (n, n). This proves a conjecture of Russell and Tymoczko [7, Conjecture 5.8].

Let V be a complex vector space with $\dim(V) = m < \infty$. Given two bases $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ and $\mathcal{C} = \{w_1, w_2, \ldots, w_m\}$ of V, there exist unique $a_{ij} \in \mathbb{C}$ such that

$$v_i = \sum_{j=1}^{m} a_{ij} w_j. \tag{1}$$

Algebraic combinatorics gives many interesting examples of bases $\mathcal{B}$ and $\mathcal{C}$ where the transition matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant m}$ has nonnegative real entries and (with respect to an appropriate order on the bases) is *unitriangular* (that is, upper triangular with diagonal entries equal to 1).

Now let V and W be two complex vector spaces of the same dimension $m < \infty$. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of V and let $\mathcal{C} = \{w_1, w_2, \ldots, w_m\}$ be a basis of W. Assume further that a finite group G acts irreducibly on both V and W, and that V and W are isomorphic G-modules. Let $\varphi : V \to W$ be a G-module isomorphism; by Schur's Lemma, φ is unique up to a scalar. The set $\varphi(\mathcal{B}) = \{\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_m)\}$ is a basis of W, and we can again consider the transition matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant m}$ defined by

$$\varphi(v_i) = \sum_{j=1}^{m} a_{ij} w_j. \tag{2}$$

The matrix A is uniquely determined up to a nonzero scalar, and we can again ask (with respect to an appropriate order on the bases, and after an appropriate scaling) whether A is unitriangular with nonnegative real entries.

Recall that the irreducible modules S^λ for the symmetric group $\mathfrak{S}_m$ are naturally labeled by partitions $\lambda \vdash m$. We consider the case where $V = V_n$ and $W = W_n$ are two different models for the irreducible representation $S^{(n,n)}$ of the symmetric group $\mathfrak{S}_{2n}$ on $2n$ letters corresponding to the partition $(n, n) \vdash 2n$. The dimension of V_n and W_n is given by the nth Catalan number $\mathrm{Cat}(n) = (1/(n+1))\binom{2n}{n}$. The space V_n is the module $S^{(n,n)}$ taken with respect to the *polytabloid basis* and the space W_n is the module $S^{(n,n)}$ taken with respect to the basis of *webs* (attached to the Lie algebra $\mathfrak{sl}_2$).

The $\mathfrak{S}_{2n}$-module V_n is defined as follows. Recall that an (n, n)-*tableau* is a filling of the $2 \times n$ Ferrers shape (n, n) with the numbers $1, 2, \ldots, 2n$. If T an (n, n)-tableau, the *(row) tabloid* $\{T\}$ is the set of all $(n!)^2$ tableaux obtainable from T by permuting its entries within rows. Let $M^{(n,n)}$ be the $\mathbb{C}$-vector space formally spanned by these tabloids:

$$M^{(n,n)} := \mathrm{span}_{\mathbb{C}}\{\{T\} : T \text{ an } (n, n)\text{-tableau}\}. \tag{3}$$

The symmetric group $\mathfrak{S}_{2n}$ acts on (n, n)-tableaux by letter permutation; this induces an action of $\mathfrak{S}_{2n}$ on $M^{(n,n)}$ given by $\sigma.\{T\} := \{\sigma.T\}$. The module $M^{(n,n)}$ is isomorphic to the induction of the trivial representation from $\mathfrak{S}_n \times \mathfrak{S}_n$ to $\mathfrak{S}_{2n}$.

Given any (n, n)-tableau T, let $C_T \subseteq \mathfrak{S}_{2n}$ be the subgroup of permutations in $\mathfrak{S}_{2n}$ that stabilize the columns of T. The *Young antisymmetrizer* is the group algebra element

$$\varepsilon_T := \sum_{\sigma \in C_T} \mathrm{sign}(\sigma) \cdot \sigma \in \mathbb{C}[\mathfrak{S}_{2n}] \tag{4}$$

and the associated *polytabloid* is

$$v_T := \varepsilon_T.\{T\} \in M^{(n,n)}. \tag{5}$$

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For any $\sigma \in \mathfrak{S}_{2n}$, we have $\sigma.v_T = v_{\sigma(T)}$, so the linear space

$$V_n := \mathrm{span}_{\mathbb{C}}\{v_T : T \text{ a } (n, n)\text{-tableau}\} \subseteq M^{(n,n)} \tag{6}$$

spanned by these polytabloids carries the structure of a $\mathfrak{S}_{2n}$-module. It turns out that V_n is irreducible of shape (n, n).

