J. Austral. Math. Soc. (Series A) 43 (1987), 398–419

ON TOPOLOGICAL ALGEBRA BUNDLES

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(Received 15 April 1986)

Communicated by J. H. Rubinstein

Abstract

Topological algebra bundles whose fibre (-algebras) admit functional representations constitute a category, antiequivalent with that of (topological) fibre bundles having completely regular bundle spaces and locally compact fibres.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 46 H 05, 46 M 20; secondary 55 R 05, 55 R 10.
Keywords and phrases: topological algebra bundle, algebraic chart, algebraic atlas, system of transition functions, $\mathscr{C}(X)$-algebra, spectrum bundle, spectrum functor, locally m-convex algebra bundle, functional representations, category antiequivalence, spectrum of a topological algebra, Waelbroeck algebra, Gel′fand-Mazur (topological) algebra.

In [5] B. R. Gelbaum has proved the following. We are given a fibre bundle E on a space X with fibre a commutative or Q-uniform Banach algebra A and group-bundle the group of isometric automorphisms of A. Then, the maximal ideal space of the Banach algebra of sections of E is a fibre bundle over the same base as E, fibre the maximal ideal space of A and group-bundle the group of self-homeomorphisms of the latter space.

In this paper we enrich the previous picture by getting a kind of an inverse of Gelbaum's result. Furthermore, the whole stage is put within a more general framework, that of "topological algebra bundles". So, one considers a triplet $\xi = (E, \pi, X)$, where $\pi\colon E \to X$ is a given map of a set E into a topological space X, the latter map being further specified, up to equivalence, by "algebraic" atlases. The preceding is patterned after certain current results of A. Mallios [11, 12], as well as some ideas in the above quoted work of B. R. Gelbaum.

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https://doi.org/10.1017/S1446788700029670 Published online by Cambridge University Press

More particularly, in Section 1 we deal with standard facts of the general theory of topological algebra bundles getting thus results analogous to those of the classical theory of vector bundles [7] (see Theorem 1.3, Remark 1.1). More specifically, we give a characterization of a topological algebra bundle through a system of transition functions on X (Theorem 1.2), a technique which is systematically applied throughout the rest of the paper. Then in Section 2 we exhibit the set of continuous sections of E as a topological algebra (see (2.3)ff), in effect, as an algebra of continuous functions with values in the algebras-fibres of the bundle E (Theorem 2.1). The above makes it possible to get our main results, as these are presented in the subsequent Sections 3 and 4.

Thus, more precisely, let $\mathscr{E}(X)$ be the category of topological algebra bundles over X and $\mathfrak{M}(\xi)$ the "spectrum bundle" of an object ξ in $\mathscr{E}(X)$, defined through the "spectrum functor" $\mathfrak{M}$ (see (3.9) and Theorem 3.1). Moreover, let $\mathscr{F}(X)$ be the category of fibre bundles over X. Then, one defines a contravariant functor $\mathscr{C}$: $\mathscr{F}(X) \to \mathscr{E}(X)$ (Theorem 3.2), which is fully faithful [7], if the objects of $\mathscr{F}(X)$ have completely regular bundle spaces and locally compact fibres (we denote this subcategory of $\mathscr{F}(X)$ by $^{\cdot}\mathscr{F}(X)$). Thus, if $^{\cdot}\mathscr{E}(X)$ denotes the category of topological algebra bundles whose fibres ($M_\alpha, \alpha \in K$) admit functional representations (viz. $\mathscr{C}(\mathfrak{M}(M_\alpha)) \cong M_\alpha$), then the categories $^{\cdot}\mathscr{E}(X)$ and $^{\cdot}\mathscr{F}(X)$ are antiequivalent (Corollary 3.1). In particular, one concludes that $\xi \in {}^{\cdot}\mathscr{F}(X)$ and $\mathscr{C}(\xi) \in {}^{\cdot}\mathscr{E}(X)$ (Theorem 3.2) are in a kind of "canonical dualtiy"; namely, ξ is isomorphic to the bundle obtained from $\mathscr{C}(\xi)$ (Theorem 3.1) via the application of the "spectrum functor" $\mathfrak{M}$. Naturally, this constitutes one further application of the classical concept of numerical spectrum (Gel′fand space) of a topological algebra. Analogous results are valid for holomorphic (algebra) bundles (Scholium 3.1). The above are mainly based on the following fact: Taking a topological fibre bundle $\xi = (E, \pi, X)$ from $\mathscr{F}(X)$, $\mathscr{C}_c(E)$ is identified (as a locally m-convex algebra) with $\Gamma(\mathscr{C}(\xi))$ (see Theorem 3.3). This same result exhibits a generalization in the context of the theory of fibre bundles of the well known identity $\mathscr{C}_c(X \times Y) = \mathscr{C}_c(X, \mathscr{C}_c(Y))$, with Y a locally compact space [4].

