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Minimal geodesics†

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Abstract. Motivated by the close relation between Aubry–Mather theory and minimal geodesics on a 2-torus we study the existence and properties of minimal geodesics in compact Riemannian manifolds of dimension ≥ 3. We prove that there exist minimal geodesics with certain rotation vectors and that there are restrictions on the rotation vectors of arbitrary minimal geodesics. A detailed analysis of the minimal geodesics of the 'Hedlund examples' shows that – to a certain extent – our results are optimal.

1. *Introduction*

1.1. *Motivation*

This paper is motivated by the wish to understand the possibilities and limitations of a higher-dimensional version of Aubry–Mather theory. This theory constructs and studies invariant sets for monotone twist mappings of a 2-dimensional annulus which are natural generalizations of the invariant curves from KAM theory. The orbits on these Aubry–Mather sets can be characterized as 'orbits of minimal action' and it is precisely this property which makes them so manageable and useful, cf., [**1**], [**14**] and [**15**]. In [**2**] it is shown that Aubry–Mather theory and the study of minimal geodesics on a 2-torus are so closely related that there exists a unifying theory encompassing both. In this paper we investigate the existence and properties of minimal geodesics on compact Riemannian manifolds of dimension greater than two. The ideas and methods presented here apply to the variational principles arising from convex Hamiltonians, see [**11**].

1.2. *Main results*

A non-constant geodesic $c:\mathbb{R}\to M$ in a Riemannian manifold M is called *minimal* if a lift $\tilde{c}$ of c to the universal Riemannian cover $\tilde{M}$ of M minimizes arclength between any two of its points, i.e. if for all $t_1 < t_2$ we have

$$L(\tilde{c}|[t_1, t_2]) = \inf\{L(\tilde{\gamma}) \mid \tilde{\gamma}:[t_1, t_2]\to \tilde{M},\ \tilde{\gamma}(t_i) = \tilde{c}(t_i) \quad \text{for } i = 1, 2\}. \tag{1.1}$$

Note that a compact M carries a minimal geodesic if and only if M has infinite fundamental group.

† Dedicated to Wilhelm Klingenberg with best wishes on his 65th birthday.

Although we treat general compact Riemannian manifolds the most important cases are manifolds with an infinite abelian fundamental group. For this introduction we restrict our attention to this class of manifolds. Obvious examples are tori $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Our main tool is the stable norm $\| \ \|$ on $H_1(M, \mathbb{R})$. For integral classes $v \in H_1(M, \mathbb{R})$ this norm is defined by

$$\|v\| = \inf n^{-1}\{L(\gamma) \mid \gamma \text{ is a closed curve representing } nv, n \in \mathbb{N}\},$$

see e.g. [6], [7] or [8]. This stable norm describes the geometry of the universal cover $\tilde{M}$ of M from a point of view from which fundamental domains look arbitrarily small. It turns out that existence and properties of minimal geodesics are closely related to convexity properties of the unit ball B of the stable norm. In contrast to the 2-dimensional case ∂B can contain flat parts if $\dim M > 2$. Depending on the choice of closed 1-forms $\omega^1, \ldots, \omega^k$ whose cohomology classes $[\omega^i]$, $1 \le i \le k$, form a basis of $H^1(M, \mathbb{R})$ we define a rotation vector $R(\gamma) \in H_1(M, \mathbb{R})$ for every curve $\gamma : [a, b] \to M$:

$$R(\gamma) = \|v(\gamma)\|^{-1} v(\gamma)$$

where

$$[\omega^i](v(\gamma)) = \int_\gamma \omega^i \quad \text{for } 1 \le i \le k.$$

The main results are:

THEOREM 3.2. *For every minimal geodesic c there exists a supporting hyperplane H of B such that the points of accumulation of $R(c \mid [s_0, s_1])$ for $s_1 - s_0 \to \infty$ are contained in $H \cap B$. In particular, if $H \cap B = \{v\}$ then the rotation vectors $R(c \mid [s_0, s_1])$ converge to v.*

THEOREM 4.4. *For every supporting hyperplane H of B there exists a minimal geodesic c such that the limits of the rotation vectors of c are contained in $H \cap B$.*

Since there exists a basis of $H_1(M, \mathbb{R})$ which consists of exposed points of ∂B, i.e. points v with supporting hyperplane H satisfying $H \cap B = \{v\}$, Theorem 4.4 implies:

THEOREM 4.8. *If* $\dim H_1(M, \mathbb{R}) = k$ *then M carries at least k minimal geodesics.*

In § 5 we define a class of Riemannian 3-tori by imposing certain C^0-conditions on their metrics. These 3-tori are modelled on an example by Hedlund [9] and will be called 'Hedlund examples'. We are able to analyze their minimal geodesics in great detail. Thus we can show that the results stated above are rather sharp. In particular, the unit ball B of the stable norm of a Hedlund example is an octahedron and there exist only three periodic minimal geodesics each corresponding to a pair of opposite vertices of the octahedron. Moreover these are the only recurrent minimal geodesics. So the Hedlund examples show that any attempt to generalize Aubry-Mather theory to higher dimensions has to cope with the difficulty that minimal orbits may become very rare.

1.3. *Historical comments*

Under the name of 'geodesics of class A' minimal geodesics on compact surfaces M of genus greater than one have been investigated by Morse [**17**]. He showed that every pair of different points on the ideal boundary S^1 of the universal cover $\tilde{M}$ can be joined by a minimal geodesic. Morse's results were generalized to higher dimensional 'manifolds of hyperbolic type' by Klingenberg [**12**]. In particular, in these cases $\pi_1(M)$ is strongly non-commutative and one obtains uncountably many disjoint closed sets of minimal geodesics. In a much more general setting these ideas turn up in Gromov's theory of hyperbolic groups. Minimal geodesics on 2-tori have been investigated by Hedlund [**9**]. The interest in minimal geodesics has recently been revived by the work of Aubry-LeDaeron [**1**] on minimal energy configurations of a 1-dimensional model in solid state physics and by the work of Mather on orbits of minimal action for area-preserving monotone twist maps, see e.g. [**14**], [**15**]. Different versions of Aubry-Mather theory and the relations between them have been discussed in [**2**], [**4**], [**5**], [**18**] and [**19**]. In higher dimensions there is the work of Bernstein-Katok [**3**], Katok [**11**] and Herman [**10**] treating perturbations of integrable systems with convex Hamiltonian. The methods of [**11**] are close to the ones presented here.

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2. *The stable norm on* $H_1(M, \mathbb{R})$

In this section we define the stable norm on $H_1(M, \mathbb{R})$ and state some of its properties. The proofs are elementary. Since they are not easily accessible in the literature they are given in the Appendix. Our presentation is closely related to [7, Ch. 4.C]. At the end of this section we give a simplified version for the case $M \cong T^n$.

Before we start we fix some notation. We consider a compact connected Riemannian manifold M. We denote by d the distance induced on M and by $L(\gamma)$ the length of a curve γ in M. The diameter diam (M) of M is defined by diam $(M) = \max\{d(p,q) \mid p, q \in M\}$. We fix an arbitrary norm $|\ |$ on $H_1(M, \mathbb{R})$. The lattice of integral homology classes in $H_1(M, \mathbb{R})$ is denoted by $H_1(M, \mathbb{Z})_{\mathbb{R}}$.

Federer's stable norm on $H_q(M, \mathbb{R})$, $q > 0$, can be defined by setting for $v \in H_q(M, \mathbb{R})$:

$$\|v\| = \inf\{\textstyle\sum |r_i| \mathrm{vol}_q(\delta_i) \mid \sum r_i \delta_i \text{ is a Lipschitz } q\text{-cycle representing } v\},$$

cf. [7, p. 50]. Using the surjectivity of the Hurewicz homomorphism $\pi_1(M, p_0) \to H_1(M, \mathbb{Z})$ one can give an equivalent but more geometric definition in the case $q = 1$. We will only use this second definition so that the equivalence with Federer's which is proved in [7] is actually irrelevant for us.

Since the Hurewicz homomorphism is surjective the function $f: H_1(M, \mathbb{Z})_{\mathbb{R}} \to \mathbb{R}^+ \cup \{0\}$, $f(v) := \inf\{L(\gamma) \mid \gamma \text{ is a closed curve representing } v\}$ is well-defined.

PROPOSITION 2.1. *There exists a norm* $\|\ \|$ *on* $H_1(M, \mathbb{R})$ *with the following property*:

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for all $\varepsilon>0$, $A>0$ there exists $C_1=C_1(A,\varepsilon)>0$ such that

$$|f(v)-\|w\||<\varepsilon\|w\|$$

whenever $v\in H_1(M,\mathbb{Z})_{\mathbb{R}}$, $w\in H_1(M,\mathbb{R})$, $|v-w|<A$ and $|w|>C_1$. In particular, if

$$v_m\in H_1(M,\mathbb{Z})_{\mathbb{R}} \text{ and } \lim m^{-1}v_m=w \text{ then } \lim(f(v_m)/m)=\|w\|.$$

This norm $\|\ \|$ is called the stable norm on $H_1(M,\mathbb{R})$.

Next we define a rotation vector $v(\gamma)\in H_1(M,\mathbb{R})$ for every Lipschitz curve $\gamma:[a,b]\to M$. Choose 1-forms $\omega^1,\dots,\omega^k$ such that the cohomology classes $[\omega^1],\dots,[\omega^k]$ form a basis of $H^1(M,\mathbb{R})$. Since $H^1(M,\mathbb{R})$ is the dual of $H_1(M,\mathbb{R})$ the class $v(\gamma)$ can be uniquely defined by

$$[\omega^i](v(\gamma))=\int_\gamma \omega^i.$$

Notation 2.2. The vector $R(\gamma)=\|v(\gamma)\|^{-1}v(\gamma)$ will be called the *rotation vector* of γ. Note that the map $\gamma\to v(\gamma)$ depends on the choice of the representatives $\omega^1,\dots,\omega^k$. However, for any two such maps the difference $|v(\gamma)-\tilde{v}(\gamma)|$ is uniformly bounded. If γ is closed then $v(\gamma)$ coincides with the class $[\gamma]\in H_1(M,Z)_{\mathbb{R}}$ given by Hurewicz homomorphism and is thus naturally defined. For a similar concept see [16, § 2].

