BULL. AUSTRAL. MATH. SOC. 20-04, 20F05
VOL. 15 (1976), 297-305.

Computer aided determination of a Fibonacci group

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The Fibonacci group $F(2, 7)$ has been known to be cyclic of order 29 for about five years. This was first established by computer coset enumerations which exhibit only the result, without supporting proofs. The working in a coset enumeration actually contains proofs of many relations that hold in the group. A hand proof that $F(2, 7)$ is cyclic of order 29, based on the working in computer coset enumerations, is presented here.

1. Introduction

Interest in Fibonacci groups was aroused by Conway [3]. The groups have been studied in detail by Johnson, Wamsley, and Wright [4]. The Fibonacci groups $F(2, n)$ may be presented

$$F(2, n) = \langle x_1, x_2, \ldots, x_n \mid x_1x_2 = x_3, \ldots, x_{n-2}x_{n-1} = x_n, x_{n-1}x_n = x_1, x_nx_1 = x_2 \rangle .$$

A first question about these groups is whether they are finite or not; this has been resolved (see Brunner [1]) for all bar $F(2, 9)$. ($F(2, 9)$ is still unknown although it is known to have a largest nilpotent quotient of order 152.)

The groups $F(2, n)$ which are known to be finite have all been identified. In spite of information to the contrary (in [4] and [7]), $F(2, 7)$ is cyclic of order 29. This has been established by coset

Received 15 June 1976.

enumeration on computer, but till now no hand proof has been available.

Given a coset enumeration which shows that a subgroup is of finite index in a certain group, Leech [5] describes a technique for expressing a word in the generators of the group which is in the subgroup as a word in the generators of the subgroup. He amplifies this method in [6] for the proof of relations which hold in the group and describes a computer implementation. This method is the basis of the proof that $F(2, 7)$ is cyclic of order 29 presented here.

2. Notation

For brevity x_i is sometimes denoted by i and x_i^{-1} by $-i$. The group identity is denoted by e. Thus $F(2, 7)$ may be presented

$$(1) \quad F(2, 7) = \langle 1, 2, 3, 4, 5, 6, 7 \mid 12\text{-}3 = 23\text{-}4 = 34\text{-}5 = 45\text{-}6 = 56\text{-}7 = 67\text{-}1 = 71\text{-}2 = e \rangle .$$

The terminology and notation used in discussion and description of coset enumerations follows [2].

3. Computer considerations

It is moderately easy to establish that $F(2, 7)$ is cyclic of order 29 by coset enumeration. The easiest way is to observe that the quotient of $F(2, 7)$ obtained by abelianizing the presentation is cyclic of order 29, so it suffices to show that any one of the x_i alone generates $F(2, 7)$. Each of the coset enumerations $F(2, 7) | \langle x_i \rangle$, though pathological, is easy enough by machine and gives index 1 as required.

In the context of the Leech method of relation proof from coset enumeration working the most important consideration is the avoidance of coincidences (hence the minimization of total cosets defined). It follows that the Felsch method of coset enumeration is preferred for machine approach to this problem. A Canberra implementation, developed by W.A. Alford, is used for all coset enumerations mentioned here, and is the basis of a computer implementation of Leech's procedure used for relation proof.

The Felsch method yields the following statistics for coset enumerations in $F(2, 7)$ presented as in (1).

https://doi.org/10.1017/S0004972700022668 Published online by Cambridge University Press

Subgroup	Index	Total cosets defined
$\langle x_1 \rangle$	1	404
$\langle x_2 \rangle$	1	615
$\langle x_3 \rangle$	1	332
$\langle x_4 \rangle$	1	742
$\langle x_5 \rangle$	1	336
$\langle x_6 \rangle$	1	327
$\langle x_7 \rangle$	1	366
$\langle e \rangle$	29	> 33000

It is easy to rewrite the presentation for $F(2, 7)$ in terms of two generators by the use of Tietze transformations. A two generator presentation for $F(2, 7)$ is:

(2) $F(2, 7) = \langle 1, 2 \mid 1121\text{-}2\text{-}21\text{-}2\text{-}21\text{-}2 = 12\text{-}12\text{-}122\text{-}12\text{-}122\text{-}122 = e \rangle$.

This presentation is much harder for coset enumeration over $\langle 1 \rangle$ and $\langle 2 \rangle$ as the following statistics indicate. (Other easily obtained two generator presentations are better, but all I have tried give significantly worse enumerations than the best of the 7 generator enumerations.)

