POWER SERIES REPRESENTING CERTAIN RATIONAL FUNCTIONS

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1. Let $\mathfrak{A}$ denote the set of functions of a complex variable z, regular at $z = 0$, and let I denote the set of non-negative integers. For $f \in \mathfrak{A}$ put

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \; \phi_f(z) = \sum_{n=0}^{\infty} \operatorname{sgn}|f_n|z^n, \; I_f = \{n|n \in I, f_n = 0\}.$$

For a given subset $\mathfrak{A}_0$ of $\mathfrak{A}$ there arises the problem of characterizing the admissible gap sets I_f of functions f in $\mathfrak{A}_0$. When $\mathfrak{A}_0$ is the set $\mathfrak{R}$ of rational functions a complete solution in given by the following theorem:

(A) *Let* $f \in \mathfrak{R}$ *and let* I_f *be infinite. Then there exist integers* $L, L_1, L_2, \ldots, L_s$, *such that* $0 \leqslant L_1 < L_2 \ldots < L_s < L$, *and* $I_f = \{n|n \in I, \; n \equiv L_j \pmod{L}, \; j = 1, \ldots, s\} \cup I'$, *where* I' *is a finite exceptional set.*

As in **(2)**, this is simply deduced from the theorem

(B) *Let* $f \in \mathfrak{R}$ *and let* I_f *be infinite. Then there exist integers* L, L_1, n_0, *such that* $0 \leqslant L_1 < L$, $n_0 \geqslant 0$, *and* $\{n|n_0. \leqslant n, \; n \equiv L_1 \pmod{L}\} \subset I_f$.

Theorem (A) was proved in 1934 by Mahler for the case when f has algebraic coefficients. This was extended to the general case by Lech in 1953; later, in 1957 Mahler gave another proof of the general case. For references see **(1)** and **(2)**.

We shall prove first

LEMMA 1. *Theorem* (A) *is equivalent to the proposition*: *if* $f \in \mathfrak{R}$ *then* $\phi_f \in \mathfrak{R}$.

In view of this one may ask the following question: let

$$f = \sum_{n=0}^{\infty} f_n z^n \in \mathfrak{R}$$

and let the coefficients f_n be all real, put

$$\chi_f(z) = \sum_{n=0}^{\infty} \operatorname{sgn} f_n z^n;$$

under what conditions is $\chi_f \in \mathfrak{R}$? Our main result proves the existence of a large class of such functions f and indicates some of its properties.

2. There are several descriptions of $\mathfrak{R}$ which we shall use. Their well-known equivalence is stated formally as

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LEMMA 2. *The following are equivalent:*

(a) $\mathfrak{R}$ *is the set of quotients* $P(z)/Q(z)$ *of polynomials with complex coefficients and with* $Q(0) \neq 0$,

(b) $\mathfrak{R}$ *is the set of sums of the form*

$$P(z) + \sum_{k=1}^{N} \sum_{j=1}^{M} A_{jk} (\alpha_k - z)^{-j}$$

where P *is a polynomial,* A_{jk} *and* α_k *are complex constants, and* $\alpha_k \neq 0$,

(c) $\mathfrak{R}$ *is the set of power series*

$$\sum_{n=0}^{\infty} f_n z^n,$$

regular at $z = 0$, *whose coefficients satisfy a linear recurrence relation:*

$$\sum_{j=0}^{N} c_j f_{n+j} = 0, \qquad n \geqslant n_0,$$

(d) $\mathfrak{R}$ *is the set of power series*

$$\sum_{n=0}^{\infty} f_n z^n,$$

regular at $z = 0$, *whose coefficients are values of an exponential polynomial:*

$$f_n = \sum_{k=1}^{N} P_k(n) \alpha_k^{-n}, \qquad n \geqslant n_0,$$

where P_k *is a polynomial and* $\alpha_k \neq 0$.

Here and in the sequel "$T(n)$, $n \geqslant n_0$" will mean that the property T holds for all non-negative integers greater than or equal to n_0. The bound n_0 will vary from case to case.