Finally, we consider in an Appendix (Section 4) another realization of the spectrum bundle $\mathfrak{M}(\xi)$ of a suitable $\xi \in \mathscr{E}(X)$, proving that the (numerical) spectrum of the algebra of sections of ξ ($\mathfrak{M}(\Gamma(\xi))$) defines a fibre bundle over the compact base X of ξ (Theorem 4.1). The latter extends, within the present framework, an analogous result of Gelbaum in [5]. The technique applied is both different from that in [5] and simpler from some points of view. In addition the preceding specializes to the standard result that $\mathfrak{M}(\mathscr{C}_c(X))$ and X are homeomorphic, when X is a completely regular space [9]. A similar result has been given in [6] by considering fibre tensor product bundles. In a future publication we hope to extend Theorem 4.1 to the case of a non compact base space X.

https://doi.org/10.1017/S1446788700029670 Published online by Cambridge University Press

It is a pleasant duty to express my sincere thanks to Professor Anastasios Mallios for generous help and advice, as well as lively interest, during the preparation of this work. Among other things it was his idea to set the whole stage in the context of topological algebras in general (not locally convex or yet locally m-convex ones), and also to apply a categorical language to my initial main result.

1.

Given a set E, a (Hausdorff) topological space X and a map $\pi\colon E \to X$, an *algebraic chart* (or simply *chart*) of E is a triplet $s \equiv (U, \varphi_U, M)$, where U is an open subset of X, M a topological algebra and φ_U a bijection

$$(1.1) \qquad \varphi_U\colon U \times M \to \pi^{-1}(U)$$

such that $\pi(\varphi_U(x, a)) = x$, for any $(x, a) \in U \times M$; thus, for every $x \in U$, the map

$$(1.2) \qquad \varphi_{U,x} := \left.\varphi_U^{-1}\right|_{\pi^{-1}(x)}\colon \pi^{-1}(x) \to M$$

is a bijection. Now, given two charts $s_1 \equiv (U, \varphi_U, M)$ and $s_2 \equiv (V, \psi_V, N)$ we say that they are *compatible*, if the next two conditions are satisfied:

$$(1.3a) \qquad \psi_{V,s} \circ \varphi_{U,x}^{-1} \in \operatorname{Hom}(M, N)$$

for every $x \in U \cap V$, and moreover the map

$$(1.3b) \qquad x \mapsto \psi_{V,x} \circ \varphi_{U,x}^{-1}\colon U \cap V \to \operatorname{Hom}_s(M, N)$$

is continuous, where $\operatorname{Hom}_s(M, N)$ is the set of continuous algebra morphisms of M into N equipped with the simple convergence topology induced on it by $\mathscr{L}_s(M, N)$. If the algebras involved have identities, the elements of $\operatorname{Hom}(M, N)$ are assumed to respect identities. Moreover, since

$$(1.4) \qquad \left(\psi_{V,x} \circ \varphi_{U,x}^{-1}\right)^{-1} = \varphi_{U,x} \circ \psi_{V,x}^{-1} \in \operatorname{Hom}(N, M),$$

for every $x \in U \cap V$, we observe that the range of (1.3b) is, in fact, the set of topological algebraic isomorphisms of M onto N, denoted by $\operatorname{Iso}(M, N)$.