Recall that a (n, n)-tableau T is *standard* if its entries increase going down columns and across rows. Let $\mathrm{SYT}(n, n)$ denote the set of all standard (n, n)-tableaux; we have $|\mathrm{SYT}(n, n)| = \mathrm{Cat}(n)$. The five tableaux of $\mathrm{SYT}(3, 3)$ are shown below.

1	3	5
2	4	6

1	3	4
2	5	6

1	2	5
3	4	6

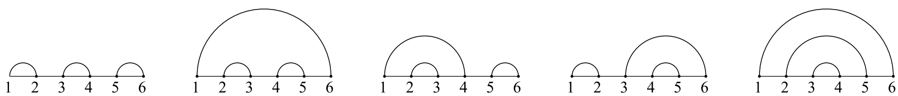
1	2	4
3	5	6

1	2	3
4	5	6

We let $T_0 \in \mathrm{SYT}(n, n)$ be the tableau whose first row entries are $1, 3, 5, \ldots$; when $n = 3$ this is the leftmost tableau shown above.

It can be shown that the set $\{v_T : T \in \mathrm{SYT}(n, n)\}$ of all standard polytabloids forms a basis of V_n. This basis, first discovered by Young [9] and studied by Specht [8], will be called the *polytabloid basis*. (Russell and Tymoczko refer to this basis as the *Specht basis*. We avoid this terminology to prevent confusion with other 'Specht bases' in symmetric group representation theory (see, for example, [1]).) Its definition extends readily to any partition λ, yielding a basis for the corresponding irreducible symmetric group module S^λ. It is perhaps the most well-known basis of S^λ.

Whereas the module V_n is defined in terms of tableaux, the module W_n is defined in terms of matchings. A *perfect matching* M on the set $[2n] := \{1, 2, \ldots, n\}$ is a set partition of $[2n]$ into n blocks in which each block has size 2. We use the notation $i \sim j$ to mean that $\{i, j\}$ is a block of M. A *web* of order n (attached to the Lie algebra $\mathfrak{sl}_2$) is a perfect matching M on $[2n]$ which is *noncrossing*: for any $a < b < c < d$, we do not simultaneously have $a \sim c$ and $b \sim d$ in M. Let $\mathcal{W}_n$ be the set of all webs of order n. The five webs of $\mathcal{W}_3$ are shown below.



Let

$$W_n := \mathrm{span}_{\mathbb{C}}\{w_M : M \in \mathcal{W}_n\} \tag{7}$$

be the vector space with basis $\mathcal{W}_n$; we have $\dim(W_n) = \mathrm{Cat}(n)$. We let $M_0 \in \mathcal{W}_n$ be the web given by $2i - 1 \sim 2i$ for all $1 \leqslant i \leqslant n$; when $n = 3$ this is the leftmost web shown above.

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The symmetric group $\mathfrak{S}_{2n}$ acts on W_n as follows. Given $1 \leqslant i \leqslant 2n-1$, let $s_i = (i, i+1) \in \mathfrak{S}_{2n}$ be the corresponding adjacent transposition. If M is an order n web, the action of s_i on w_M is given by

$$s_i.w_M = \begin{cases} -w_M & \text{if } i \sim i+1 \text{ in } M, \\ w_M + w_{M'} & \text{otherwise,} \end{cases} \tag{8}$$

where in the second line M' is the web obtained from M by replacing the pairs $a \sim i$ and $b \sim i+1$ with the pairs $a \sim b$ and $i \sim i+1$. This action respects the Coxeter relations of $\mathfrak{S}_{2n}$, and so endows W_n with the structure of an $\mathfrak{S}_{2n}$-module (see [5]). The module W_n is irreducible of shape (n, n) (see [5]); the basis $\{w_M : M \in \mathcal{W}_n\}$ is the *web basis* of W_n. The web basis is related to the invariant theory of the Lie group SL_2.