The length of a curve and the stable norm of its rotation vector satisfy the following inequality:

LEMMA 2.3. *For all $\varepsilon>0$ there exists $C_2=C_2(\varepsilon)>0$ such that $|v(\gamma)|\ge C_2$ implies $L(\gamma)\ge(1-\varepsilon)\|v(\gamma)\|$.*

Finally we describe how all this simplifies if M is diffeomorphic to a torus, say $M=T^n=\mathbb{R}^n/\mathbb{Z}^n$. In this case we can choose $\omega^i=dx^i$ where $(x^1,\dots,x^n)$ denote the coordinates of a point in $\mathbb{R}^n$. Choosing the basis in $H_1(M,\mathbb{R})$ dual to $[dx^1],\dots,[dx^n]$ we identify $H_1(M,\mathbb{R})$ with $\mathbb{R}^n$ and $H_1(M,\mathbb{Z})$ with $\mathbb{Z}^n\subset\mathbb{R}^n$. With respect to this identification $f:\mathbb{Z}^n\to\mathbb{R}^+\cup\{0\}$ is given by

$$f(k)=\inf_{x\in\mathbb{R}^n}\tilde{d}(x,x+k),$$

where $\tilde{d}$ denotes the distance d lifted to $\mathbb{R}^n$. In this case the distance induced by $\|\ \|$ is a homogenized version of $\tilde{d}$, i.e.

$$\|v-w\|=\lim_{m\to\infty}\frac{1}{m}\tilde{d}(x+mv,x+mw)$$

for all $x,v,w\in\mathbb{R}^n$. To see this choose a sequence $v_m\in\mathbb{Z}^n$ such that $\lim m^{-1}v_m=v$. From Proposition 2.1 we know that $\lim(f(v_m)/m)=\|v\|$. Since $\tilde{d}(x,y)=\tilde{d}(x+k,y+k)$ for all $k\in\mathbb{Z}^n$ we see that $\lim(f(v_m)/m)=\|v\|$ implies

$$\|v\|=\lim\frac{1}{m}\tilde{d}(x,x+v_m)=\lim\frac{1}{m}\tilde{d}(x,x+mv)\quad\text{for all } x\in\mathbb{R}^n.$$

Hence

$$\|v-w\|=\lim\frac{1}{m}\tilde{d}(x,x+m(v-w))=\lim\frac{1}{m}\tilde{d}(x+mv,x+mw).$$

Finally suppose $\gamma:[a,b]\to M$ is a curve and $\tilde{\gamma}$ is a lift of γ to $\mathbb{R}^n$. Then $\int_\gamma \omega^i = \tilde{\gamma}^i(b)-\tilde{\gamma}^i(a)$. Hence, with respect to the above identification we have

$$v(\gamma)=\tilde{\gamma}(b)-\tilde{\gamma}(a),$$

i.e. the rotation vector of γ is simply the normalized vector $\tilde{\gamma}(b)-\tilde{\gamma}(a)$.

To conclude this section we consider the case $M=T^2$. Then we have the special situation that $f(mk)=mf(k)$ for all $k\in\mathbb{Z}^2$, $m\in\mathbb{N}$; hence $f(k)=\|k\|$. To the author's knowledge M. Morse was the first to notice this fact, cf, [**17**, Theorem 9]. If c_i, $i=1,2$, are closed geodesics on M with $L(c_i)=f([c_i])>0$ and $[c_2]\notin\mathbb{Q}\cdot[c_1]$ then c_1 and c_2 intersect transversely, say $c_1(0)=c_2(0)$. Hence we can form the closed curve $c_1 * c_2$ representing $[c_1]+[c_2]$ and $L(c_1 * c_2)=L(c_1)+L(c_2)$. Since $\dot{c}_1(0)\neq\dot{c}_2(0)$ the curve $c_1 * c_2$ is not a geodesic. Hence

$$\|[c_1]+[c_2]\|=f([c_1]+[c_2])<L(c_1)+L(c_2)=\|[c_1]\|+\|[c_2]\|.$$

So in this case we have a strict triangle inequality. This implies that the unit ball $B=\{v\in\mathbb{R}^2 \mid \|v\|\le 1\}$ is strictly convex, i.e. ∂B does not contain straight line segments. This contrasts with the higher dimensional case: in the Hedlund examples, cf. § 5, the unit ball B is a centrally symmetric polyhedron. In a different context the strict convexity in the 2-dimensional case is proved and used in [**16**].

3. *Restrictions on the rotation vectors of minimal geodesics*

If M is diffeomorphic to a 2-torus and $\tilde{c}:\mathbb{R}\to\mathbb{R}^2$ is a lift of a minimal geodesic then the rotation vectors

$$R(c\,|\,[t_0,t_1])=\|\tilde{c}(t_1)-\tilde{c}(t_0)\|^{-1}(\tilde{c}(t_1)-\tilde{c}(t_0))$$

converge for $t_1-t_0\to\infty$. This follows from Hedlund's result [**9**] that $\tilde{c}$ lies within bounded distance from a straight line. It can also be deduced from the above-mentioned strict convexity of the unit ball of the stable norm, cf. Theorem 3.2 below. The Hedlund examples which will be discussed in § 5 show that a similar statement is not always true for n-tori, $n\ge 3$. However, we will prove that in general the limits of the rotation vectors of a minimal geodesic are restricted to lie in the intersection of B with a supporting hyperplane.

If the fundamental group of M is non-abelian the situation is more complicated. If Γ is the group of deck-transformations of the universal cover $\tilde{M}\to M$ we consider $\bar{M}=\tilde{M}/[\Gamma,\Gamma]$. So $\tilde{M}=\bar{M}$ if $\pi_1(M)$ is abelian. We denote by $\bar{\pi}:\bar{M}\to M$ the natural projection and by $\bar{d}$ the induced distance on $\bar{M}$. In the non-abelian case we can only restrict the rotation vectors of those minimal geodesics whose lifts to $\bar{M}$ are arclength-minimizing, i.e. whose lifts to $\bar{M}$ satisfy (1.1).

Notation 3.1. We let $\mathcal{M}$ denote the set of minimal geodesics in M.

$\bar{\mathcal{M}}\subseteq\mathcal{M}$ denotes the subset of those geodesics whose lifts to $\bar{M}$ satisfy (1.1).

In particular, we have $\bar{\mathcal{M}}=\mathcal{M}$ if $\pi_1(M)$ is abelian. As above we denote by B the unit ball with respect to the stable norm, $B=\{v\in H_1(M,\mathbb{R}) \mid \|v\|\le 1\}$. In this section *all curves will be parametrized by arclength.*

THEOREM 3.2. *For every $c\in\bar{\mathcal{M}}$ there exists a supporting hyperplane H to B with the* *following property: for every neighborhood U of $H \cap B$ there exists $C > 0$ such that $R(c|[s_0, s_1]) \in U$ whenever $s_1 - s_0 \geq C$.*

First we describe the idea underlying the proof of 3.2 in the case $M \simeq T^n = \mathbb{R}^n/\mathbb{Z}^n$. If $\tilde{c}: \mathbb{R} \to \mathbb{R}^n$ is a lift of c then $\tilde{c}_m(s) := m^{-1}\tilde{c}(ms)$ is a minimal geodesic with respect to the distance $\tilde{d}_m(x, y) = m^{-1}d(mx, my)$. Hence every convergent subsequence of $(\tilde{c}_m)_{m \in \mathbb{N}}$ converges to a minimal geodesic of the normed vector space $(\mathbb{R}^n, \| \ \|)$. It is not difficult to see that a curve $\bar{c}: \mathbb{R} \to \mathbb{R}^n$ is minimal with respect to $\| \ \|$ if and only if there exists a supporting hyperplane H to B such that all rotation vectors

$$\|\bar{c}(s_1) - \bar{c}(s_0)\|^{-1}(\bar{c}(s_1) - \bar{c}(s_0)), \quad s_0 < s_1,$$

of $\bar{c}$ are contained in $H \cap B$. To be precise we note that '$\bar{c}$ minimal with respect to $\| \ \|$' means that $\bar{c}$ satisfies (1.1) where the length of a curve is defined with respect to the metric $\rho(x, y) = \|x - y\|$, cf. [**20**, § 13]. This makes our claim plausible and our proof mimics a proof in the case $(\mathbb{R}^n, \| \ \|)$. However, there is a serious obstacle, cf. the remark preceding Lemma 3.6. We overcome this difficulty by an application of Carathéodory's Theorem on the generation of convex hulls which reduces to an order argument in the 2-dimensional case, cf. Lemma 3.8 below.

The rigorous proof of 3.2 is preceded by some lemmas. It is convenient to introduce the following notion.

DEFINITION 3.3. A curve $\bar{\gamma}: \mathbb{R} \to \bar{M}$ is A-almost minimizing if $A > 0$ is a real number such that for all $s_0 < s_1$:

$$L(\bar{\gamma}|[s_0, s_1]) = s_1 - s_0 \leq \bar{d}(\bar{\gamma}(s_0), \bar{\gamma}(s_1)) + A.$$

Remark. In Theorem 3.2 it suffices to assume that a lift of c to $\bar{M}$ is almost minimizing (for some $A > 0$).

LEMMA 3.4. *Suppose $\gamma: \mathbb{R} \to M$ has an A-almost minimizing lift $\bar{\gamma}: \mathbb{R} \to \bar{M}$. Then for every $\varepsilon > 0$ there exists $C_3 = C_3(\varepsilon, A)$ such that*

$$(1 - \varepsilon)\|v(\gamma|[s_0, s_1])\| \leq s_1 - s_0 \leq (1 + \varepsilon)\|v(\gamma|[s_0, s_1])\|$$

whenever $s_1 - s_0 \geq C_3$.

Proof. The left hand inequality holds for every sufficiently long curve. This follows from Lemma 2.3. To prove the right hand inequality we proceed as in the proof of 2.3. We consider the closed curve $\beta = \gamma_1 * (\gamma|[s_0, s_1])$ where γ_1 is a shortest geodesic from $\gamma(s_1)$ to $\gamma(s_0)$. In particular we have $L(\beta) \leq s_1 - s_0 + \operatorname{diam}(M)$ and $\|v(\beta) - v(\gamma|[s_0, s_1])\| < A_1$ for some constant $A_1 > 0$ depending on $\operatorname{diam}(M)$ and $\omega^1, \ldots, \omega^k$. We want to show that β has almost minimal length in its homology class, i.e. there exists $A_2 > 0$ independent of s_0 and s_1 such that $L(\beta) \leq f(v(\beta)) + A_2$. Suppose the closed curve α has the same initial point p as β and $v(\alpha) = v(\beta)$. Then $v(\alpha^{-1} * \beta) = 0$, i.e. $\alpha^{-1} * \beta$ represents an element in the kernel of the (extended) Hurewicz homomorphism $\pi_1(M, p) \to H_1(M, \mathbb{Z})_{\mathbb{R}}$. Hence the endpoints of lifts $\bar{\alpha}, \bar{\beta}$ of α, β to $\bar{M}$ with the same initial point $\bar{p}$ are mapped to each other by an element of the torsion subgroup $\bar{T}$ of the deck transformation group $\bar{\Gamma} \simeq H_1(M, \mathbb{Z})$ of $\bar{M}$. Since $\bar{\Gamma}$ is abelian and $\bar{M}/\bar{\Gamma} \simeq M$ is compact the displacement function $x \in \bar{M} \to \bar{d}(x, Fx)$ of every $F \in \bar{\Gamma}$ is bounded. Since $\bar{T}$ is finite there exists a constant A_3 such that the endpoints $\bar{q}, \bar{r}$ of $\bar{\alpha}, \bar{\beta}$ have distance less than A_3. Since $\bar{\gamma}$ is A-almost minimizing we conclude:

$$s_1 - s_0 \leq \bar{d}(\bar{p}, \bar{q}) + \operatorname{diam}(M) + A.$$

The preceding arguments show:

$$\bar{d}(\bar{p}, \bar{q}) \leq \bar{d}(\bar{p}, \bar{r}) + A_3 \leq L(\alpha) + A_3.$$

Hence $L(\beta) \leq s_1 - s_0 + \operatorname{diam}(M) \leq L(\alpha) + 2\operatorname{diam}(M) + A + A_3$. Finally, if α is an arbitrary closed curve with $v(\alpha) = v(\beta)$ we obtain $L(\beta) \leq L(\alpha) + 4\operatorname{diam}(M) + A + A_3$, hence

$$L(\beta) \leq f(v(\beta)) + A_2,$$

where $A_2 = 4\operatorname{diam}(M) + A + A_3$. Now we apply Proposition 2.1 to $v(\beta)$ and $v(\gamma\|[s_0, s_1])$ and conclude

$$|f(v(\beta)) - \|v(\gamma|[s_0, s_1])\|| < \delta \|v(\gamma|[s_0, s_1])\|$$

provided $\delta > 0$ and $|v(\gamma|[s_0, s_1])| > C_1(A_1, \delta)$. Then

$$s_1 - s_0 \leq f(v(\beta)) + A_2 \leq (1+\delta)\|v(\gamma|[s_0, s_1])\| + A_2.$$

Since A_2 is independent of s_0 and s_1 this implies the right hand inequality of our claim.