Furthermore, the compatibility of s_1, s_2 is, in effect, equivalent with the existence of a continuous map, say λ, such that

$$(1.5) \qquad \lambda\colon U \cap V \to \operatorname{Hom}_s(M, N)\colon x \mapsto \lambda(x) := \psi_{V,x} \circ \varphi_{U,x}^{-1}.$$

Thus, for any $x \in U \cap V$ and $a \in M$, one has

$$(1.6) \qquad \left(\psi_V^{-1} \circ \varphi_U\right)(x, a) = \left(x, [\lambda(x)](a)\right) \equiv \left(x, \lambda(x)a\right).$$

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A family $\mathscr{A} = \{(U_\alpha, \varphi_\alpha, M_\alpha)\}_{\alpha \in K}$ of charts of E is called an *algebraic atlas* (or simply *atlas*) of E, if the charts of $\mathscr{A}$ are mutually equivalent and $X = \bigcup_{\alpha \in K} U_\alpha$. Moreover, two atlases $\mathscr{A}$, $\mathscr{B}$ of E are said to be equivalent ($\mathscr{A} \sim \mathscr{B}$) if $\mathscr{A} \cup \mathscr{B}$ is also an atlas of E. The last notion provides an equivalence relation in the set of all atlases of E, in such a way that one sets

DEFINITION 1.1. (A. Mallios). A triplet $\xi = (E, \pi, X)$, as above, is said to be a *topological algebra bundle* over X, if we are given an equivalence class of algebraic atlases, say $[\mathscr{A}]$, of E.

Locally (m-) convex algebra bundles will have the obvious meaning by specializing to the pertinent class of topological algebras for the fibres.

Now, given an atlas $\mathscr{A}$ of E, and hence its equivalence class $[\mathscr{A}]$, we may always consider the associated to $\mathscr{A}$ *maximal atlas* of E say $\mathscr{A}^*$, consisting of all charts of E compatible with the charts of $\mathscr{A}$, that is,

$$\mathscr{A}^* = \bigcup_{\mathscr{B} \in [\mathscr{A}]} \mathscr{B}. \tag{1.7}$$

As we shall presently see, the last atlas defines E, in a unique way, as a topological algebra bundle.

We first give the following lemma, a useful tool for the realization of a topological algebra bundle (see Theorems 1.1 and 1.2 below). It also extends within the present context an analogous result of [11, Lemma 2.1], [7, Theorem 1.12, page 3].

LEMMA 1.2. *Given the topological algebras M, N and a (Hausdorff) topological space X, consider the following two assertions*:

(1) *The map h: $X \to \operatorname{Hom}_s(M, N)$ is continuous.*

(2) *The map $\tilde{h}$: $X \times M \to X \times N$: $(x, a) \mapsto \tilde{h}(x, a) = (x, [h(x)](a))$ is continuous.*

Then, (2) $\Rightarrow$ (1). *Moreover, if $\operatorname{Hom}_s(M, N)$ is a locally equicontinuous subset of $\mathscr{L}_s(M, N)$, then the previous two assertions are equivalent. In this respect, one defines $x \mapsto h(x) := \tilde{h}_x$ ($\equiv \tilde{h}(x, \cdot)$), $x \in X$.*

PROOF. (2) $\Rightarrow$ (1): The map $h(x) = (p_2 \circ \tilde{h})_x$ (partial map of $p_2 \circ \tilde{h}$), where p_2 is the projection p_2: $X \times N \to N$, is continuous for every $x \in X$. Thus, the relation

$$[h(x)](a) = p_2(\tilde{h}(x, a)),$$

with $(x, a) \in X \times M$, proves the assertion.

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Now, if $\mathrm{Hom}(M, N) \subseteq \mathscr{L}_s(M, N)$ is locally equicontinuous, one still proves that (1) ⇒ (2): Indeed, the map $\tilde{h}$ is the composition of the maps

$$X \times M \xrightarrow{\delta} X \times \mathrm{Hom}_s(M, N) \times M \xrightarrow{\epsilon} X \times N,$$

where $(x, a) \mapsto \delta(x, a) := (x, h(x), a)$ is a continuous map by the continuity of h; besides, $(x, h(x), a) \mapsto \varepsilon(x, h(x), a) := (x, h(x)a)$ is also continuous by hypothesis for $\mathrm{Hom}_s(M, N)$ (see [3, Chapter X, Section 2.2, Corollary 4]).