Subgroup	Index	Total cosets defined
$\langle 1 \rangle$	1	1143
$\langle 2 \rangle$	1	3103
$\langle e \rangle$	29	75498

Using Leech's method it is theoretically possible to obtain a proof that $F(2, 7)$ is cyclic of order 29 from any one of these enumerations. However, as already mentioned, it is desirable to minimize coincidences so the enumeration $F(2, 7)|\langle x_6 \rangle$ looks the most attractive of the above enumerations. As a starting point for application of Leech's method of relation proof I am unable to find any substantially better enumeration which yields the desired result.

Unfortunately direct application of Leech's method to this enumeration, $F(2, 7)|\langle 6 \rangle$, would lead to a tremendously long proof. The problem arises from the large number of coincidences involved. By studying the coset enumeration I have been able to find a lemma which leads to a better coset enumeration (fewer coincidences). The lemma is reasonably easily proved using Leech's method.

The lemma arises this way. In the enumeration $F(2, 7)|\langle 6\rangle$ the first coset coincidence (which precipitates the total collapse) implies $-1\text{-}34464 \in \langle 6\rangle$. Let $h_1 = -1\text{-}34464$. Promisingly, the enumeration $F(2, 7)|\langle 6, h_1\rangle$ involves the definition of only 177 cosets, as against 327 for $F(2, 7)|\langle 6\rangle$.

The proof, by Leech's method, of the relation corresponding to the first coincidence involves only 29 of the 327 cosets defined, and of course no other coincidence. The proof of Lemma 1 presented below is essentially that given by Leech's method, abbreviated by the combination of the 110 separate substitutions or reductions given by the method.

Still 177 cosets and coincidences would yield a terribly long proof of the desired result, so I looked for another lemma. In the performance of $F(2, 7)|\langle 6, h_1\rangle$ the first coincidence, after the definition of 102 cosets, leads to a three coset collapse, and is not of great value. However the fourth coincidence precipitates the total collapse and provides a valuable lemma, $-4\text{-}6\text{-}232464 \in \langle 6, h_1\rangle$. Let $h_2 = -4\text{-}6\text{-}232464$. The enumeration $F(2, 7)|\langle 6, h_1, h_2\rangle$ involves the definition of 36 cosets.

Again proof of the corresponding relation is easy enough. The proof involves 46 of the 177 cosets. (The subproof that

$$-16^2h_1^{-1}6^{-2}h_16^2\text{-}1\text{-}344\text{-}2 = 63$$

involves the third coincidence, the only time one of the first three coincidences is used.)

The processing of the 36 cosets and coincidences by Leech's method is still too long, so I found one further lemma analogously (from the first coincidence, which precipitates the total collapse) namely $-47\text{-}34464 \in \langle 6, h_1, h_2\rangle$. Let $h_3 = -47\text{-}34464$. The proof of the corresponding relation involves 13 cosets.

The enumeration $F(2, 7)|\langle 6, h_1, h_2, h_3\rangle$ needs 8 cosets. From the

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deduction that generator 5 applied to coset 1 yields coset 1, the method gives a proof that $5 \in \langle 6, h_1, h_2, h_3 \rangle$, involving all 8 cosets.

The method used here for lemma production looks nicely mechanical. Find the first important coincidence and add the corresponding subgroup generator to the initial subgroup to get an easier enumeration. Unfortunately it does not always work with the enumeration method used.

The index 2 enumeration $F(2, 9) + (1, 2)^2 \mid \langle 1, -23, 2\text{-}9 \rangle$ requires the definition of 39 cosets. The first coincidence, which precipitates the collapse to two cosets, implies that $4\text{-}7\text{-}25 \in \langle 1, -23, 2\text{-}9 \rangle$. However the enumeration $F(2, 9) + (1, 2)^2 \mid \langle 1, -23, 2\text{-}9, 4\text{-}7\text{-}25 \rangle$ requires the definition of 44 cosets.

Note that the relations corresponding to the coincidences and the deduction actually give expressions for h_1, h_2, h_3, and 5 in terms of the corresponding subgroup generators, which is rather more than is required for the proof that $F(2, 7)$ is cyclic.

4. Theorem and proof

THEOREM. *The group* $F(2, 7)$ *is cyclic of order* 29.

Proof. The proof follows directly from four lemmas.

LEMMA 1. $-1\text{-}34464\ (= h_1) \in \langle 6 \rangle$.