The next lemma relies on a cut-and-paste technique which has been used in [**3**, Lemma 2], in a different context.

LEMMA 3.5. *Let $\bar{\gamma}: \mathbb{R} \to \bar{M}$ be a lift of a curve $\gamma: \mathbb{R} \to M$ and let $[s_0, s_0 + a]$ and $[s_1, s_1 + b]$ be real intervals with $s_0 + a \leq s_1$. Then there exists a curve $\beta: \mathbb{R} \to M$ with lift $\bar{\beta}: \mathbb{R} \to \bar{M}$ such that*:

(a) $\bar{\gamma}(s) = \bar{\beta}(s)$ *if* $s \leq s_0 + a$ *or* $s \geq s_1 + b + 4\operatorname{diam}(M)$.

(b) *There exists* $s_2 \in [s_0 + a,\ s_0 + a + \operatorname{diam}(M)]$ *such that* $\beta(s) = \gamma(s + s_1 - s_2)$ *for all* $s \in [s_2, s_2 + b]$.

Remark. According to (a) the curve $\bar{\beta}$ is $(A + 4\operatorname{diam}(M))$-almost minimizing if $\bar{\gamma}$ is A-almost minimizing. The important point in 3.5 is that the segments $\gamma([s_0, s_0 + a])$, $\gamma([s_1, s_1 + b])$ which may lie far apart on the curve γ follow each other within distance $\leq \operatorname{diam}(M)$ on β.

Proof. Choose deck transformations $G, H \in \bar{\Gamma}$ such that

$$\bar{d}(G(\bar{\gamma}(s_1)), \bar{\gamma}(s_0 + a)) \leq \operatorname{diam}(M)$$

and

$$\bar{d}(H(\bar{\gamma}(s_0 + a)), G(\bar{\gamma}(s_1 + b))) \leq \operatorname{diam}(M).$$

Since $\bar{\Gamma}$ is commutative and consists of isometries we have

$$\bar{d}(H(\bar{\gamma}(s_1)), \bar{\gamma}(s_1 + b)) \leq \bar{d}(H(\bar{\gamma}(s_1)), HG^{-1}(\bar{\gamma}(s_0 + a))) + \bar{d}(G^{-1}H(\bar{\gamma}(s_0 + a)), \bar{\gamma}(s_1 + b)) \leq 2\operatorname{diam}(M).$$

We choose curves $\bar{\beta}_1$ from $\bar{\gamma}(s_0 + a)$ to $G(\bar{\gamma}(s_1))$, $\bar{\beta}_2$ from $G(\bar{\gamma}(s_1 + b))$ to $H(\bar{\gamma}(s_0 + a))$ and $\bar{\beta}_3$ from $H(\bar{\gamma}(s_1))$ to $\bar{\gamma}(s_1 + b)$ such that $L(\bar{\beta}_1) \leq \operatorname{diam}(M)$, $L(\bar{\beta}_2) \leq \operatorname{diam}(M)$ and $L(\bar{\beta}_3) \leq 2\operatorname{diam}(M)$. Then we define

$$\bar{\beta} = (\bar{\gamma}|(-\infty, s_0 + a]) * \bar{\beta}_1 * (G \circ \bar{\gamma}|[s_1, s_1 + b]) * \bar{\beta}_2 * (H \circ \bar{\gamma}|[s_0 + a, s_1]) * \bar{\beta}_3 * (\bar{\gamma}|[s_1 + b, \infty)).$$

Then $\bar{\beta}$ satisfies (a) and $\beta = \bar{\pi} \circ \bar{\beta}$ satisfies (b) with $s_2 = s_0 + a + L(\bar{\beta}_1)$. □

The crux in the proof of 3.2 is that - a priori - we can prove the following lemma only for segments of equal (or comparable) lengths.

LEMMA 3.6. *Suppose $\tilde{\gamma}:\mathbb{R}\to\tilde{M}$ is an A-almost minimizing lift of $\gamma:\mathbb{R}\to M$. Let $\varepsilon>0$ and $k\in\mathbb{N}$ be given. There exists $C_4=C_4(\varepsilon, A, k)>0$ such that the following is true for all $n\le k$, $a\ge C_4$: if the intervals $[s_0, s_0+a],\dots,[s_n, s_n+a]$ have disjoint interiors then*

$$\left\|\sum_{i=0}^{n} R(\gamma|[s_i, s_i+a])\right\|\ge(n+1)-\varepsilon.$$

Proof. Increasing the constant A by $4k\operatorname{diam}(M)$ we may assume that we have $s_i+a\le s_{i+1}\le s_i+a+\operatorname{diam}(M)$ for $0\le i<n$. This follows by repeated application of Lemma 3.5. We set

$$r=n+1-\left\|\sum_{i=0}^{n} R(\gamma|[s_i, s_i+a])\right\|.$$

We will derive an estimate for r which shows that r converges to zero if a gets arbitrarily large. According to 3.4 there exists $\delta=\delta(a)$ with $\lim_{a\to\infty}\delta(a)=0$ such that for all s_0, s_1 with $s_1-s_0\ge a$:

$$(1-\delta)(s_1-s_0)\le\|v(\gamma|[s_0, s_1])\|\le(1+\delta)(s_1-s_0). \tag{3.7}$$

Note that in any normed vector space the equalities $\|\sum_{i=0}^{n} v_i\|=n+1-r$ and $\|v_i\|=1$ imply: if $\rho_i\in\mathbb{R}^+$ and $0<\rho\le\min_{0\le i\le n}\rho_i$ then

$$\left\|\sum_{i=0}^{n}\rho_i v_i\right\|\le\rho(n+1-r)+\sum_{i=0}^{n}(\rho_i-\rho).$$

Hence the definition of r and the preceding inequalities imply

$$\left\|\sum_{i=0}^{n} v(\gamma|[s_i, s_i+a])\right\|\le(1-\delta)(n+1-r)a+2\delta(n+1)a.$$

Since

$$v(\gamma|[s_0, s_n+a])=\sum_{i=0}^{n} v(\gamma|[s_i, s_i+a])+\sum_{i=0}^{n-1} v(\gamma|[s_i+a, s_{i+1}])$$

and

$$\|v(\gamma|[s_i+a, s_{i+1}])\|<A_1$$

for some constant A_1 depending on $\operatorname{diam}(M)$ and $\omega^1,\dots,\omega^k$, we obtain:

$$\|v(\gamma|[s_0, s_n+a])\|\le(1-\delta)(n+1-r)a+2\delta(n+1)a+nA_1.$$

Using (3.7) again we obtain

$$s_n+a-s_0\le(n+1-r)a+(1-\delta)^{-1}(2\delta(n+1)a+nA_1).$$

On the other hand $s_{i+1}\ge s_i+a$ implies that $s_n+a-s_0\ge(n+1)a$. The last two inequalities yield the following estimate on r:

$$r\le a^{-1}(1-\delta)^{-1}nA_1+(1-\delta)^{-1}2\delta(n+1).$$

Remembering that $n\le k$ and that $\lim_{a\to\infty}\delta(a)=0$ we see that r can be made arbitrarily small by choosing a sufficiently large.

The following lemma will be used to overcome the restriction in Lemma 3.6 that the intervals have equal lengths. Remember that B denotes the unit ball in $(H_1(M,\mathbb{R}), \|\ \|)$.

LEMMA 3.8. *Let $(V_m)_{m\in\mathbb{N}}$ be a sequence of subsets of ∂B such that V_{m+1} is contained in the convex cone $C(V_m)$ generated by V_m. Let $(\varepsilon_m)_{m\in\mathbb{N}}$ be a sequence with* $\lim \varepsilon_m = 0$ *such that $\|v_0 + \cdots + v_n\| \ge n+1-\varepsilon_m$ whenever $n \le k = \dim H_1(M,\mathbb{R})$ and $v_0, \ldots, v_n$ are pairwise different elements in V_m. Then there exists a supporting hyperplane H to B such that for every neighborhood U of $H \cap B$ there exists $m_0 \in \mathbb{N}$ such that $(C(V_m) \cap \partial B) \subseteq U$ for all $m \ge m_0$.*

Proof. Suppose $\sum_{i=0}^{n} t_i v_i$ lies in the simplex generated by the v_i, i.e. $t_i \ge 0$ and $\sum_{i=0} t_i = 1$. Using $\|v_i\| = 1$ we obtain:

$$\left\| \sum_{i=0}^{n} v_i \right\| \le \left\| \sum_{i=0}^{n} t_i v_i \right\| + \sum_{i=0}^{n} \|(1-t_i)v_i\| \le \left\| \sum_{i=0}^{n} t_i v_i \right\| + n.$$

If $\{v_0, \ldots, v_n\} \subseteq V_m$ our assumption implies:

$$\left\| \sum_{i=0}^{n} t_i v_i \right\| \ge 1 - \varepsilon_m. \tag{3.9}$$

By Carathéodory's Theorem, cf. [13, Satz 2.4], the convex hull of V_m in $H_1(M,\mathbb{R})$ is the union of all n-simplices with vertices in V_m and $n \le k$. Hence we can use (3.9) to conclude that the closure of the convex hull of V_m is disjoint from

$$B(1-2\varepsilon_m) = \{v \in H_1(M,\mathbb{R}) \mid \|v\| \le 1 - 2\varepsilon_m\}.$$

Then there exists a supporting hyperplane H_m to $B(1-2\varepsilon_m)$ such that the interior of $B(1-2\varepsilon_m)$ and V_m are separated by H_m. Let H be a hyperplane which is a limit of a subsequence of the H_m. Since $(\varepsilon_m)_{m\in\mathbb{N}}$ converges to zero H is a supporting hyperplane to B. Let U be a neighborhood of $H \cap B$. To complete the proof of Lemma 3.8 we will show that there exists $m_0 \in \mathbb{N}$ such that for $m \ge m_0$ we have $(C(V_m) \cap \partial B) \subseteq U$. Otherwise we can find a sequence $v_i \in (C(V_{m_i}) \cap \partial B)$ with $\lim m_i = \infty$ such that $\lim v_i = v$ exists and $v \notin H \cap B$. Now we use our assumption that $C(V_{m+1}) \subseteq C(V_m)$. Since H_m separates the interior of $B(1-2\varepsilon_m)$ from the convex hull of V_m the set $C(V_m) \cap \partial B$ is disjoint from the open halfspace of H_m containing $0 \in H_1(M,\mathbb{R})$. Now

$$v_i \in (C(V_{m_i}) \cap \partial B) \subseteq (C(V_m) \cap \partial B) \quad \text{for all } m \le m_i.$$

Hence $v = \lim v_i$ does not lie in the open halfspace of H containing 0. Since $v \in \partial B$ and H is a supporting hyperplane to B we obtain $v \in H \cap B$. This contradicts our assumption on v and completes the proof of Lemma 3.8. □

After all this preparation we can prove Theorem 3.2: we assume that $c \in \bar{\mathcal{M}}$, i.e. a lift $\bar{c}: \mathbb{R} \to \bar{M}$ of c is arclength-minimizing. Actually it suffices that $\bar{c}$ is almost minimizing. For every $m \in \mathbb{N}$ we consider the set

$$V_m = \{R(c \mid [i2^m, (i+1)2^m]) \mid i \in \mathbb{Z}\} \subseteq H_1(M,\mathbb{R}).$$