The local equicontinuity of $\mathrm{Hom}_s(M, N)$ was, of course, a crucial assumption for the proof of the previous lemma, and this is also the case for the next Theorems 1.1, 1.2. This condition is valid, for example, in case of Fréchet locally convex algebras, when by $\mathrm{Hom}_s(M, N)$ one should mean, of course, continuous morphisms of the whole structure [3, Chapter X, Section 3.1, Example 2)].

Now, given a topological algebra bundle $\xi = (E, \pi, X)$, we topologize E as follows: A subset A of E is *open*, if $\varphi_\alpha^{-1}(A \cap \pi^{-1}(U_\alpha)) \subseteq U_\alpha \times M_\alpha$ is open, for every $\alpha \in K$, where $\mathscr{A} = \{(U_\alpha, \varphi_\alpha, M_\alpha)\}_{\alpha \in K}$ is a given atlas of ξ. Thus, one gets a topology $\mathscr{T}_{\mathscr{A}}$ of E (not necessarily Hausdorff), which is, of course, independent of the atlas $\mathscr{A}$ considered.

The next result is analogous to that of [11, Lemma 2.2] for **A**-vector bundles.

THEOREM 1.1. *Let $\xi = (E, \pi, X)$ be a topological algebra bundle and $\mathscr{A} = \{(U_\alpha, \varphi_\alpha, M_\alpha)\}_{\alpha \in K}$ an atlas of ξ such that the sets $\mathrm{Isom}_s(M_\alpha, M_\beta)$ are locally equicontinuous. Then, $\mathscr{T}_{\mathscr{A}}$ is the unique topology on E making π continuous and the maps φ_α: $U_\alpha \times M_\alpha \to \pi^{-1}(U_\alpha)$, $\alpha \in K$, homeomorphisms. Moreover, each fibre $E_x \equiv \pi^{-1}(x)$, $x \in X$, of E is uniquely defined as a topological algebra whose topology is the relative one induced on it by E.*

PROOF. For any charts $(U_\alpha, \varphi_\alpha, M_\alpha)$, $(U_\beta, \varphi_\beta, M_\beta)$ of $\mathscr{A}$, the set

$$(1.8)\quad \varphi_\alpha^{-1}\big(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)\big) = \varphi_\alpha^{-1}\big(\pi^{-1}(U_\alpha \cap U_\beta)\big) = \big(U_\alpha \cap U_\beta\big) \times M_\alpha$$

is open, for every $\alpha \in K$, such that $\pi^{-1}(U_\alpha) \in \mathscr{T}_{\mathscr{A}}$, for any $\alpha \in K$, hence the continuity of π. Moreover, by Lemma 1.1 and the continuity of (1.3b) (see also (1.6)),

$$(1.9)\qquad \varphi_\beta^{-1} \circ \varphi_\alpha : \big(U_\alpha \cap U_\beta\big) \times M_\alpha \to \big(U_\alpha \cap U_\beta\big) \times M_\beta$$

is continuous, and hence its inverse map $\varphi_\alpha^{-1} \circ \varphi_\beta$ too, so that the map φ_α: $U_\alpha \times M_\alpha \to \pi^{-1}(U_\alpha)$, $\alpha \in K$ is a homeomorphism. Furthermore, for every $x \in U_\alpha$, $\pi^{-1}(x) \subseteq E$ is endowed by (1.2) with the structure of M_α. The last assumption is independent of the choice of chart, since, for every $x \in U_\alpha \cap U_\beta$, the map $(\varphi_\beta^{-1} \circ \varphi_\alpha)_x \equiv \varphi_{\beta,x}^{-1} \circ \varphi_{\alpha,x} \in \mathrm{Isom}(M_\alpha, M_\beta)$ (see (1.3b)). Thus, the resulting topology of $\pi^{-1}(x) \cong M_\alpha$, coincides with the relative one induced on it from the open set $\pi^{-1}(U_\alpha) \subseteq E$, since φ_α, $\alpha \in K$, are homeomorphisms. On the other hand, the

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uniqueness of the topology of E satisfying the preceding conditions follows by a standard argument.