LEMMA 2. $-4\text{-}6\text{-}232464\ (= h_2) \in \langle 6 \rangle$.

LEMMA 3. $-47\text{-}34464\ (= h_3) \in \langle 6 \rangle$.

LEMMA 4. $5 \in \langle 6 \rangle$.

Hence $F(2, 7)$ is cyclic, and the result follows from abelianizing the defining relations.

In the proofs of the lemmas underlined subwords in one line are replaced by equivalents in the next. Where this is not by direct application of one defining relation or by free reduction the annotations describe how the equivalent is calculated. A dot in a word indicates that a subword with freely trivial value will be inserted in the corresponding position in the following line.

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Proof of Lemma 1. We prove $-1-34464 = 6^{10}$ by showing $-4316^{10}-4-6-4 = e$:

$$-431.66\underline{6}.66\underline{6}.66\underline{6}.6-4-6-4 \qquad (1)$$
$$= -43\underline{12}-24\underline{-4661}.1-24\underline{-4661}.1-24\underline{-4661}.1-24\underline{-46}-4-6-4 \qquad (2)$$
$$= -433\underline{-2423}-\underline{31}\underline{-2423}\underline{-31}\underline{-2423}-3.1\underline{-24}\underline{5-4}-6-4 \qquad (3)$$
$$= -43.35-2.5-2.5\underline{-35}\underline{-5-4}\underline{4-5}5\underline{6}\underline{-61}.33-6-4 \qquad (4)$$
$$= -4\underline{3-22}\underline{35}.\underline{-27}-\underline{75}.\underline{-27}-\underline{754}-6-37\underline{71}\underline{-133}.-6-4 \qquad (5)$$
$$= -41.\underline{67}-\underline{72}\underline{-2-1}\underline{-61}\underline{-1-77-1-5-6}.-3\underline{724-5}\underline{5-6}-4 \qquad (6)$$
$$= \underline{-41-7}7\underline{11}.-3.7.\underline{-3-2}\underline{2-3}.-6-4-4 \qquad (7)$$
$$= 5\underline{212}-2\underline{-35}\underline{-57-5}5-4\underline{-1-7}\underline{7-6-4-4} \qquad (8)$$
$$= 5\underline{4-244}\underline{5-4-2-2} \qquad (9)$$
$$= 56-7 \qquad (10)$$
$$= e \ ;$$

(1) $6 = 1\underline{-7} = 11-2$

(2) $\underline{-46}61 = \underline{56}1 = 71 = 2$

(3) $\underline{-24}23 = 3\underline{23} = 34 = 5$

(5) $\underline{23}5 = 45 = 6$, $\underline{-75}4 = -64 = -5$, $\underline{-13}3 = 23 = 4$

(6) $\underline{-1-7}\underline{7-1}-5-6 = -2\underline{-6-5}-6 = -2\underline{-7-6} = -2-1 = -3$, $72\underline{4-5} = 7\underline{2-3} = 7-1 = -6$

(7) $-4\underline{1-7} = -46 = 5$

(8) $2\underline{12} = 23 = 4$, $\underline{-57}-5 = 6-5 = 4$, $\underline{7-6}-4-4 = \underline{5-4}-4 = 3-4 = -2$

(9) $4\underline{-24}4 = 4\underline{34} = 45 = 6$, $\underline{5-4}-2-2 = \underline{3-2}-2 = 1-2 = -7$

(10) defining relation 5 .

Proof of Lemma 2. Let $x = 6^2h_1^{-1}6^{-2}h_16^2$ $(= 6^2)$. We prove $-4-6-232464 = 6^{-35}$ by showing $2464\left(6^4h_1-6xh_16^{-2}xh_1\right)-4-6-23 = e$:

$$24646666h_1\underline{-6}x\underline{h_1}-6.\underline{-6}x\underline{h_1-4-6}-23 \qquad (1)$$
$$= 24646666h_17\underline{-1x-1-344-2}264-62-\underline{27}\underline{-1x-1-344-2}3 \ . \qquad (2)$$
$$\text{Consider } -1\underline{x}-1-344-2 \qquad (3)$$
$$= \underline{-16}.6-4-6-4-43\underline{1-6-6-1}.-3446466\underline{6-1}-344-2 \qquad (4)$$
$$= 1\underline{-24}\underline{-46-4}.-6-4-4\underline{34}\underline{3-27}\underline{4-4-3}446.\underline{4-3}\underline{-4-3}44.-2 \qquad (5)$$
$$= 1\underline{33-4}\underline{4}5\underline{-53}\underline{-3-7}\underline{7-6-4-4}.51723-5.\underline{446-7}\underline{7-1}3-5.\underline{44-5}5.-2 \qquad (6)$$
$$= 116-4-2\underline{-2-27}-\underline{751}723\underline{-57}-\underline{72}-63\underline{-57}-\underline{72}\underline{5-445}\underline{-53}\underline{-3-2} \qquad (7)$$
$$= 116-4\underline{-2-3772}.\underline{36}1-6.\underline{36}1.14-31-7-4-4 \qquad (8)$$
$$= 116-4\underline{33-4}4\underline{5}51-6.-4\underline{4551-771}4-31-7-4-4 \qquad (9)$$

$= 116\underline{-41651-6-5}5\underline{-4}3\underline{4-31}-7-4-4$ (10)

$= .1.1\underline{65235}-2-7-4-4$ (11)

$= 6\underline{-61-66-556-611-2-7-4-4}$ (12)

$= 63$. (13)

Substitute (13) for (3) in (2)

$(1) = 2464.\underline{6666h_1}.76326\underline{4-62}-1633$ (14)

$= 246.\underline{4-3347-1}-7\underline{2-3}\underline{556-1-3}446.\underline{45}-\underline{57}6326.-3-3-1633$ (15)

$= \underline{246-77-1}-2.-7-1-5.\underline{446-771}-\underline{16}6.632.\underline{6-55-44-3-3}-1633$ (16)

$= -1-6-\underline{2-11-7-1-57}-\underline{722}-7.\underline{6-55}6\underline{32-33}4\underline{3-1-16}.33$ (17)

$= \underline{-1-6-36-73-72-24}73.-1.\underline{51-6-55}63.3$ (18)

$= .-73\underline{73-22-1-66773-223}$ (19)

$= -66\underline{-7325124}$ (20)

$= -645$ (21)

$= e$;

(1) $-6 = 7-1,\ h_1 = -1-344.64 = -1-344-2264,$

$\underline{h_1}-4-6 = -1-344\underline{64-4-6} = -1-344$

(4) $-16 = -7 = 1-2,\ 1-6\underline{-6} = 1\underline{-6-5-4} = \underline{1-7}-53 = \underline{6-5}3 = 43,$

similarly $66-1 = -3-4,\ -1 = -27$

(5) $\underline{-46}-4 = 5-4 = 3,\ \underline{4-4-3} = 23-5,\ 4-3 = 2 = -13$

(6) $3\underline{3-4} = 3-2 = 1,\ \underline{-3}-7 = -2\underline{-1-7} = -2-2,\ \underline{7-6}-4-4 = \underline{5-4}-4 = 3-4 = -2,$

similarly $446-7 = 2,\ 4\underline{4-5} = 4-3 = 2$

(7) $-2\underline{-27} = -2-1 = -3,\ \underline{-75}1 = -61 = 7,\ \underline{5-4}.\underline{45} = \underline{3-223}-31\underline{-16} = 14-31-7$

(8) $-2-37\underline{72} = -2-37-6\underline{1-13} = -2-3\underline{7-63} = -2\underline{-353} = \underline{-24}3 = 33,$

$\underline{36} = 5\underline{-44}5 = 55$

(9) $3\underline{3-4} = 3-2 = 1,\ \underline{455}\underline{1-77}1 = 6\underline{56}71 = \underline{6771} = 12 = 3$

(10) $-41651\underline{-6-5} = -4165\underline{1-7} = -4.16\underline{56} = \underline{-46}-\underline{6167} = 5\underline{71} = 52$

(11) $65\underline{235} = 65\underline{45} = 6\underline{56} = 67 = 1$

(12) $\underline{-61}-66\underline{-556}\underline{-61}1-2-7-4-4 = \underline{7-64}7\underline{71}-2-7-4-4 = 5\underline{472-2-7-4}-4 = 5-4 = 3$

(14) $\underline{66} = 4\underline{56} = 47,\ 6 = .45 = \underline{-22}-33\underline{45} = -1-72-355,\ 4\underline{-62} = \underline{4-5}\underline{-42} = -3-3$

(15) $\underline{4-3347-1} = 2\underline{5-6} = -1\underline{3-4} = -1-2,\ 5\underline{56}-1-3 = 5\underline{7-1}-3 = \underline{5-6}-3 = -4-3 = -5$