Since V_n and W_n are isomorphic and irreducible $\mathfrak{S}_{2n}$-modules, there exists an isomorphism $\varphi : V_n \to W_n$ which is uniquely determined up to a scalar. Russell and Tymoczko used combinatorial techniques to show that, after scaling, the map φ sends v_{T_0} to w_{M_0} and relates the polytabloid basis to the web basis in a unitriangular way.

THEOREM 1.1 (Russell–Tymoczko [7, Corollary 5.4, Theorem 5.5]). *There exists a unique $\mathfrak{S}_{2n}$-module isomorphism $\varphi : V_n \to W_n$ satisfying $\varphi(v_{T_0}) = w_{M_0}$. For any $T \in \mathrm{SYT}(n, n)$, expand $\varphi(v_T)$ in the web basis as*

$$\varphi(v_T) = \sum_{M \in \mathcal{W}_n} a_{TM} w_M. \tag{9}$$

The transition matrix (a_{TM}) between the polytabloid and web bases is unitriangular with respect to an appropriate order on the bases.

Russell and Tymoczko conjectured [7, Conjecture 5.8] that the entries a_{TM} are nonnegative; we prove their conjecture here.

THEOREM 1.2. *The entries a_{TM} of the transition matrix of Theorem 1.1 are nonnegative integers.*

Theorem 1.2 will be proven in Section 2. Our strategy is to use an alternative model for W_n in terms of products of matrix minors.

We conclude this section with open problems and directions for future research. Our proof of Theorem 1.2 is not combinatorial and leaves open the following problem.

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PROBLEM 1.3. *Give a combinatorial interpretation of the nonnegative integers a_{TM} appearing in Theorem 1.2.*

Given any partition $\lambda \vdash m$, the irreducible representation S^λ of the symmetric group $\mathfrak{S}_m$ has a number of interesting bases, suggesting possible generalizations of Theorems 1.1 and 1.2. The polytabloid basis $\{v_T : T \in \mathrm{SYT}(\lambda)\}$ can be constructed as above, and so may be compared with various extensions of the web basis.

For any $\lambda \vdash m$, the *Kazhdan–Lusztig cellular basis* [3] is a basis of the $\mathfrak{S}_m$-irreducible S^λ. It can be shown that the web and KL cellular bases coincide when $\lambda = (n, n)$ and $m = 2n$. Garsia and McLarnan proved that the transition matrix between the polytabloid and KL cellular bases is unitriangular, but that its entries can be negative for general shapes λ [2]. Theorem 1.2 shows that this transition matrix *does* have nonnegative entries when λ is a $2 \times n$ rectangle.

PROBLEM 1.4. *Determine the set of partitions $\lambda \vdash n$ for which the transition matrix from the polytabloid basis to the Kazhdan–Lusztig cellular basis has nonnegative entries.*

When $\lambda = (k, k, 1^{m-2k})$ is a *flag-shaped partition*, the *skein basis* of S^λ has entries labeled by noncrossing set partitions of $\{1, 2, \ldots, m\}$ with k total blocks and no singleton blocks. This basis was introduced in [6] to give algebraic proofs of cyclic sieving results and coincides with the web and KL cellular bases when $m = 2k$ (but differs from the KL cellular basis in general). Again, the transition basis between the polytabloid and skein bases can have negative entries for general flag-shaped partitions λ.

When $\lambda = (n, n, n)$ is a $3 \times n$ rectangle, there is a basis of $S^{(n,n,n)}$ indexed by order n webs attached to the Lie algebra $\mathfrak{sl}_3$. These are graphs embedded in a disk with $3n$ boundary points satisfying certain conditions; see [5] for details on these webs, and how $\mathfrak{S}_{3n}$ acts on the vector space spanned by them. This web basis is related to the invariant theory of SL_3.