The above theorem yields, in particular, that a basis of $\mathcal{T}_{\mathscr{A}}$ consists of the open sets $\pi^{-1}(U)$ for any given chart $(U, \varphi, M) \in \mathscr{A}^*$ (see (1.7)).

On the other hand, in analogy with the standard notion of a (complex) vector bundle (see, for instance, [1] and/or [7]), we can define topological algebra bundles as follows:

Let X be a topological space and $(E_x)_{x \in X}$ a family of topological algebras. Moreover, if $E = \sum_x E_x$ is the respective vector space direct sum of the family $(E_x)_{x \in X}$, we consider E topologized so that the relative topology on each E_x is the initial one of the algebra E_x, $x \in X$, and the canonical projection π: $E \to X$ is continuous.

Now, a triplet $\xi = (E, \pi, X)$ as above is said to be a *topological algebra bundle*, if there exists an open covering $\mathscr{U} = \{U\}$ of X, such that $\xi|_U$ is a *trivial topological algebra bundle*; that is, there exists a topological algebra M and a homeomorphism φ_U: $U \times M \to \pi^{-1}(U) \equiv E|_U$, in such a way that, for every $x \in U$, the map $\varphi_{U,x} \equiv \varphi_U^{-1}|_{\pi^{-1}(x)}$: $\pi^{-1}(x) \to M$ is an algebraic isomorphism, hence an algebraic homeomorphism.

Consequently, Theorem 1.1 implies that *the notion of a topological algebra bundle through a family of topological algebras over a topological space X is equivalent to that of Definition* 1.1.

Now, given a topological algebra bundle $\xi = (E, \pi, X)$ and an atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha, M_\alpha)\}_{\alpha \in K}$ of E, one always gets a family of continuous functions

$$\lambda_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \operatorname{Hom}_s(M_\beta, M_\alpha), \tag{1.10}$$

by the relation $\lambda_{\alpha\beta}(x) := \varphi_{\alpha,x} \circ \varphi_{\beta,x}^{-1}$ (see (1.5)) such that the relative "cocycle condition"

$$\lambda_{\alpha\beta}(x) \cdot \lambda_{\beta\gamma}(x) = \lambda_{\alpha\gamma}(x), \qquad x \in U_\alpha \cap U_\beta \cap U_\gamma, \tag{1.11}$$

to be valid. Therefore, the relation

$$\lambda_{\alpha\alpha}(x) = \mathrm{id}_{M_\alpha} \tag{1.12}$$

for any $x \in U_\alpha$, $\alpha \in K$, is also true. By (1.11) (see also (1.5)) one has

$$\lambda_{\alpha\beta}(x) \in \operatorname{Isom}_s(M_\beta, M_\alpha) \tag{1.13}$$

for every $x \in U_\alpha \cap U_\beta$.

A triplet $\Lambda \equiv \{(U_\alpha), (\lambda_{\alpha\beta}), (M_\alpha)\}$ consisting of an open covering (U_α) of a topological space X, a family of topological algebras (M_α) and a family of continuous maps $\lambda_{\alpha\beta}$: $U_\alpha \cap U_\beta \to \operatorname{Hom}_s(M_\beta, M_\alpha)$ satisfying (1.11), is said to be a *system of transition functions* on X.

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The preceding characterizes, in fact, topological algebra bundles according to the next

THEOREM 1.2. *Let X be a (Hausdorff) topological space and $\Lambda \equiv \{(U_\alpha), (\lambda_{\alpha\beta}), (M_\alpha)\}$ a system of transition functions on X such that the sets* $\mathrm{Isom}_s(M_\beta, M_\alpha)$ *are locally equicontinuous. Then, there exists a topological algebra bundle $\xi = (E, \pi, X)$, having Λ as a system of transition functions.*

PROOF. On the disjoint union $T \equiv \biguplus_\alpha (U_\alpha \times M_\alpha)$ we define an equivalence relation as follows

$$(1.14) \quad (x_\alpha, a_\alpha) \sim (x_\beta, a_\beta) \quad \text{if and only if} \quad \begin{cases} x_\alpha = x_\beta \equiv x \in U_\alpha \cap U_\beta, \\ a_\alpha = \lambda_{\alpha\beta}(x) a_\beta. \end{cases}$$

