ON A CLASS OF NEAR-RINGS

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In [3], Ligh proved that every distributively generated Boolean near-ring is a ring, and he gave an example to which the above fact can not be extended. That is, let G be an additive group and let the multiplication on G be defined by $xy = y$ for all x, y in G. Ligh called this Boolean near-ring G a general Boolean near-ring. Then in [4], Ligh called R a β-near-ring if for each x in R, $x^2 = x$ and $xyz = yxz$ for all x, y, z in R, and he proved that the structure of a β-near-ring is "very close" to that of a usual Boolean ring. We note that general Boolean near-rings and Boolean semirings as defined in [5] are β-near-rings. The purpose of this paper is to generalize the structure theorem on β-near-rings given by Ligh in [4] to a broader class of near-rings.

First, let us recall some definitions.

DEFINITION 1. A (left) *near-ring* is an algebraic system $(R, +, \cdot)$ such that (1). $(R, +)$ is a group, (2). $(R, \cdot)$ is a semigroup and (3). $x(y + z) = xy + xz$ for all x, y, z in R.

DEFINITION 2. We call K an *ideal* of a near-ring R if and only if (1). K is a normal subgroup of $(R, +)$, (2). RK is contained in K and (3). $(m + k)n - mn$ is in K for all m, n in R and k in K. We know that K is an ideal of a near-ring R if and only if K is the kernel of a near-ring homomorphism (see [2]).

DEFINITION 3. (Ligh) A near-ring R is called a *β-near-ring* if for each x in R, $x^2 = x$ and $xyz = yxz$ for all x, y, z in R.

DEFINITION 4. A near-ring $(R, +, \cdot)$ is said to be *small* if in the multiplication table of $(R, \cdot)$ there are at most two distinct rows, not counting duplicates, such that either $ab = b$ for all a, b in R or one row is determined by 0 and the other by a left identity (Ligh).

We know that the Pierce decomposition theorem holds true in near-rings; that is,

PROPOSITION. *If R is a near-ring then $R \cong xR \oplus S_x$ where x is an idempotent of R, x is a left identity of xR and $xs = 0$ for all s in S_x.*

DEFINITION 5. *A near-ring R is called a weak β-near-ring if $xyz = yxz$ for all x, y, z in R. We know that any near-ring with commutative multiplication is a weak β-near-ring but not necessarily a β-near-ring.*

It is not hard to prove the following lemma:

LEMMA. *If R is a weak β-near-ring, then S_x is an ideal of R where $R \cong xR \oplus S_x$, a Pierce decomposition of R.*

PROOF. For any r in R and s in S_x,

$$\begin{aligned} & x(-r+s+r) \\ &= x(-r) + xs + xr \\ &= -(xr) + 0 + xr = 0, \end{aligned}$$

so that $-r+s+r$ is an element of S_x. Thus $(S_x, +)$ is a normal subgroup of $(R, +)$. Also,

$$\begin{aligned} x(rs) &= (xr)s = (rx)s \text{ (for } R \text{ is a weak } \beta\text{-near-ring)}, \\ &= r(xs) = r0 = 0, \text{ so that } (rs) \text{ is in } S_x. \end{aligned}$$

Thus RS_x is a subset of S_x. Finally, for any m and n in R, s in S_x,

$$\begin{aligned} x[(m+s)n - mn] &= (xm + xs)n - xmn \\ &= (xm + 0)n - xmn \\ &= xmn - xmn = 0, \text{ so that} \end{aligned}$$

$(m+s)n - mn$ is in S_x. Therefore S_x is an ideal of R.

THEOREM. *Every weak β-near-ring is isomorphic to a subdirect sum of subdirectly irreducible near-rings $\{R_i\}$ where R_i is one of the following types:*

(a) *R_i is a small β-near-ring,*

(b) *the intersection of all proper ideals of R_i has no nonleft identity-idempotents,*

(c) *if there are nonzero idempotents in R_i then they are left identities of R_i.*

PROOF. Since R is isomorphic to a subdirect sum of subdirectly irreducible near-rings R_i ([4], Theorem 3.1), it suffices to show that each R_i is one of (a), (b) and (c). The proof divides into four cases.

Case 1. The set $\{0r \text{ for all } r \text{ in } R_i\} = 0R_i$ is a proper subset of R_i. Since R_i is subdirectly irreducible the intersection, I, of all proper ideals of R_i is non-trivial. Considering I again we have $0a = 0$ for all a in I. This follows because $R_i \cong 0R_i \oplus S_0$ and I is contained in S_0 by the Pierce decomposition theorem on R_i. Furthermore we claim that I has no nonleft identity-idempotents. In fact, let x be a nonleft identity-idempotent in I; then

$$R_i \cong xR_i \oplus S_x.$$

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Noting that S_x is a proper ideal of R_i and x is not in S_x we conclude that x is not in I. This is a contradiction. Thus I contains no nonzero idempotents. Therefore R_i is type (b).

Case 2. The set $0R_i = R_i$. We claim that R_i is type (a). Let $0r$, $0r'$ and $0r''$ be in R_i; then

$$(0r)^2 = (0r)(0r) = 0r, \quad (0r)(0r') = 0r'$$

and $((0r)(0r'))(0r'') = 0r'' = ((0r')(0r))(0r'')$. Thus R_i is a small β-near-ring by definitions 3 and 4.

Case 3. The set $0R_i = 0$ and there are no proper ideals in R_i. We claim that R_i is type (c). In this case $xR_i = R_i$ for all nonzero idempotents x in R_i since $xR_i \neq 0$ and S_x is improper. Thus x is a left identity of R_i. Therefore R_i is type (c).

Case 4. The set $0R_i = 0$ and there are proper ideals in R_i. Then R_i is type (b) by the same arguments as in case 1.

COROLLARY. (Ligh) *Every β-near-ring is isomorphic to a subdirect sum of subdirectly irreducible near-rings $\{R_i\}$ where each R_i is either a two-element field or a small near-ring.*

PROOF. Since $x^2 = x$ for any x in R_i and R_i is a weak β-near-ring, all cases are small. Next we consider cases 2 and 3 only. If there exists a nonzero right distributive element r in R_i then case 2 is impossible. In fact, for any element x in R_i,

$$(0 + x)r = 0r + xr = r + r, \ r = r + r, \ r = 0.$$

This leads to a contradiction that $r \neq 0$. So, we are left with case 3 only. In this case, let r be a nonzero right distributive element and suppose that there are two different nonzero elements, x and y, in R_i. Then,

$$(x - y)r = (x + (-y))r = xr + (-y)r = r + r.$$

Since $x - y$ is not zero, the above expression implies that $r + r = r$, $r = 0$. This is a contradiction that $r \neq 0$. Thus all nonzero elements in R_i are equal. Therefore R_i is a two-element field.

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