CONJECTURE 1.5. *The transition matrix from the polytabloid basis to the $\mathfrak{sl}_3$-web basis of $S^{(n,n,n)}$ is unitriangular with nonnegative entries.*

2. Proof of Theorem 1.2

The main idea is to recast the vector space W_n using the work of Kung and Rota on binary forms [4]. To this end, let $\mathbf{x} = (x_{ij})_{1 \leqslant i \leqslant 2, 1 \leqslant j \leqslant 2n}$ be a $2 \times 2n$ matrix of variables. We work in the polynomial ring over $\mathbb{C}$ generated by these variables

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and denoted $\mathbb{C}[\mathbf{x}]$. For any two-element subset $\{a < b\} \subseteq [2n]$, let

$$\Delta_{ab} := x_{1a}x_{2b} - x_{1b}x_{2a} \tag{10}$$

be the maximal minor of $\mathbf{x}$ with column set $\{a, b\}$. These minors satisfy the following syzygy: for $a < b < c < d$ we have

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc}. \tag{11}$$

If $M = \{I_1, I_2, \ldots, I_n\}$ is any perfect matching on $[2n]$ (noncrossing or otherwise), so that $I_1, I_2, \ldots, I_n$ are 2-element subsets of the column set of x, we set

$$\Delta_M := \Delta_{I_1}\Delta_{I_2}\cdots\Delta_{I_n}. \tag{12}$$

LEMMA 2.1. *Let $\mathcal{W}_n$ be the set of noncrossing perfect matchings on $[2n]$. For any perfect matching M on $[2n]$, there are nonnegative integers c_{MN} such that $\Delta_M = \sum_{N\in\mathcal{W}_n} c_{MN}\Delta_N$.*

Proof. If the matching M does not have any crossings, we are done. Otherwise, there exist $a < b < c < d$ such that $a \sim c$ and $b \sim d$ in M; this quadruple of indices is said to form a *crossing pair*. Applying the syzygy relation of Equation (11) gives

$$\Delta_M = \Delta_{M'} + \Delta_{M''}, \tag{13}$$

where the perfect matchings M' and M'' are identical to M except that ($a \sim b$ and $c \sim d$) in M' and ($a \sim d$ and $b \sim c$) in M''. Observe that the crossing pair (a, b, c, d) of M is present in neither M' nor M'', and that the transitions $M \rightsquigarrow M'$ and $M \rightsquigarrow M''$ may eliminate other crossing pairs of M, as well. Since both M' and M'' have strictly fewer total crossing pairs than M, we are done by induction. $\square$

The symmetric group $\mathfrak{S}_{2n}$ acts on the matrix $\mathbf{x}$ of variables by column permutation, and hence on the polynomial ring $\mathbb{C}[\mathbf{x}]$. For any $\sigma \in \mathfrak{S}_{2n}$ and any perfect matching $M = \{I_1, I_2, \ldots, I_n\}$, let $\sigma(M)$ be the perfect matching $\{\sigma(I_1), \sigma(I_2), \ldots, \sigma(I_n)\}$. This action of $\mathfrak{S}_{2n}$ on perfect matchings is different from the previously defined actions of $\mathfrak{S}_{2n}$ on V_n and W_n. A relationship between the action $M \mapsto \sigma(M)$ and matrix minors reads as follows. Recall that an *inversion* in a permutation $\sigma \in \mathfrak{S}_{2n}$ is a pair $\{i < j\}$ with $\sigma(i) > \sigma(j)$.

LEMMA 2.2. *Let $\sigma \in \mathfrak{S}_{2n}$ and let $M = \{I_1, I_2, \ldots, I_n\}$ be a perfect matching on $[2n]$. We have*

$$\sigma.\Delta_M = \pm\Delta_{\sigma(M)} \tag{14}$$

where the sign is $(-1)^c$ where c is the number of pairs $I_1, I_2, \ldots, I_n$ that are inversions of the permutation σ.