Let $E \equiv T/\sim$ be the corresponding quotient set and $q\colon T \to E$ the canonical quotient map. We consider T equipped with the direct sum topology and E with the respective quotient topology. Now, we consider the map

$$(1.15) \qquad \pi\colon E \to X\colon [(x_\alpha, a_\alpha)] \mapsto x_\alpha,$$

which is uniquely defined via (1.14) and also continuous: Indeed, π is continuous if and only if the maps $\pi \circ q \circ j_\alpha$ are continuous ($j_\alpha\colon U_\alpha \times M_\alpha \hookrightarrow T$ is the canonical injection), which is, of course, true since $\pi \circ q \circ j_\alpha = pr_\alpha \circ (i_\alpha \times id_{M_\alpha})$, where $pr_\alpha\colon X \times M_\alpha \to X$ is the projection and $i_\alpha\colon U_\alpha \hookrightarrow X$ the inclusion map.

Moreover, for any $\alpha \in K$, we consider the map

$$(1.16) \qquad \varphi_\alpha = q|_{U_\alpha \times M_\alpha},$$

whose range is $\pi^{-1}(U_\alpha)$, since (1.15) implies $\pi(q(x_\alpha, a_\alpha)) = x_\alpha$ and therefore $(\pi \circ \varphi_\alpha)(x_\alpha, a_\alpha) = x_\alpha \in U_\alpha$. Besides, (1.16) is a homeomorphism: Indeed, φ_α is continuous by the continuity of q and moreover "onto" since for $b \equiv [(x_\beta, a_\beta)] \in \pi^{-1}(U_\alpha)$, one has $x_\beta \in U_\alpha \cap U_\beta$ and $(x_\alpha, a_\alpha) \sim (x_\alpha, \lambda_{\alpha\beta}(x_\beta) a_\beta)$, such that $b = \varphi_\alpha(x_\alpha, \lambda_{\alpha\beta}(x) a_\beta)$ (see (1.14)). Furthermore, the relation $(x_\alpha, a_\alpha) \sim (x'_\alpha, a'_\alpha)$ implies that $x_\alpha = x'_\alpha$ and $a'_\alpha = \lambda_{\alpha\alpha}(x) a_\alpha = \mathrm{id}_{M_\alpha}(a_\alpha) = a_\alpha$, which shows that φ_α is 1-1. Finally, φ_α^{-1} is continuous, by the continuity of the composition of the following continuous maps

$$(U_\alpha \cap U_\beta) \times M_\beta \to (U_\alpha \cap U_\beta) \times M_\alpha \to U_\alpha \times M_\alpha$$

(see Lemma 1.1, (1) $\Rightarrow$ (2)).

Now, by considering the map $\varphi_{\alpha,x}^{-1} \circ \varphi_{\beta,x}\colon M_\beta \to M_\alpha$, $x \in U_\alpha \cap U_\beta$, for each $a_\beta \in M_\beta$, let $a_\alpha = (\varphi_{\alpha,x}^{-1} \circ \varphi_{\beta,x})(a_\beta)$. Then, $\varphi_\beta(x, a_\beta) = \varphi_\alpha(x, a_\alpha)$ which means $(x, a_\beta) \sim (x, a_\alpha)$ (see (1.14)), and therefore $a_\alpha = \lambda_{\alpha\beta}(x) a_\beta$. Thus, one gets $(\varphi_{\alpha,x}^{-1} \circ \varphi_{\beta,x})(a_\beta) = \lambda_{\alpha\beta}(x) a_\beta$, which proves the assertion.