Now we use Lemma 3.6 for $k = \dim H_1(M,\mathbb{R})$. We obtain a sequence ε_m converging to zero such that $\|v_0 + \cdots + v_n\| \ge n+1-\varepsilon_m$ whenever $n \le k$ and $v_0, \ldots, v_n$ are pairwise different elements in V_m. Since $v(c|[i2^{m+1},(i+1)2^{m+1}])$ is the sum of two vectors of the type $v(c|[j2^m,(j+1)2^m])$ we conclude that V_{m+1} is contained in the convex cone $C(V_m)$ generated by V_m. Now Lemma 3.8 provides a supporting hyperplane H to B such that our claim is true if we only consider rotation vectors of type $R(c|[i2^m,(i+1)2^m])$ for $m\in\mathbb{N}$, $i\in\mathbb{Z}$. To obtain the general case we choose a neighborhood U' of $H\cap B$ with compact closure in the interior of U. Our preceding arguments show that there exists $m\in\mathbb{N}$ such that $(C(V_m)\cap\partial B)\subseteq U'$. We set

$$A=\sup\{\|v(c|[s_0,s_1])\|\,|\,0\le s_1-s_0\le 2^m\}.$$

We can find $\tilde{C}>0$ such that the following is true: if $\|w\|^{-1}w\in U'$, $z\in H_1(M,\mathbb{R})$ and $\|w\|>\tilde{C}$, $\|z\|\le 2A$ then $\|w+z\|^{-1}(w+z)\in U$. Finally we choose $C>2^{m+2}$ such that $s_1-s_0>C-2^{m+1}$ implies $\|v(c|[s_1,s_0])\|>\tilde{C}$, cf. Lemma 3.4. We will show that $R(c|[s_1,s_0])\in U$ if $s_1-s_0>C$: choose $i<j$ in $\mathbb{Z}$ such that

$$[s_0,s_1]=[s_0,i2^m]\cup[i2^m,j2^m]\cup[j2^m,s_1]$$

where

$$0\le i2^m-s_0<2^m,\,0\le s_1-j2^m<2^m \quad\text{and hence}\quad j2^m-i2^m>C-2^{m+1}.$$

Then $v(c|[s_0,s_1])=w+z$ with $w=v(c|[i2^m,j2^m])\in C(V_m)$, $\|w\|>\tilde{C}$ and

$$z=v(c|[s_0,i2^m])+v(c|[j2^m,s_1]),\ \|z\|\le 2A.$$

Hence $R(c|[s_0,s_1])=\|w+z\|^{-1}(w+z)\in U$ by the choice of $\tilde{C}$. This proves Theorem 3.2.

At this stage one might hope that the set of those minimal geodesics for which the statement in Theorem 3.2 is satisfied with a fixed supporting hyperplane H is closed since this is true if M is homeomorphic to T^2. However, in § 5 we shall see that such statement is not true for any supporting hyperplane in the Hedlund examples. Maybe this indicates that for some purposes 'minimality' should be replaced by a stronger condition, cf, Remark 1 preceding Theorem 4.5. Nevertheless, if c_j is a sequence in $\bar{\mathcal{M}}$ converging to $c\in\bar{\mathcal{M}}$, i.e. $\lim \dot{c}_j(0)=\dot{c}(0)$, then the rotation vectors of c_j and c are not completely independent of each other as will be shown below. We choose a supporting hyperplane H to B such that $H\cap B$ is the smallest 'face' of B containing the set of limits of rotation vectors $R(c|[s_0,s_1])$ for $s_1-s_0\to\infty$, i.e. if H' supports B and contains the above set then $(H\cap B)\subseteq(H'\cap B)$. This 'face' $H\cap B$ is uniquely determined by c and will be denoted by $F(c)$. Note that such 'faces' may very well be 0-dimensional. Finally we let $W(c)$ denote the union of all straight line segments on ∂B which contain a point of $F(c)$. Put differently we have $w\in W(c)$ if and only if there exists $v\in F(c)$ such that

$$\|v+w\|=\|v\|+\|w\|=2.$$

THEOREM 3.10. *Suppose the minimal geodesics $c_j\in\bar{\mathcal{M}}$ converge to $c\in\bar{\mathcal{M}}$. Then for every neighborhood U of $W(c)$ there exist $j_0\in\mathbb{N}$ and $C>0$ such that $R(c_j|[s_0,s_1])\in U$ whenever $j\ge j_0$ and $s_1-s_0>C$.*

Remark. In particular, if $v\in\partial B$ does not lie on any non-trivial straight line segment in ∂B then the set of $c\in\bar{\mathcal{M}}$ with $F(c)=\{v\}$ is closed. However, such points v need not exist.

Proof. We use the same idea and the same tools as in the proof of Theorem 3.2. There exists $\varepsilon > 0$ such that $\|v+w\| < 2-\varepsilon$ for all $v \in F(c)$, $w \in \partial B \setminus U$. For this $\varepsilon > 0$ we choose $\delta > 0$ such that the following is true: if $L^+ \subseteq H_1(M, \mathbb{R})$ is a halfspace disjoint from

$$B(1-2\delta) = \{v \in H_1(M, \mathbb{R}) \mid \|v\| \leq 1-2\delta\},$$

then $\|v+w\| > 2-\varepsilon$ for all $v, w \in (L^+ \cap \partial B)$. According to Lemma 3.6 and the first few lines in the proof of Lemma 3.8 there exists $m(\delta) \in \mathbb{N}$ such that for every $\tilde{c} \in \bar{\mathcal{M}}$ the closed convex hull $K_m(\tilde{c})$ of

$$V_m(\tilde{c}) = \{R(\tilde{c} \mid [i2^m, (i+1)2^m] \mid i \in \mathbb{Z}\}$$

is disjoint from $B(1-\delta)$ for all $m \geq m(\delta)$. Using Theorem 3.2 we fix some $m \geq m(\delta)$ such that $\|v+w\| < 2-\varepsilon$ for all $v \in V_m(c)$, $w \in \partial B \setminus U$. Since

$$\lim_{j\to\infty} R(c_j \mid [0, 2^m]) = R(c \mid [0, 2^m]),$$

since $K_n(c_j) \cap B(1-\delta) = \emptyset$ for $n \geq m$ and since the convex cones $C(V_n(c_j)) = C(K_n(c_j))$ generated by $V_n(c_j)$ decrease monotonically with n (cf., the proof of Theorem 3.2) there exists $j_0 \in \mathbb{N}$ such that for $j \geq j_0$ we can find halfspaces L_j^+ with the following properties:

(a) $L_j^+ \cap B(1-2\delta) = \emptyset$,

(b) $R(c \mid [0, 2^m]) \in L_j^+$,

(c) $K_n(c_j) \subseteq L_j^+$ for all $n \geq m$.

By (a)-(c) and by the choice of δ we have

$$\|R(c \mid [0, 2^m]) + w\| > 2-\varepsilon,$$

for all $w \in \bigcup_{j \geq j_0} (\bigcup_{n \geq m} K_n(c_j))$. Since $R(c \mid [0, 2^m]) \in V_m(c)$ our choice of m implies $\bigcup_{j \geq j_0} (\bigcup_{n \geq m} K_n(c_j)) \subseteq U$. Now our claim follows from the arguments used at the end of the proof of Theorem 3.2.

Remark. If we only consider limit geodesics of a fixed $c \in \bar{\mathcal{M}}$ then the situation is much simpler than in the general case discussed in Theorem 3.10. We let $\bar{\mathcal{M}}(c)$ denote the set of geodesics $\tilde{c}: \mathbb{R} \to M$ such that there exists a sequence t_i in $\mathbb{R}$ with $\dot{\tilde{c}}(0) = \lim_{i\to\infty} \dot{c}(t_i)$. Then $\bar{\mathcal{M}}(c) \subseteq \bar{\mathcal{M}}$ is a compact set of minimal geodesics. Theorem 3.2 shows that the face $F(\tilde{c}) \subseteq \partial B$ determined by an arbitrary $\tilde{c} \in \bar{\mathcal{M}}(c)$ is contained in $F(c)$. In particular there exists a minimal set of the geodesic flow such that the face of ∂B determined by the geodesics in this minimal set is contained in $F(c)$.

Finally we specialize Theorem 3.2 to the case $M = T^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $\tilde{c}: \mathbb{R} \to \mathbb{R}^n$ be a lift of $c \in \mathcal{M}$. Then we can identify $v(c \mid [s_0, s_1]) \in H_1(T^n, \mathbb{R})$ with $\tilde{c}(s_1) - \tilde{c}(s_0) \in \mathbb{R}^n$. Using Lemma 3.4 we can state Theorem 3.2 in the following form: there exists a supporting hyperplane H to B such that for every neighborhood U of $H \cap B$ there exists $C > 0$ such that

$$(s_1 - s_0)^{-1}(\tilde{c}(s_1) - \tilde{c}(s_0)) \in U,$$

whenever $s_1 - s_0 > C$.

If g_0 and g_1 are Riemannian metrics on T^n with

$$|g_1(X, X) - g_0(X, X)| < \varepsilon g_1(X, X) \quad \text{for all } X \in T(T^n)$$

then the corresponding stable norms $\| \ \|_0$ and $\| \ \|_1$ satisfy

$$(1-\varepsilon)\| \ \|_1 \le \| \ \|_0 \le (1+\varepsilon)\| \ \|_1$$

and hence $(1-\varepsilon)B_0 \subseteq B_1 \subseteq (1+\varepsilon)B_0$ for the corresponding unit balls. This follows from the fact that the induced distances $\tilde{d}_0, \tilde{d}_1$ on $\mathbb{R}^n$ satisfy such inequalities, cf. the discussion in § 2.

If g_0 is a flat metric then B_0 is an ellipsoid. Hence, if g_1 is C^0-close to a flat g_0 then the intersections of B_1 with its supporting hyperplanes will have small diameter. For such g_1 the restrictions on the rotation vectors of minimal geodesics $c \in \mathcal{M}$ are pretty strong.

4. *Existence of minimal geodesics*

In this section we show that for every supporting hyperplane H to B there exists a minimal geodesic $c \in \bar{\mathcal{M}}$ such that the limits of the rotation vectors $R(c|[s_0, s_1])$ for $s_1 - s_0 \to \infty$ are contained in $H \cap B$. From this one can easily derive the existence of at least k geometrically distinct minimal geodesics where $k = \dim H_1(M, \mathbb{R})$. The geodesics that we construct have a minimality property which is - at least a priori - stronger than the one defining $\bar{\mathcal{M}}$, cf., 3.1. However, we do not know of an example of a $c \in \bar{\mathcal{M}}$ which does not satisfy this stronger condition.

Let ω denote a closed 1-form such that $0 \ne [\omega] \in H^1(M, \mathbb{R})$.

DEFINITION 4.1. A geodesic $c \in \bar{\mathcal{M}}$ is called $[\omega]$-minimal if there exists $A > 0$ such that for all $s_1 > s_0$

$$L(c|[s_0, s_1]) \le \inf\left\{L(\gamma) \,\middle|\, \gamma:[a, b] \to M, \quad \int_\gamma \omega = \int_{c|[s_0, s_1]} \omega\right\} + A.$$

Note that this is well-defined since the choice of a different representative of $[\omega]$ will only affect the constant A. Moreover $[\omega]$-minimality implies $[\lambda\omega]$-minimality for all $\lambda \in \mathbb{R}\backslash\{0\}$.