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Proof. Let $I = \{i < j\}$ be any 2-element subset of $[2n]$ and let $\sigma(I) = \{\sigma(i), \sigma(j)\}$. Observe that $\{i < j\}$ is an inversion of σ if and only if the action of σ reverses columns i and j of the matrix $\mathbf{x}$. Therefore $\sigma.\Delta_I = \pm\Delta_{\sigma(I)}$ where the sign is $+$ if $\{i < j\}$ is not an inversion of σ and $-$ if $\{i < j\}$ is an inversion of σ. The result follows from Equation (12). □

Equation (14) shows that the $\mathbb{C}$-vector subspace

$$\text{span}_{\mathbb{C}}\{\Delta_M : M \text{ a perfect matching on } [2n]\} \tag{15}$$

of $\mathbb{C}[\mathbf{x}]$ is closed under the action of $\mathfrak{S}_{2n}$ and thus carries the structure of a $\mathfrak{S}_{2n}$-module. By Lemma 2.1 the $\mathfrak{S}_{2n}$-module (15) is in fact spanned by the minor products Δ_M corresponding to *noncrossing* perfect matchings. In symbols, we have

$$\text{span}_{\mathbb{C}}\{\Delta_M : M \text{ a perfect matching on } [2n]\} = \text{span}_{\mathbb{C}}\{\Delta_M : M \in \mathcal{W}_n\}. \tag{16}$$

Let us describe the action of adjacent transpositions on the module (16). If $M \in \mathcal{W}_n$ is a noncrossing perfect matching on $[2n]$ and $1 \leqslant i \leqslant 2n-1$, we claim that

$$s_i.\Delta_M = \begin{cases} -\Delta_M & \text{if } i \sim i+1 \text{ in } M, \\ \Delta_M + \Delta_{M'} & \text{otherwise,} \end{cases} \tag{17}$$

where in the second line M' is obtained from M by replacing the pairs $a \sim i$ and $b \sim i+1$ with the pairs $a \sim b$ and $i \sim i+1$. The first line of Equation (17) follows from Lemma 2.2 and the second line follows from the syzygy (11).

Comparing Equations (8) and (17), we see that the linear map

$$\psi : W_n \to \text{span}_{\mathbb{C}}\{\Delta_M : M \in \mathcal{W}_n\} \tag{18}$$

defined by $\psi(w_M) = \Delta_M$ for all $M \in \mathcal{W}_n$ is a surjective map of $\mathfrak{S}_{2n}$-modules. Since W_n irreducible, the map ψ is an isomorphism which sends the web basis $\{w_M : M \in \mathcal{W}_n\}$ to the minor product basis $\{\Delta_M : M \in \mathcal{W}_n\}$. We therefore identify $W_n = \text{span}_{\mathbb{C}}\{\Delta_M : M \in \mathcal{W}_n\}$ and $w_M = \Delta_M$. We are ready to prove our main result.

Proof of Theorem 1.2. Let $\varphi : V_n \to W_n$ be the $\mathfrak{S}_{2n}$-module isomorphism of Theorem 1.1 satisfying

$$\varphi(v_{T_0}) = w_{M_0} = \Delta_{M_0}. \tag{19}$$

Let $T \in \text{SYT}(n, n)$ be an arbitrary standard tableau and let $\sigma \in \mathfrak{S}_{2n}$ be any permutation satisfying $\sigma(T_0) = T$. Then $\sigma(M_0)$ is a perfect matching on $[2n]$ (which is not necessarily noncrossing). Applying σ to Equation (19) gives

$$\varphi(v_T) = \varphi(v_{\sigma(T_0)}) = \varphi(\sigma.v_{T_0}) = \sigma.\varphi(v_{T_0}) \tag{20}$$

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where the second equality used the action of $\mathfrak{S}_{2n}$ on polytabloids. Since $\varphi(v_{T_0}) = \Delta_{M_0}$, Lemma 2.2 implies the further equalities

$$\sigma.\varphi(v_{T_0}) = \sigma.\Delta_{M_0} = \pm\Delta_{\sigma(M_0)} \tag{21}$$

where the sign could *a priori* depend on T.

By Lemma 2.1, the polynomial $\Delta_{\sigma(M_0)}$ appearing in Equation (20) expands nonnegatively and with integer coefficients into the web basis $\{\Delta_M : M \in \mathcal{W}_n\}$. Theorem 1.2 will follow if we can show that the sign appearing in Equation (21) is always positive. The unitriangularity result Theorem 1.1 of Russell and Tymoczko [7] implies that $+1$ is among the coefficients in the expansion of $\varphi(v_T) = \pm\Delta_{\sigma(M_0)}$ into the web basis. By Lemma 2.1, the sign appearing in Equation (21) must be positive. □

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