Let

$$f = \sum_{n=0}^{\infty} f_n z^n \in \mathfrak{R}, \; g = \sum_{n=0}^{\infty} g_n z^n \in \mathfrak{R},$$

and put

$$f \circ g = \sum_{n=0}^{\infty} f_n g_n z^n. \tag{1}$$

By Hadamard's Multiplication Theorem **(3)**,

$$(f \circ g)(z) = 1/2\pi i \int_C f(w)\, g(z/w)\, dw/w \tag{2}$$

where C is a sufficiently small simple contour about the origin. By Lemma 2, (d), or directly by (2), $f \circ g \in \mathfrak{R}$ if $f, g \in \mathfrak{R}$. It follows that under the ordinary addition and the multiplication of (1) $\mathfrak{R}$ becomes a commutative algebra over the complex numbers, with the identity $e(z) = 1/(1 - z)$.

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3. We prove now Lemma 1. Let

$$f = \sum_{n=0}^{\infty} f_n z^n \in \mathfrak{R};$$

without loss of generality let I_f be infinite. By Theorem (A)

$$\phi_f(z) = e(z) - e(z^L)P(z) + Q(z)$$

where P and Q are polynomials and

$$P(z) = \sum_{j=1}^{S} z^{L_j}.$$

Therefore $\phi_f \in \mathfrak{R}$. Suppose now that $\phi_f \in \mathfrak{R}$. By Lemma 2, (c)

$$\sum_{j=0}^{N} c_j \operatorname{sgn}|f_{n+j}| = 0, \qquad n \geqslant n_0.$$

However, there are exactly 2^N different sequences

$$\operatorname{sgn}|f_n|, \operatorname{sgn}|f_{n+1}|, \ldots, \operatorname{sgn}|f_{n+N-1}|.$$

It follows that the sequence $\{\operatorname{sgn}|f_n|\}$, $n = 0, 1, \ldots,$ is periodic, $n \geqslant n_0$. Since

$$I_f = I_{\phi_f},$$

this implies at once Theorem (B), and therefore also Theorem (A).

4. Let $f \in \mathfrak{R}$, by Lemma 2, (b) f is a sum of a polynomial and a finite number of partial fractions corresponding to the distinct poles $z = \alpha_k$, $k = 1, 2, \ldots, N$. A pole at α_k will be called pseudo-rational if $\alpha_k/|\alpha_k|$ is a root of unity, otherwise it will be called pseudo-irrational. We have now a unique decomposition

$$f = P + f_1 + f_2 \tag{3}$$

where P is a polynomial, all the poles of f_1 are pseudo-rational, and those of f_2 are all pseudo-irrational. A function $f \in \mathfrak{R}$ is called itself pseudo-rational if in its decomposition (3) $f_2 \equiv 0$.

Let

$$f = \sum_{n=0}^{\infty} f_n z^n \in \mathfrak{R}, \qquad g = \sum_{n=0}^{\infty} g_n z^n \in \mathfrak{R},$$

and let f_n and g_n be real for all n. Put

$$f \cup g = \sum_{n=0}^{\infty} \max(f_n, g_n) z^n, f \cap g = \sum_{n=0}^{\infty} \min(f_n, g_n) z^n. \tag{4}$$

We can state now our principal result.

THEOREM 1. *Let*

$$f = \sum_{n=0}^{\infty} f_n z^n \in \mathfrak{R}$$

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and let f_n be real for all n. If f is pseudo-rational then $\chi_f \in \mathfrak{R}$. The set $\mathfrak{P}$ of all pseudo-rational functions with real coefficients is a sub-algebra of $\mathfrak{R}$, over the real numbers, under the ordinary addition and the multiplication of (1), *and it is also a lattice under the operations of* (4).

5. We need first a preliminary

LEMMA 3. *Let*

$$E(n) = \sum_{k=1}^{N} P_k(n)\alpha_k^{-n}$$

be an exponential polynomial, real for $n = 0, 1, \ldots$. Let the α_k be roots of unity. Then $\{\text{sgn } E(n)\}$, $n = 0, 1, \ldots$, *is a periodic sequence, $n \geqslant n_0$, and* $\min\{|E(n)| \, |E(n) \neq 0\} \geqslant c > 0$.