To construct minimal geodesics for a given supporting hyperplane H to B we proceed as follows: choose a closed 1-form ω such that

$$H = \{v \in H_1(M, \mathbb{R}) \,|\, [\omega](v) = 1\}. \tag{4.2}$$

Then we prove the existence of $[\omega]$-minimal geodesics $c \in \bar{\mathcal{M}}$. Finally we show that the rotation vectors of an $[\omega]$-minimal c converge to points in $H \cap B$ if H and ω are related by (4.2).

Our first step is to analyze the function

$$g_\omega(t) = g(t) := \inf\left\{L(\gamma) \,\middle|\, \gamma:[a, b] \to M, \quad \int_\gamma \omega = t\right\}.$$

Note that g is defined for all $t \in \mathbb{R}$ since $[\omega] \ne 0$. Obviously we have $g(t) = g(-t)$.

LEMMA 4.3. (a) *There exists $A = A(\omega) > 0$ such that*

$$g(t) + g(s) \le g(t+s) \le g(t) + g(s) + A$$

for all $t \ge 0$, $s \ge 0$.

(b) *We have* $\lim_{t\to\infty}(t/g(t)) = \|[\omega]\|$ *where*

$$\|[\omega]\| = \max_{\|v\|=1} [\omega](v).$$

Proof of (a): the left hand inequality follows from the fact that every γ with $\int_\gamma \omega = t+s$ can be written as $\gamma = \gamma_1 * \gamma_2$ with $\int_{\gamma_1}\omega = t$, $\int_{\gamma_2}\omega = s$. To obtain the right hand inequality we try to reverse this procedure. Suppose γ_1, γ_2 are curves with $\int_{\gamma_1}\omega = t$, $\int_{\gamma_2}\omega = s$. We choose a shortest curve γ_3 from the endpoint of γ_1 to the initial point of γ_2 and consider $\tilde\gamma = \gamma_1 * \gamma_3 * \gamma_2$. Then

$$L(\tilde\gamma) \le L(\gamma_1) + L(\gamma_2) + \operatorname{diam}(M)$$

and

$$\left|\int_{\tilde\gamma} \omega - (s+t)\right| \le C \operatorname{diam}(M) \quad \text{where } C = C(\omega) > 0.$$

Finally we choose a closed curve β in M with $[\omega]([\beta]) = \int_\beta \omega \neq 0$, say $[\omega]([\beta]) = r > 0$. Then through every $p \in M$ there exists a closed curve β_p with $[\beta_p] = [\beta]$ and $L(\beta_p) \le L(\beta) + 2\operatorname{diam}(M)$. Hence for every $u \in \mathbb{R}$, $p \in M$ there exists a curve $\alpha = \alpha(u,p)$ with initial point p satisfying $\int_\alpha \omega = u$ and $L(\alpha) \le (|u| r^{-1} + 1)L(\beta_p)$. Thus we can find a curve γ_4 whose initial point is the endpoint of γ_2 and which satisfies $\int_{\gamma_4}\omega = s + t - \int_{\tilde\gamma}\omega$ and

$$L(\gamma_4) \le (C \operatorname{diam}(M) r^{-1} + 1)(L(\beta) + 2\operatorname{diam}(M)).$$

Now $\gamma = \gamma_1 * \gamma_3 * \gamma_2 * \gamma_4$ satisfies $\int_\gamma \omega = s+t$ and $L(\gamma) \le L(\gamma_1) + L(\gamma_2) + A$, where A only depends on ω and the geometry of M. This implies $g(t+s) \le g(t) + g(s) + A$.

Proof of (b). We choose $v \in H_1(M, \mathbb{R})$ with $\|v\| = 1$ and $\|[\omega]\| = [\omega](v)$. Since $H_1(M, \mathbb{Z})_\mathbb{R}$ is a lattice in $H_1(M, \mathbb{R})$ there exists a constant $B > 0$ and a sequence $v_m \in H_1(M, \mathbb{Z})_\mathbb{R}$ with $\|v_m - mv\| < B$ for all $m \in \mathbb{N}$, in particular $|[\omega](v_m) - m\|[\omega]\|| < \|[\omega]\| B$. From Proposition 2.1 we conclude that $\lim f(v_m)/m = 1$. Hence we can find closed curves γ_m representing v_m such that $\lim (L(\gamma_m)/m) = 1$ and

$$\lim\left(\frac{1}{m}\int_{\gamma_m}\omega\right) = \lim \frac{1}{m}[\omega](v_m) = \|[\omega]\|.$$

If we set $t_m = \int_{\gamma_m}\omega = [\omega](v_m)$ then $g(t_m) \le L(\gamma_m)$ and hence

$$\liminf_{m\to\infty} (t_m/g(t_m)) \ge \lim (t_m/L(\gamma_m)) = 1.$$

Using (a) we easily obtain $\liminf_{t\to\infty}(t/g(t)) \ge 1$. Finally we prove $\limsup_{t\to\infty}(t/g(t)) = 1$. If γ is a curve with $\int_\gamma \omega = t > 0$ then $[\omega](v(\gamma)) = t$ and hence $t \le \|[\omega]\| \, \|v(\gamma)\|$. Now Lemma 2.3 implies that for every $\delta > 0$ there exists $t_0 > 0$ such that $\|v(\gamma)\| \le (1+\delta)L(\gamma)$ for all $t \ge t_0$. Hence $t \le \|[\omega]\|(1+\delta)g(t)$ for $t \ge t_0$ and this completes the proof of (b).

THEOREM 4.4. *For every* $[\omega] \neq 0$ *in* $H^1(M, \mathbb{R})$ *there exists an* $[\omega]$*-minimal* $c \in \bar{\mathcal{M}}$.

Proof. For every $i \in \mathbb{N}$ we choose an arclength-parametrized geodesic $c_i : [-s_i, s_i] \to M$ such that $\int_{c_i}\omega = 2i$ and $2s_i = L(c_i) = g(2i)$. To find such a geodesic we choose arclength-parametrized curves γ_ε in M such that $\int_{\gamma_\varepsilon}\omega = 2i$ and $L(\gamma_\varepsilon) \le g(2i) + \varepsilon$.

Then we replace γ_ε by a shortest geodesic c_ε which is homotopic to γ_ε with fixed endpoints, hence $L(c_\varepsilon) \le L(\gamma_\varepsilon)$ and $\int_{c_\varepsilon} \omega = 2i$. Finally we let c_i be a limit geodesic of the c_ε for $\varepsilon \to 0$. Note that $2s_i = g(2i)$ and 4.3(a) imply that the sequence $(s_i)_{i\in\mathbb{N}}$ is unbounded. We let $c:\mathbb{R}\to M$ denote a limit geodesic of the c_i, i.e. $\dot c(0) = \lim_{n\to\infty} \dot c_{i(n)}(0)$. First we show that $c \in \bar{\mathcal{M}}$. Obviously it suffices to prove that a lift $\bar c_i : [-s_i, s_i] \to \bar M$ of c_i is a shortest connection between its endpoints. Since $\bar M = \tilde M/[\Gamma, \Gamma]$ every curve $\bar\gamma : [a, b] \to \bar M$ with $\bar\gamma(a) = \bar c_i(-s_i)$, $\bar\gamma(b) = \bar c_i(s_i)$ satisfies $\int_{\bar\gamma} \bar\omega = \int_{\bar c_i} \bar\omega = 2i$ where $\bar\omega = \bar\pi^* \circ \omega$. For $\gamma = \bar\pi \circ \bar\gamma$ this implies $\int_\gamma \omega = 2i$, hence

$$L(\gamma) = L(\bar\gamma) \ge g(2i) = L(c_i) = L(\bar c_i).$$

This proves that $c \in \bar{\mathcal{M}}$. Finally we have to show that c is $[\omega]$-minimal. Using $\int_{c_i} \omega = 2i$, $L(c_i) = g(2i)$ and Lemma 4.3(a) we easily obtain for all $[a, b] \subseteq [-s_i, s_i]$:

$$L(c_i|[a, b]) \le \inf\left\{ L(\gamma) \,\middle|\, \int_\gamma \omega = \int_{c_i|[a,b]} \omega \right\} + 2A,$$

where $A = A(\omega)$ is the constant appearing in 4.3(a). Since a subsequence of the c_i converge to c we obtain an analogous estimate for c, i.e. c is $[\omega]$-minimal.

Remark 1. If we fix ω and A then the set of $c \in \bar{\mathcal{M}}$ which satisfy the condition in the Definition 4.1 of $[\omega]$-minimality is closed.

Remark 2. The constant A above only depends on ω and the geometry of M. If we only choose representatives ω in a fixed finite-dimensional subspace of the space of closed 1-forms - say we choose only harmonic forms - then the constant A can be chosen uniformly for all ω such that $H = \{v \in H_1(M, \mathbb{R}) \,|\, [\omega](v) = 1\}$ is a supporting hyperplane to B.

THEOREM 4.5. *Suppose the supporting hyperplane H to B and $[\omega] \in H^1(M, \mathbb{R})$ are related by $H = \{v \in H_1(M, \mathbb{R}) \,|\, [\omega](v) = 1\}$. Let $c \in \bar{\mathcal{M}}$ be $[\omega]$-minimal. Then for every neighborhood U of $H \cap B$ there exists $C > 0$ such that $R(c|[s_0, s_1]) \in U$ whenever $s_1 - s_0 > C$.*

Proof. There exists $\delta = \delta(U) > 0$ such that $[\omega](v) \le 1 - \delta$ for every $v \in \partial B \setminus U$. Hence

$$[\omega](v(c|[s_0, s_1])) = \int_{c|[s_0, s_1]} \omega \le (1-\delta)\|v(c|[s_0, s_1])\|,$$

whenever $R(c|[s_0, s_1]) \notin U$. Moreover Lemma 3.4 implies that for every $\delta > 0$ there exists $C_3 = C_3(2\delta)$ such that

$$\|v(c|[s_0, s_1])\| \le \left(1 + \frac{\delta}{2}\right)(s_1 - s_0) \quad \text{if } s_1 - s_0 > C_3.$$

Here we assume that c is parametrized by arclength. Hence

$$\int_{c|[s_0,s_1]} \omega \le \left(1 - \frac{\delta}{2}\right)(s_1 - s_0), \tag{4.6}$$

if $R(c|[s_0, s_1]) \notin U$ and $s_1 - s_0 > C_3$. On the other hand we will now show that the $[\omega]$-minimality implies that $\int_{c|[s_0,s_1]} \omega$ grows asymptotically like $s_1 - s_0$. Note that

$\|[\omega]\|=1$ by our assumption. Hence Lemma 4.3(b) implies $\lim g(t)^{-1}t=1$. Since c is $[\omega]$-minimal we have

$$g\left(\int_{c|[s_0,s_1]}\omega\right)\le s_1-s_0\le g\left(\int_{c|[s_0,s_1]}\omega\right)+A.$$

This implies

$$\lim_{t\to\infty}\left(\frac{1}{t}\int_{c|[s,s+t]}\omega\right)=1 \tag{4.7}$$

uniformly in s. Now (4.7) contradicts (4.6) unless our claim is true.

Theorems 4.4 and 4.5 prove our claim that for every supporting hyperplane H of B there exists $c\in\bar{\mathcal{M}}$ such that the limits of the rotation vectors of c are contained in $H\cap B$. However, this does not mean that for different supporting hyperplanes H_1, H_2 we get different geodesics c_1, c_2 in $\bar{\mathcal{M}}$. Only if $H_1\cap H_2\cap B=\varnothing$ and $H_1\cap(-H_2)\cap B=\varnothing$ are we sure that we obtain geometrically distinct geodesics $c_1, c_2\in\bar{\mathcal{M}}$, i.e. $c_1(\mathbb{R})\neq c_2(\mathbb{R})$.