(16) $24\underline{6-7} = 2\underline{4-5} = 2-3 = -1,\ 44\underline{6-7} = 4\underline{4-5} = 4-3 = 2,\ \underline{4-3}-3 = 2-3 = -1$

(17) $\underline{1-7}-1\underline{-57} = 6\underline{-16} = 6-7,\ \underline{-72}2 = 12 = 3,\ 3-1\underline{-16} = 3\underline{-1-7} = 3-2 = 1$

(18) $-1-6-3\underline{6-73-72} = -1-6-3\underline{-53}.1 = -1-6-3\underline{-46-61} = -1-6\underline{-357} = -1\underline{-64}7$

$= -1\underline{-57} = -16 = -7,\ 51\underline{-6-5} = 5\underline{1-7} = 56 = 7$

(19) $7\underline{3-2} = 71 = 2,\ \underline{2-1}-6 = 7-6 = 5$

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(20) 6-7325 = -5325 = -425 = -35 = 4, 124 = 34 = 5

(21) defining relation 4 .

Proof of Lemma 3. We prove -47-34464 = 6^{-35} by showing $-34464h_2^{-1}$-47 = e ;

$-34464h_2^{-1}$-47	(1)
= -34.464-4-6-4-2-32.64-47	(2)
= -345-53-3-2-32-11-7767	(3)
= -36-4-5762	(4)
= -312	(5)
= e ;	

(3) -53-3-2-3 = -53-4-3 = -4-5, 767 = 71 = 2

(4) 6-4-576 = 6-466 = 656 = 67 = 1

(5) defining relation 1 .

Proof of Lemma 4. Let $y = h_1h_3^{-1}$. We prove 5 = 6^{255} by showing $-5\left[-6y^2h_2y^{-2}6y^26y^2-6y^{-1}-6y^2h_2y^{-2}6y^2-6y^{-1}-6y^2h_2y^{-2}6y^26y^2\right] = e$.

-5-6yyh_2-y-y6yy6yy-6-y-6yyh_2-y-y6yy-6-y-6yyh_2-y-y6yy6yy	(1)
= -5-63-24h_2-42-363-2463-24-6-3-63-24h_2-42-363-24-6-3-63-24h_2-42-363 -2463-24	(2)
= -5-61-6-23246-163-24.614-6-3-61-6-23246-163-24-6-3-61-6-23246-163 -24.614	(3)
= -232-24-55614-5-232-24-5-232-24-55614	(4)
= -2714-5-2-2714	(5)
= -3-24	(6)
= e ;	

(1) $y = h_1h_3^{-1}$ = -1-34464-4-6-4-43-74 = -1-74 = -24 = 3, -y = -3 = -42

(2) 4h_2-4 = 4-4-6-232464-4 = -6-23246

(3) -5-61-6 = -57-6 = -55 = e, 46-163 = 46-73 = 4-53 = -33 = e,
-6-3-61-6 = -6-37-6 = -6-35 = -64 = -5

(4) 32-24-5 = 34-5 = 5-5 = e

(5) -271 = -22 = e

(6) defining relation 2 .

References

[1] A.M. Brunner, "The determination of Fibonacci groups", *Bull. Austral. Math. Soc.* **11** (1974), 11-14.

[2] John J. Cannon, Lucien A. Dimino, George Havas and Jane M. Watson, "Implementation and analysis of the Todd-Coxeter algorithm", *Math. Comp.* **27** (1973), 463-490.

[3] J.H. Conway, "Advanced problem 5327", *Amer. Math. Monthly* **72** (1965), 915.

[4] D.L. Johnson, J.W. Wamsley, and D. Wright, "The Fibonacci groups", *Proc. London Math. Soc.* (3) **29** (1974), 577-592.

[5] John Leech, "Coset enumeration", *Computational problems in abstract algebra*, 21-35 (Proc. Conf. Oxford, 1967. Pergamon, Oxford, London, Edinburgh, New York, Toronto, Sydney, Paris, Braunschweig, 1970).

[6] John Leech, "Computer proof of relations in groups", *Topics in group theory and computation* (Proc. Royal Irish Acad. Summer School on Group Theory and Computation, Galway 1973. Academic Press, to appear).

[7] Various Authors, "Generators and relations for cyclic groups" (Solution to advanced problem 5327, see [3] above), *Amer. Math. Monthly* **74** (1967), 91-93.

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