We have

$$E(n) = \sum_{k=1}^{N} \sum_{j=0}^{M} a_{kj}\, n^j \alpha_k^{-n} \tag{5}$$

where $M = \max_k \deg P_k$; M is called the degree of E. One can write (5) as

$$E(n) = \sum_{j=0}^{M} F_j(n) n^j \tag{6}$$

where

$$F_j(n) = \sum_{k=1}^{N} a_{kj}\alpha_k^{-n}.$$

By the hypothesis $\alpha_k = \exp 2\pi i\, p_k/q_k$, $0 \leqslant p_k < q_k$, $(p_k, q_k) = 1$. Let $Q =$ l.c.m. $\{q_k\}$, then $F_j(n) = F_j(n + Q)$ for all n and j. We can also show that $F_j(n)$ is real for all n and j; this follows by observing that with each pair α_k, $P_k = \sum a_{kj}n^j$ in E there is associated the conjugate pair $\bar{\alpha}_k$, $\bar{P}_k = \sum \bar{a}_{kj}\, n^j$.

The lemma will be proved by induction on the degree M of E. Suppose first that $M = 0$, then

$$E(n) = F_0(n) = \sum_{k=1}^{N} a_{k_0}\alpha_k^{-n}$$

so that $\{E(n)\}$, $n = 0, 1, \ldots$, is a periodic sequence of real numbers with period Q. Therefore the lemma holds here. Suppose now that the lemma has been established for exponential polynomials of degree $\leqslant M$, and let deg $E = M + 1$. Then

$$E(n) = F_{M+1}(n)\, n^{M+1} + E_1(n) \tag{7}$$

where $F_{M+1}(n)$ is real for all n and not identically zero, and deg $E_1 \leqslant M$. Let Q be the common period of $F_0, F_1, \ldots, F_{M+1}$ and consider the set

$$S = \{F_{M+1}(0), F_{M+1}(1), \ldots, F_{M+1}(Q)\}.$$

If no member of S vanishes then

$$(8)\qquad \min_n |F_{M+1}(n)| = \min_{0\leqslant n\leqslant Q-1} |F_{M+1}(n)| = c > 0,$$

and the first term on the right in (7) dominates the whole right-hand side since $|E_1(n)| = 0(n^M)$. Now the periodicity of $F_{M+1}(n)$ and the condition (8) imply that the lemma holds in this case.

Suppose now that some members of S vanish. For $n \in I$ let $n \in A$ if $n \equiv n_1 \pmod Q$ and $F_{M+1}(n_1) = 0$, $0 \leqslant n_1 < Q$; otherwise let $n \in B$. When n is restricted to B the lemma holds as before; when $n \in A$, $E(n) = E_1(n)$ and the lemma holds by the induction assumption since deg $E_1 < M$. This concludes the proof.

6. We prove now Theorem 1. Let $f = \sum_0^\infty f_n z^n$ be a pseudo-rational function and let f_n be real for all n. By Lemma 2, (b) we have

$$(9)\quad f(z) = P(z) + \sum_{r=1}^{R} \sum_{k=1}^{N} \sum_{j=1}^{M} A_{rkj}(\alpha_{rk} - z)^{-j} = P(z) + \sum_{r=1}^{R} g_r(z)$$

where $|\alpha_{rk}| = a_r$ and $0 < a_1 < a_2 < \ldots < a_R$. That is, we order the partial fractions according to the increasing absolute value of the poles. R will be called the order of f. Since the presence of P in (9) influences only a finite number of coefficients we assume without loss of generality that $P \equiv 0$.

We show first that $\chi_f \in \mathfrak{R}$. The proof will proceed by induction on the order R of f. Let $R = 1$, then $f = g_1(z)$ and so

$$(10)\qquad f_n = a_1^{-n} E_1(n)$$

where $E_1(n)$ satisfies the conditions of Lemma 3. It follows that $\{\text{sgn } E_1(n)\}$, n $= 0, 1, \ldots$, is a periodic sequence, $n \geqslant n_0$, which implies immediately that $\chi_f \in \mathfrak{R}$. Suppose now $\chi_f \in \mathfrak{R}$ for any function f of order $\leqslant R$, satisfying the conditions. Let f be a function of order $R + 1$, then

$$f(z) = g_1(z) + h(z)$$

where the order of h is $\leqslant R$ and the absolute value a_1 of the poles of g_1 is less than that of any pole of h. Let

$$g_1(z) = \sum_{n=0}^{\infty} g_{n1} z^n,\ h(z) = \sum_{n=0}^{\infty} h_n z^n,$$

then $f_n = g_{n1} + h_n$. Suppose that $g_{n1} \neq 0$ for all n. By Lemma 3 it follows easily that $h_n = 0(g_{n1})$ for large n and therefore sgn f_n = sgn g_{n1}, $r \geqslant n_0$. However, by the induction assumption $\{\text{sgn } g_{n1}\}$, $n = 0, 1, \ldots$, is a periodic sequence, $n \geqslant n_0$. Hence $\{\text{sgn } f_n\}$, $n = 0, 1, \ldots$, is a periodic sequence, $n \geqslant n_0$, and $\chi_f \in \mathfrak{R}$.