THEOREM 4.8. *There exist at least $k=\dim H_1(M,\mathbb{R})$ geometrically distinct geodesics $c_1,\ldots,c_k\in\bar{\mathcal{M}}$ with the following additional properties: the limits*

$$\lim\nolimits_{t\to\infty} R(c_i|[s,s+t])=v_i,\ 1\le i\le k,$$

exist uniformly in s and $v_1,\ldots,v_k$ form a basis of $H_1(M,\mathbb{R})$. The c_i, $1\le i\le k$, are $[\omega^i]$-minimal where $[\omega^i]\in H^1(M,\mathbb{R})$ and $[\omega^i](v)=1$ defines a supporting hyperplane H_i of B with $H_i\cap B=\{v_i\}$.

Proof. A point $v\in\partial B$ is an exposed point of B if there exists a supporting hyperplane H of B in v such that $H\cap B=\{v\}$. It is well-known that B is the closed convex hull of its exposed points, cf, [**10**, Satz 4.5]. Since B is centrally symmetric the exposed points of B comes in pairs $v, -v$. Hence we can find exposed points $v_1,\ldots,v_k$ of B which form a basis of $H_1(M,\mathbb{R})$. We choose $[\omega^i]\in H^1(M,\mathbb{R})$ such that $[\omega^i](v)=1$ defines a supporting hyperplane H_i of B with $H_i\cap B=\{v_i\}$. Now our claim follows from Theorems 4.4 and 4.5.

Remark. Actually we proved slightly more than we stated in (4.6): for every $i\in\{1,\ldots,k\}$ there exists a minimal set of the geodesic flow such that the corresponding geodesics c are $[\omega^i]$-minimal (and hence $F(c)=\{v_i\}$), cf, the remark following Theorem 3.10 or Remark 1 following Theorem 4.4.

The Hedlund examples which we discuss in the next section show that Theorem 4.6 is - to a certain extent - optimal. The discussion at the end of the previous section shows that we can obtain arbitrarily large lower bounds for the number of exposed points and hence of distinct minimal geodesics if we consider metrics on T^n which are sufficiently C^0-close to a flat metric g_0.

5. *The Hedlund examples*

At the end of [**9**] Hedlund presents an example of a Riemannian metric on the 3-torus with a closed geodesic c which has minimal length in its homology class while sufficiently high iterates of c do not have minimal length in their homology classes. The fact that this phenomenon cannot occur on orientable surfaces was fundamental for Hedlund's results on minimal geodesics on a 2-torus. So this example indicates that these results do not generalize to higher dimensions. Here we will discuss the minimal geodesics of this example in detail. This will illustrate the results obtained in §§ 3 and 4 in the sense that it shows that the worst possible case can occur.

Although the coefficients of Hedlund's metric are explicit trigonometric polynomials his arguments are qualitative and so will be ours. For this reason we believe that the ideas will become clearer if we start by formulating the C^0-conditions on the metrics which are crucial for our estimates. Metrics satisfying these estimates will be called 'Hedlund examples'. They are far away from the flat metrics even in the C^0-topology. Following Hedlund we only treat the 3-dimensional case. The generalization to higher dimensions is obvious.

In $\mathbb{R}^3$ we consider the straight lines $l_1 = \mathbb{R} \times \{0\} \times \{0\}$, $l_2 = \{0\} \times \mathbb{R} \times \{\frac{1}{2}\}$, $l_3 = \{\frac{1}{2}\} \times \{\frac{1}{2}\} \times \mathbb{R}$. For $i = 1, 2, 3$ we set

$$L_i = \bigcup_{k \in \mathbb{Z}^3} (l_i + k)$$

and $L = \bigcup_{i=1}^3 L_i$. The standard euclidean scalar product on $\mathbb{R}^3$ with orthonormal basis e_1, e_2, e_3 will be denoted by $\langle\ ,\ \rangle$. The euclidean ε-neighborhoods $U_\varepsilon(L_i)$, $U_\varepsilon(L)$ of L_i and L consist of tubes $U_\varepsilon(l)$ around straight lines l in L_i, respectively L. The fact that these tubes are pairwise disjoint if $\varepsilon < \frac{1}{4}$ is of fundamental importance and it is precisely this point which would fail in the 2-dimensional case.

Definition 5.1. A Hedlund example is a $\mathbb{Z}^3$-periodic Riemannian metric g on $\mathbb{R}^3$ satisfying the following C^0-conditions for some $\varepsilon \in (0, 10^{-2})$:

(P1) $g_x(v, v) \leq (1+\varepsilon)^2 \langle v, v\rangle$ for all $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$.

(P2) $\varepsilon_i^2 := \min\{g_x(v, v) \mid x \in U_\varepsilon(L_i), \langle v, v\rangle = 1\} < \varepsilon^2$ for $i = 1, 2, 3$. This minimum is attained for $x \in L_i$, $v = \pm e_i$. Moreover $g_x(v, v) > \varepsilon_i^2$ if $x \in U_\varepsilon(L_i) \backslash L_i$ and $\langle v, v\rangle = 1$.

(P3) $g_x(v, v) \geq \langle v, v\rangle$ if $x \notin U_\varepsilon(L)$.

Remark. Maybe the simplest Hedlund examples are those conformally equivalent to $\langle\ ,\ \rangle$:

$$g_x(v, w) = \phi^2(x)\langle v, w\rangle,$$

where $\phi: \mathbb{R}^3 \to (0, 1+\varepsilon)$ is $\mathbb{Z}^3$-periodic, $\phi \geq 1$ outside $U_\varepsilon(L)$, $\min(\phi \mid U_\varepsilon(L_i)) =: \varepsilon_i < \varepsilon$ and $\phi(x) = \varepsilon_i$ for $x \in U_\varepsilon(L_i)$ if and only if $x \in L_i$.

Properties (P2) and (P3) have the following immediate consequences:

(1) The straight lines in L are (g-) minimal geodesics and they are the only ones contained in $U_\varepsilon(L)$. The g-length of a segment on such a straight line is smaller than ε times its euclidean length.

(2) The g-length of a curve outside $U_\varepsilon(L)$ is not smaller than its euclidean length. In particular the g-distance between ε-tubes around different lines in L is at least $\frac{1}{2} - 2\varepsilon$.

Now we present the results on the minimal geodesics of the Hedlund examples. The proofs will be given at the end of this section.

PROPOSITION 5.2. *For every arclength-parametrized minimal geodesic segment* $c:[a,b]\to\mathbb{R}^3$ *the length* $\lambda(A)$ *of* $A=A(c)=\{s\in[a,b]\,|\,c(s)\notin U_\varepsilon(L)\}$ *is smaller than* 4.

So every minimal geodesic spends at most a fixed finite amount of time outside the tubes $U_\varepsilon(L)$, i.e. most of the time it runs inside the tubes. We say that a curve $\gamma: I\to\mathbb{R}^3$ changes tubes n times if there are parameter values $t_0<t_1<\cdots<t_n$ in I such that $\gamma(t_{i-1})$ and $\gamma(t_i)$ lie in different components (=tubes) of $U_\varepsilon(L)$.

PROPOSITION 5.3. *A minimal geodesic segment can change tubes at most three times.*

As a consequence of (5.2), (5.3) and (P2) we obtain:

COROLLARY 5.4. *Every minimal geodesic in* $(\mathbb{R}^3, g)$ *is asymptotic in each of its senses to one of the lines in* L.

If $p=(p^1,p^2,p^3)\in L_i$, $q=(q^1,q^2,q^3)\in L_j$ and $i\neq j$, say $\{i,j,k\}=\{1,2,3\}$, then the 'standard path' connecting x and y is defined as follows: we first follow the line $p+te_i$ to the point z with $z^i=q^i$, then the line $z+te_k$ to the point w with $w^k=q^k$ and then the line $w+te_j$ until we reach q.

PROPOSITION 5.5. *A minimal geodesic segment with initial point* $p\in L_i$ *and endpoint* $q\in L_j$, $i\neq j$, *is contained in the euclidean neighborhood of radius* 2 *around the standard path connecting* p *and* q.

Proposition 5.5 determines rather precisely the trace of a minimal geodesic which is asymptotic to a line $l\subseteq L_i$ for $t\to-\infty$ and to a line $l'\subseteq L_j$, $i\neq j$, for $t\to\infty$. Moreover a simple limit argument using (5.5) proves:

PROPOSITION 5.6. *For each pair of directed lines* $l\subseteq L_i$, $l'\subseteq L_j$ *with* $i\neq j$ *there exists a minimal geodesic* c *which is asymptotic to* l *for* $t\to-\infty$ *and to* l' *for* $t\to\infty$.

That c is asymptotic to a line l with direction vector v for $t\to\infty$ (respectively for $t\to-\infty$) means that $\lim_{t\to\infty} d(c(t),l)=0$ (respectively $\lim_{t\to-\infty} d(c(t),l)=0$) and $\lim_{t\to\infty}\dot c(t)=\lambda v$ (respectively $\lim_{t\to-\infty}\dot c(t)=\lambda v$) for some $\lambda>0$. If l and l' are parallel lines in L with opposite orientations there obviously does not exist a minimal geodesic which is asymptotic to l for $t\to-\infty$ and to l' for $t\to\infty$. Using an argument due to Morse [17] we prove:

PROPOSITION 5.7. *For each pair of parallel lines* l, l' *in* L *with the same orientation there exists a minimal geodesic asymptotic to* l *for* $t\to-\infty$ *and to* l' *for* $t\to\infty$.

There are infinitely many (in $\mathbb{R}^3$) minimal geodesics asymptotic to parallel lines $l\neq l'$ in L - say in L_i - since the translation by ne_i, $n\in\mathbb{Z}$, will map such a minimal geodesic to a different one with the same properties. In this case we do not have as explicit information on the trace of such minimal geodesics as in the case of non-parallel asymptotes. In the course of our estimates we see:

PROPOSITION 5.8. *The stable norm* $\| \;\|$ *on* $\mathbb{R}^3$ *corresponding to a Hedlund example* g *with* $g_x(e_i, e_i) = \varepsilon_i$ *for* $x \in L_i$ *is given by*:

$$\|v\| = \varepsilon_1 |v^1| + \varepsilon_2 |v^2| + \varepsilon_3 |v^3|,$$

where $v = (v^1, v^2, v^3)$. *Hence the unit ball* B *of* $\| \;\|$ *is a centrally symmetric octahedron with vertices on the coordinate axes.* B *is a regular octahedron if* $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$.