Suppose now that $g_{n1} = 0$ for infinitely many n, and let $n \in A$ if $g_{n1} = 0$, $n \in B$ otherwise. Much in the same way as in the proof of Lemma 3 we show that $\{\operatorname{sgn} f_n\}$ is a periodic sequence when n is restricted to A, and also when n is restricted to B, which again implies that $\chi_f \in \mathfrak{R}$.

Furthermore, it is easy to show that χ_f must have the following form

$$\chi_f(z) = P(z) + e(z) - e(z^L)Q(z)$$

where

$$Q(z) = \sum_{j=0}^{L-1} \epsilon_j z^j$$

and $\epsilon_j = 0, 1$ or -1. It follows that not only $\chi_f \in \mathfrak{R}$ but actually $\chi_f \in \mathfrak{P}$.

We proceed now with the rest of the proof. It is clear that $\mathfrak{P}$ is closed under addition and multiplication by real numbers. We show next that $f \circ g \in \mathfrak{P}$ if $f, g \in \mathfrak{P}$. Although this follows immediately from Lemma 2, (d) the following proof supplies a closed explicit representation for $f \circ g$. As in Lemma 2, (b) let

$$f(z) = P(Z) + \sum_{k=1}^{M} \sum_{j=1}^{N} A_{jk}(\alpha_k - z)^{-j},$$

$$g(z) = P_1(z) + \sum_{k=1}^{M_1} \sum_{j=1}^{N_1} B_{jk}(\beta_k - z)^{-j},$$

then

$$(11) \quad f \circ g = Q(z) + \sum_{k=1}^{M} \sum_{j=1}^{N} \sum_{k_1=1}^{M_1} \sum_{j_1=1}^{N_1} A_{jk}B_{j_1k_1}(\alpha_k - z)^{-j} \circ (\beta_{k_1} - z)^{-j_1}$$

where Q is a polynomial. Now

$$(12) \quad (\alpha_k - z)^{-j} \circ (\beta_{k_1} - z)^{-j_1} = \sum_{n=0}^{\infty} \binom{n+j-1}{n}\binom{n+j_1-1}{n} z^n / \alpha_k^{n+j}\beta_{k_1}^{n+j_1}.$$

Let constants γ_{pqs} $s = 1, 2, \ldots, p + q - 1$, be determined so that

$$\binom{n+p-1}{n}\binom{n+q-1}{n} = \sum_{s=1}^{p+q-1} \gamma_{pqs}\binom{n+s-1}{n}$$

identically in n. Then by (12)

$$(13) \qquad (\alpha_k - z)^{-j} \circ (\beta_{k_1} - z)^{-j_1} = \sum_{s=1}^{j_1+j-1} \gamma_{jj_1s}\alpha_k^{s-j}\beta_{k_1}^{s-j_1}(\alpha_k\beta_{k_1} - z)^{-s}.$$

By putting together (11) and (13) we obtain an explicit representation of $f \circ g$ and see at once that $f \circ g \in \mathfrak{P}$, since

$$\frac{\alpha_k\beta_{k_1}}{|\alpha_k\beta_{k_1}|} = \frac{\alpha_k}{|\alpha_k|}\cdot\frac{\beta_{k_1}}{|\beta_{k_1}|}.$$

By (4)

$$f \cup g = 1/2 \sum_{n=0}^{\infty} [f_n + g_n + (f_n - g_n)\operatorname{sgn}(f_\epsilon - g_n)]z^n$$
$$= 1/2[f + g + (f - g)\chi_{f-g}],$$

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and $f \cap g = f + g - f \cup g$. Since $f - g \in \mathfrak{P}$ implies $\chi_{f-g} \in \mathfrak{P}$, it follows that if $f, g \in \mathfrak{P}$ then $f \cup g \in \mathfrak{P}$ and $f \cap g \in \mathfrak{P}$. This completes the proof.

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