We now compare the results on the minimal geodesics of the Hedlund examples with the results in the general case obtained in §§ 3 and 4. Proposition 5.6 provides examples of minimal geodesics $\tilde{c}: \mathbb{R} \to \mathbb{R}^3$, covering a $c \in \mathcal{M}(T^3)$, such that $F(c)$ is an edge of ∂B and consists precisely of the limit vectors of $(s-t)^{-1}(\tilde{c}(s) - \tilde{c}(t))$ for $(s-t) \to \infty$. However, there are no minimal geodesics such that $F(c)$ is a 2-dimensional face of ∂B. On the other hand Propositions 5.6 together with 5.5 show: if e is an edge of ∂B and v a vertex of ∂B such that $e \cup \{v\}$ is contained in a face of ∂B then there exists a sequence $c_i \in \mathcal{M}(T^3)$ with $\lim c_i = c \in \mathcal{M}(T^3)$ such that $F(c_i) = e$ while $F(c) = \{v\}$. Under the same hypotheses Proposition 5.7 together with Proposition 5.3 provide a sequence $c_i \in \mathcal{M}(T^3)$ converging to $c \in \mathcal{M}(T^3)$ such that $F(c_i) = \{v\}$ while $F(c) = \{e\}$. In particular, there is no face F of ∂B such that the set

$$\{c \in \mathcal{M}(T^3) \,|\, F(c) \subseteq F\}$$

is closed. This justifies Theorem 3.10. Finally Corollary 5.4 shows that the existence result 4.8 is optimal, cf., also the remark following Theorem 4.8: the geodesic flow on the unit tangent bundle of a Hedlund example (T^3, g) has precisely $6 = 2 \times 3$ minimal sets such that the corresponding geodesics are in $\mathcal{M}(T^3)$. These minimal sets are the periodic orbits

$$\mathscr{C}_i = \{\dot{c}_i(t) \,|\, t \in \mathbb{R}\} \quad \text{and} \ -\mathscr{C}_i,\ i \in \{1, 2, 3\},$$

where c_i is an arclength-parametrized geodesic whose lifts to $\mathbb{R}^3$ lie on the lines in L_i. Note that all the minimal geodesics in the Hedlund examples are $[\omega]$-minimal for some $[\omega] \in H^1(T^3, \mathbb{R})$. However, for every representative ω of $[\omega]$ there are $[\omega]$-minimal geodesics such that the constant A in Definition 4.1 has to be chosen larger than any prescribed value.

Next we present proofs for the statements 5.2–5.8 on a Hedlund example g. The following estimate is fundamental. It is an elementary consequence of the C^0-conditions (P2), (P3) and the condition $\varepsilon < 10^{-2}$ in Definition 5.1.

LEMMA 5.9. *For a piecewise* C^1*-curve* $\gamma: [a, b] \to \mathbb{R}^3$ *parametrized by arclength set* $A = A(\gamma) = \gamma^{-1}(\mathbb{R}^3 \setminus U_\varepsilon(L))$. *If* $x = \gamma(b) - \gamma(a)$ *then the length* $\lambda(A)$ *of* A *is bounded by*

$$\lambda(A) \le \tfrac{11}{10}\left(L(\gamma) - \sum_{i=1}^{3} \varepsilon_i |x^i|\right) + 10^{-2}.$$

Proof. We set $[a, b] = A \cup A_1 \cup A_2 \cup A_3$ where $A_i = \gamma^{-1}(U_\varepsilon(L_i))$. The proof consists in estimating the variation of γ^i on the sets A and $A_j, j \ne i$. Since $g \ge \langle\ ,\ \rangle$ on

$\mathbb{R}^3\setminus U_\varepsilon(L)$ and $g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))=1$ we have

$$\left|\int_A \dot\gamma^i(s)\, ds\right| \leq \int_A |\dot\gamma(s)|\, ds \leq \lambda(A). \tag{5.10}$$

If $\gamma(s_1)$ and $\gamma(s_2)$ lie in the same tube in $U_\varepsilon(L_j)$, $j\neq i$, then

$$|\gamma^i(s_1)-\gamma^i(s_2)|<2\varepsilon.$$

Hence for $j\neq i$

$$\left|\int_{A_j} \dot\gamma^i(s)\, ds\right| < 2n_j\varepsilon, \tag{5.11}$$

where n_j is the number of tubes in $U_\varepsilon(L_j)$ intersected by γ.

Consequence (2) of (P2), (P3) says that different tubes in $U_\varepsilon(L)$ lie at g-distance $\geq\frac{1}{2}-2\varepsilon$. Hence:

$$\lambda(A)\geq(n_1+n_2+n_3-1)(\tfrac{1}{2}-2\varepsilon). \tag{5.12}$$

Since $\int_a^b \dot\gamma^i(s)\, ds = x^i$ the inequalities (5.10) and (5.11) imply

$$\left|\int_{A_i} \dot\gamma^i(s)\, ds\right| \geq |x^i| - \lambda(A) - 2(n_j+n_k)\varepsilon,$$

where $\{i, j, k\}=\{1, 2, 3\}$. For $s\in A_i$ we have

$$1=g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))\geq\varepsilon_i^2\langle\dot\gamma(s), \dot\gamma(s)\rangle\geq\varepsilon_i^2|\dot\gamma^i(s)|^2.$$

This implies:

$$\lambda(A_i)\geq\varepsilon_i\left|\int_{A_i}\dot\gamma^i(s)\, ds\right|\geq\varepsilon_i(|x^i|-\lambda(A)-2(n_j+n_k)\varepsilon). \tag{5.13}$$

Adding these inequalities and using (5.12) and $\varepsilon_i\leq\varepsilon$ we obtain

$$L(\gamma)\geq\lambda(A)+\sum_{i=1}^{3}\varepsilon_i|x^i|-3\varepsilon\lambda(A)-4\varepsilon^2(\tfrac{1}{2}-2\varepsilon)^{-1}(\lambda(A)+\tfrac{1}{2}). \tag{5.14}$$

This implies our claim by a rough estimate since $\varepsilon<10^{-2}$ according to Definition 5.1.

Using Lemma 5.9 we can easily prove Proposition 5.2: the g-distance $d(p, p+x)$ of two points $p, p+x$ in $\mathbb{R}^3$ can be estimated by

$$d(p, p+x)\leq\sum_{i=1}^{3}\varepsilon_i|x^i|+3. \tag{5.15}$$

This can be seen by estimating the g-length of the following curve γ from p to $p+x$ using properties (P1) and (P2) of g. The curve γ first joins p to a nearby line in L_1, then follows this line until the first coordinate agrees with p^1+x^1, then changes to a nearby line in L_2 and follows this line until the second coordinate agrees with p^2+x^2, then changes to a nearby line in L_3 and follows this line until the third coordinate agrees with p^3+x^3 and then joins this point to $p+x$.

For an arclength-parametrized geodesic $c:[a, b]\to\mathbb{R}^3$ with $c(a)=p$, $c(b)=p+x$ and $L(c)=b-a=d(p, p+x)$ we obtain from Lemma 5.9 and eq. (5.15):

$$\lambda(A)\leq\tfrac{11}{10}\times 3+10^{-2}<4.$$

This proves Proposition 5.2.

Moreover (5.14) and (5.15) imply for all $p \in \mathbb{R}^3$, $x \in \mathbb{R}^3$:

$$\sum_{i=1}^{3} \varepsilon_i |x^i| - 10^{-2} \leq d(p, p+x) \leq \sum_{i=1}^{3} \varepsilon_i |x^i| + 4.$$

According to the discussion at the end of § 2 this implies that the stable norm $\| \ \|$ of g is given by $\|x\| = \sum_{i=1}^{3} \varepsilon_i |x^i|$. This proves Proposition 5.8.

Next we prove Proposition 5.3 which states that a minimal geodesic segment $c:[a, b] \to \mathbb{R}^3$ changes tubes at most three times. So we may assume that $c(a) = p$ and $c(b) = p + x$ are in $U_\varepsilon(L)$. In this case an inspection of the proof of (5.15) shows that we can improve the estimate to:

$$L(c) = d(p, p+x) \leq \sum_{i=1}^{3} \varepsilon_i |x^i| + \tfrac{3}{2} + \tfrac{1}{10}. \tag{5.16}$$

Now Lemma 5.9 implies

$$\lambda(A(c)) \leq \tfrac{11}{10}(\tfrac{3}{2} + \tfrac{1}{10}) + 10^{-2} < \tfrac{9}{5}. \tag{5.17}$$

On the other hand, consequence (2) of (P2) and (P3) implies that $\lambda(A(c)) \geq 4(\frac{1}{2} - 2\varepsilon)$ if c changes tubes four times. Since $4(\frac{1}{2} - 2\varepsilon) > 2 - \frac{1}{10} > \frac{9}{5}$ the geodesic c can change tubes at most three times.

To prove Corollary 5.4 note that a simple estimate based on (P2) shows the following: for all $\delta \in (0, \varepsilon)$ there exists $r(\delta) \geq 0$ such that every arclength-parametrized minimal geodesic segment $c:[a, b] \to \mathbb{R}^3$ with endpoints in a tube $U_\varepsilon(l)$ for some line $l \subseteq L$ satisfies $c(s) \in U_\delta(l)$ for $s \in [a + r(\delta), b - r(\delta)]$. If $c: \mathbb{R} \to \mathbb{R}^3$ is a minimal geodesic then Propositions 5.2 and 5.3 imply the existence of lines l and l' such that $c(s_i) \in U_\varepsilon(l)$ for a sequence $s_i \to \infty$ and $c(s_i') \in U_\varepsilon(l')$ for a sequence $s_i' \to -\infty$.

Now the preceding statement proves that c is asymptotic to l for $s \to \infty$ and to l' for $s \to -\infty$.

To prove Proposition 5.5 let $c:[a, b] \to \mathbb{R}^3$ be an arclength-parametrized minimal geodesic segment such that $c(a) = p \in L_i$, $c(b) = q = p + x \in L_j$ and $i \neq j$. Choosing a curve γ from p to $q = p + x$ which runs on lines in L parallel to the straight line segments of the standard path from p to q - except for one or two changes of tubes - we can improve the estimate (5.15) to:

(a) If $\{i, j, k\} = \{1, 2, 3\}$ and $|x^k| > \frac{1}{2}$ then

$$L(c) = d(p, p+x) \leq \sum_{i=1}^{3} \varepsilon_i |x^i| + 1 + \tfrac{1}{10}$$

(b) If $|x^k| = \frac{1}{2}$ then

$$L(c) = d(p, p+x) \leq \sum_{i=1}^{3} \varepsilon_i |x^i| + \tfrac{1}{2} + \tfrac{1}{10}.$$

Now Lemma 5.9 implies that c can change tubes at most twice in case (a) and at most once in case (b) and that $\lambda(A(c)) \leq \frac{5}{4}$. Using this one can easily complete the proof of Proposition 5.5.

Proposition 5.6 is a simple consequence of Proposition 5.5. Let the directed lines $l \subseteq L_i$, $l' \subseteq L_j$ be given by $l = \{q + tv \mid t \in \mathbb{R}\}$ with direction vector $v \in \{e_i, -e_i\}$ and

$l'=\{p+tw \mid t\in\mathbb{R}\}$ with direction vector $w\in\{e_j, -e_j\}$, $i\neq j$. We choose arclength-parametrized minimal geodesic segments $c_n:[0, a_n]\to\mathbb{R}^3$ with initial points $c_n(0)=p-nv$ and endpoints $c_n(a_n)=q+nw$. All these geodesic segments intersect a fixed compact set $K\subseteq\mathbb{R}^3$ since this is true for the standard paths from $p-nv$ to $q+nw$, say $c_n(s_n)\in K$. Then $\lim s_n=\infty=\lim(a_n-s_n)$ and a geodesic $c:\mathbb{R}\to\mathbb{R}^3$ such that $\dot{c}(0)$ is a limit vector of the sequence $\dot{c}_n(s_n)$ is minimal and asymptotic to l for $s\to-\infty$ and to l' for $s\to\infty$.

The proof of Proposition 5.7 is a little bit more subtle than the preceding ones. We use an estimate invented by M. Morse, cf. [**17**, Theorem 13]. Let

$$\gamma_1(t)=p+te_i,\ \gamma_2(t)=p+k+te_i,\ k\in\mathbb{Z}^3,$$

parametrize the directed parallel lines $l\neq l'$ in L_i. We choose arc-length-parametrized minimal geodesic segments $c_n:[0, d_n]\to\mathbb{R}^3$ from $c_n(0)=p-ne_i$ to $c_n(a_n)=p+k+ne_i$. Let $[0, a_n)$ respectively $(b_n, d_n]$ be maximal intervals such that

$$c_n([0, a_n))\subseteq U_\varepsilon(l) \quad\text{and}\quad c_n((b_n, d_n])\subseteq U_\varepsilon(l').$$

We first prove that b_n-a_n is bounded: the arguments in the proof of Corollary 5.4 show that c_n does not intersect $U_\varepsilon(l)\cup U_\varepsilon(l')$ on the interval $[a_n+r, b_n-r]$ for some $r=r(\varepsilon)>0$ independent of n. Since the g-distance of the ε-tubes about two different parallel lines in L is at least $1-4\varepsilon$ we conclude from (5.17) that c_n does not intersect a tube $U_\varepsilon(l'')$ for some l'' in L_i different from l and l'. Hence (5.16) and (5.17) imply that b_n-a_n is bounded, say $b_n-a_n<A$ for all $n\in\mathbb{N}$. We choose translates $\tilde{c}_n$ of c_n by integer multiples of e_i such that the sequence $\tilde{c}_n(a_n)$ is bounded. Let c be a geodesic such that $\dot{c}(0)$ is a limit vector of the sequence $\dot{\tilde{c}}_n(a_n)$, say $\dot{c}(0)=\lim\dot{\tilde{c}}_n(a_n)$. We distinguish the following three cases:

(a) The sequences a_n and d_n-b_n diverge to infinity. Then c is a minimal geodesic, $c(s)$ lies in the closure of the tube $U_\varepsilon(l)$ for $s\leq 0$ and in the closure of $U_\varepsilon(l')$ for $s\geq A$. Now the arguments used in the proof of Corollary 5.4 show that c is asymptotic to l for $s\to-\infty$ and to l' for $s\to\infty$, i.e. c has the properties stated in our claim.

(b) The sequence a_n has a bounded subsequence converging to some $a\in\mathbb{R}$. We will show that this leads to a contradiction so that case (b) cannot occur. Since $c(-a)$ is a point of accumulation of $\tilde{c}_n(0)\in l$ we have $c(-a)\in l$. On the other hand $\lim d_n=\infty$ and $d_n-b_n<A$ imply that $c\,|\,[-a,\infty)$ is minimal and that $c([A,\infty))$ is contained in the closure of $U_\varepsilon(l')$. Hence $c\,|\,[-a,\infty)$ is a minimal geodesic ray with initial point on the periodic minimal geodesic l and converging to the translate $l'=l+k\neq l$ of l. In [**17**, Theorem 13], Morse shows that this situation is impossible; his proof is formulated for the 2-dimensional case but carries over literally to the present situation.

(c) The sequence d_n-b_n has a bounded subsequence. In analogy to (b) this case cannot occur.

6. *Open problems*

There are a lot of questions related to our results. Here we formulate some of them. The minimal geodesics with respect to a stable norm $\|\ \|$ associated to a Hedlund example (T^3, g) are – up to parametrization – precisely those curves $x:\mathbb{R}\to\mathbb{R}^3$ whose component functions $x^i(t)$ are monotone. We recall the motivation for Theorem 3.2: if $\tilde{c}:\mathbb{R}\to\mathbb{R}^3$ is a lift of $c\in\mathcal{M}(T^3)$ then $\tilde{c}_m(s)=m^{-1}\tilde{c}(ms)$ converges to a minimal geodesic x of $(\mathbb{R}^3, \|\ \|)$. However, Corollary 5.4 shows that in the Hedlund examples these limits are actually very special. They are of the type

$$x(s)=\begin{cases}(s/\varepsilon_i)e_i & \text{for } s\geq 0\\ (s/\varepsilon_j)e_j & \text{for } s\leq 0,\end{cases}$$

for $\{i,j\}\subseteq\{1,2,3\}$, possibly $i=j$.

This raises the following question: are there additional restrictions on the rotation vectors of minimal geodesics $c\in\bar{\mathcal{M}}(M)$?

If they exist such restrictions would arise from some 'semi-global' effects (like Proposition 5.3) which are overlooked by the stable norm. For recurrent $c\in\bar{\mathcal{M}}$, i.e. for those $c\in\bar{\mathcal{M}}$ whose tangent vectors $\dot{c}(s)$ belong to a minimal set of the geodesic flow, one might ask: does every recurrent minimal geodesic $c\in\bar{\mathcal{M}}$ have a unique rotation vector, i.e. does $\lim_{s-t\to\infty} R(c|[t,s])$ exist? Another open question concerns the relation between $[\omega]$-minimality and minimality.

Finally we note that the existence of the minimal geodesics which do not lie on the lines in L and which are not predicted by Theorem 4.6 is not a special feature of the Hedlund examples: it is not difficult to show that a periodic minimal geodesic on a compact Riemannian manifold M with $\dim M>1$ is never isolated in the set of all minimal geodesics. One is tempted to ask if the set of tangent vectors to (unit-speed) minimal geodesics on M is always a connected subset of the unit tangent bundle.

Appendix

Here we present proofs for the results on the stable norm stated in § 2. The function $f: H_1(M,\mathbb{Z})_{\mathbb{R}}\to\mathbb{R}^+\cup\{0\}$,

$$f(v)=\inf\{L(\gamma)\,|\,\gamma \text{ is a closed curve representing } v\}$$

obviously has the following properties:

(1) $f(v)=f(-v)$ and $f(v)=0$ if and only if $v=0$.

(2) $f(nv)\leq nf(v)$.

(3) $f(v+w)\leq f(v)+f(w)+2\operatorname{diam}(M)$.

From (1)-(3) we can easily conclude that f is Lipschitz continuous. The following lemma implies Proposition 2.1:

(4) LEMMA. *Let $f_m: m^{-1}H_1(M,\mathbb{Z})_{\mathbb{R}}\to\mathbb{R}^+\cup\{0\}$ be defined by $f_m(v)=m^{-1}f(mv)$. Then f_m converges uniformly on compact sets to a norm $\|\ \|$ on $H_1(M,\mathbb{R})$. More precisely: there exists a norm $\|\ \|$ on $H_1(M,\mathbb{R})$ such that for all $\varepsilon>0$, $A>0$ there exists $m_0\in\mathbb{N}$ such that $|f_m(v)-\|v\||<\varepsilon$ whenever $m\geq m_0$, $v\in m^{-1}H_1(M,\mathbb{Z})_{\mathbb{R}}$ and $|v|\leq A$.*

Proof. This is elementary analysis. Let C be a Lipschitz constant for f. We first show: suppose $A>0$ is given and m, n are integers satisfying $m\geq 2CA\varepsilon^{-1}n$. Then for all $v\in m^{-1}H_1(M,\mathbb{Z})_{\mathbb{R}}$, $w\in n^{-1}H_1(M,\mathbb{Z})_{\mathbb{R}}$ with $|w|\leq A$:

(5) $f_m(v)\leq f_n(w)+C|v-w|+\varepsilon$.

To prove (5) we estimate for an arbitrary multiple j of n:

$$|f_m(v)-f_j(w)|\le\frac{1}{m}|f(mv)-f(jw)|+\left|\frac{m-j}{mj}\right|f(jw)$$

$$\le C|v-w|+2C\left|\frac{m-j}{m}\right||w|.$$

Now let $m=ln+k$ where $0\le k<n$. We use the preceding estimate with $j=ln$ and apply (2):

$$f_m(v)\le f_{ln}(w)+C|v-w|+2C\frac{k}{m}|w|$$

$$\le f_n(w)+C|v-w|+\varepsilon.$$

This proves (5). Now suppose $v_m\in m^{-1}H_1(M,\mathbb{Z})_{\mathbb{R}}$ is a sequence converging to some $v\in H_1(M,\mathbb{R})$. We will show that $\lim f_m(v_m)=:\|v\|$ exists. This will prove the uniform convergence of $\{f_m\}$ on compact sets. Given $\varepsilon>0$ choose n such that $f_n(v_n)\le \liminf f_j(v_j)+\varepsilon/3$, and such that $|v_m-v_n|\le\varepsilon(3C)^{-1}$ for $m\ge n$. From (5) we obtain: if $m\ge 3(2C(|v|+\varepsilon))\varepsilon^{-1}n$ then

$$f_m(v_m)\le f_n(v_n)+C|v_m-v_n|+\varepsilon/3\le\liminf f_m(v_m)+\varepsilon.$$

This proves that $\lim f_m(v_m)=\liminf f_m(v_m)$ and hence the uniform convergence of f_m on compact sets. Finally we have to show that the limit is indeed a norm: the triangle inequality is an immediate consequence of (3). The positive homogeneity can be seen as follows: if $r=p/q\in\mathbb{Q}^+$ and $v=\lim v_m$ with $v_m\in m^{-1}H(M,\mathbb{Z})_{\mathbb{R}}$ then

$$\|rv\|=\lim_{m\to\infty}f_{mq}\left(\frac{p}{q}v_m\right)=\lim_{m\to\infty}\frac{1}{mq}f(mpv_m)=r\|v\|.$$

Since $\|\ \|$ is continuous this implies positive homogeneity. From (1) we conclude that $\|v\|=\|-v\|$ and $\|v\|\ge 0$. So we are left with proving $\|v\|>0$ if $v\ne 0$: choose a closed 1-form ω on M such that the corresponding cohomology class satisfies $[\omega](v)\ne 0$, say $[\omega](v)>\delta>0$. If $v_m\in m^{-1}H_1(M,\mathbb{Z})_{\mathbb{R}}$ converge to v we have $[\omega](v_m)>\delta$ for all m greater than some m_0. Since M is compact there exists $A>0$ such that $\omega(V)\le A|V|$ for all $V\in TM$. If γ is a closed C^1-curve representing mv_m we can estimate:

$$m\delta<[\omega](mv)=\int_\gamma\omega\le AL(\gamma).$$

This implies $f_m(v_m)\ge A^{-1}\delta$ for $m\ge m_0$, hence $\|v\|\ge A^{-1}\delta>0$.

Finally we prove:

LEMMA 2.3. *For all $\varepsilon>0$ there exists $C_2=C_2(\varepsilon)>0$ such that $|v(\gamma)|\ge C_2$ implies $L(\gamma)\ge(1-\varepsilon)\|v(\gamma)\|$.*

Proof. We add to γ a shortest path from $\gamma(1)$ to $\gamma(0)$ to obtain a closed curve $\tilde\gamma$ such that $L(\tilde\gamma)\le L(\gamma)+\operatorname{diam}(M)$. There exists a constant A depending on the $\omega^1,\dots,\omega^k$ but independent of γ such that

$$|v(\tilde\gamma)-v(\gamma)|<A.$$

If we choose $C_2 \geq C_1(\varepsilon/2, A)$ we obtain from Proposition 2.1: if $|v(\gamma)| \geq C_2$ then

$$|f(v(\tilde{\gamma})) - \|v(\gamma)\|| \leq \varepsilon/2 \|v(\gamma)\|.$$

Hence

$$L(\gamma) \geq L(\tilde{\gamma}) - \operatorname{diam}(M) \geq f(v(\tilde{\gamma})) - \operatorname{diam}(M)$$
$$\geq (1-\varepsilon/2)\|v(\gamma)\| - \operatorname{diam}(M).$$

This proves our claim for $C_2 = \max(C_1(\varepsilon/2, A), (2/\varepsilon)\operatorname{diam}(M))$.

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