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On the other hand, two systems of transition functions on X, say $\Lambda = \{(U_\alpha),(\lambda_{\alpha\beta}),(M_\alpha)\}$, $\Lambda' = \{(V_i),(\mu_{ij}),(N_i)\}$, are *equivalent* if there exist continuous maps

$$(1.17) \qquad p_\alpha^i\colon U_\alpha \cap V_i \to \mathrm{Isom}_s(M_\alpha, N_i)$$

such that

$$(1.18) \qquad p_\alpha^i(x) \cdot \lambda_{\alpha\beta}(x) \cdot p_\beta^j(x)^{-1} = \mu_{ij}(x)$$

for $x \in U_\alpha \cap U_\beta \cap V_i \cap V_j$. This is an equivalence relation in the set of systems of transition functions (see [7, page 10] for an analogous proof). Thus, by extending the standard terminology pertaining to *G-cocycles* (G being a topological group [7, page 9]), we denote by $H^1(X)$ the corresponding quotient set. Now, given a topological space X, let $\mathscr{E}(X)$ be the category whose objects are topological algebra bundles over X (Definition 1.1). Thus, given $\xi = (E, \pi, X)$, $\xi' = (E', \pi', X)$ of $\mathscr{E}(X)$, a *morphism* of ξ into ξ' is a fibre preserving continuous map $a\colon E \to E'$ (that is, $\pi' \circ a = \pi$), such that

$$(1.19) \qquad a_x\left(\equiv a|_{\pi^{-1}(x)}\right) \in \mathrm{Hom}(E_x, E'_x), \qquad x \in X$$

(topological algebra morphisms).

So, by considering isomorphism classes of the objects of $\mathscr{E}(X)$ one gets the following result analogous to [7, Theorem 3.6, page 10], [12, Theorem 1.2], which provides an equivalence version of Theorem 1.2.

THEOREM 1.3. *Let $\Phi(X)$ be the set of isomorphism classes of topological algebra bundles over a topological space X having locally equicontinuous sets of fibre isomorphisms (see* (1.4)). *Moreover, let $H^1(X)$ be the set of equivalence classes of systems of transition functions on X, as above (see* (1.18)). *Then,*

$$(1.20) \qquad \Phi(X) = H^1(X)$$

within a bijection.

Theorem 1.3 yields that the topological algebra bundle constructed by Theorem 1.2 is unique up to equivalence.

REMARK 1.1. Given a topological algebra bundle $\xi = (E, \pi, X)$ of fibre type a topological algebra M, let $\mathscr{A} \equiv \{(U_\alpha, \varphi_\alpha, M);\ \alpha \in K\}$ be an atlas of E and $\Lambda = \{(U_\alpha),(\lambda_{\alpha\beta}), M;\ \alpha \in K\}$ a system of transition functions for E. Thus, one has $\lambda_{\alpha\beta}(x) \in \mathscr{A}ut(M)$, for every $x \in U_\alpha \cap U_\beta$, where $\mathscr{A}ut(M)$ is the set of topological algebraic self-isomorphisms of the topological algebra M. Thus, if *$\mathscr{A}ut(M)$ is a locally equicontinuous subset of* $\mathrm{Hom}_s(M, M)$, then it is a *topological group* (see [3, Chapter X, Section 3.5, Corollary of Proposition 10] and also [6,

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Section 2]), such that Λ is a G-cocycle (see [7, page 9]), for $G = \mathcal{A}ut(M)$. In this case (1.20) takes the more standard form

$$\Phi(X) = H^1(X, \mathcal{A}ut(M)), \tag{1.21}$$

within a bijection.

2.

Let $\xi = (E, \pi, X)$ be a topological algebra bundle, $\mathcal{A} \equiv \{(U_\alpha, \varphi_\alpha, M_\alpha)\}_{\alpha \in K}$, an atlas of ξ and $\Gamma(\xi)$ the set of (continuous) sections of ξ. Now, $\Gamma(\xi)$ is made into a topological algebra and, in fact, a $\mathcal{C}_c(X)$-algebra, where $\mathcal{C}_c(X)$ denotes the (locally m-convex) algebra of $\mathbf{C}$-valued continuous functions on X, endowed with the compact-open topology. Thus, for $\gamma, \gamma_1, \gamma_2 \in \Gamma(\xi)$, $f, f_1, f_2 \in \mathcal{C}(X)$ and $x \in U_\alpha$, we define

$$\begin{aligned} (f \cdot \gamma)(x) = (\gamma \cdot f)(x) &:= \varphi_\alpha(x, f(x) \cdot t_\alpha(\gamma(x))), \\ (f_1 \cdot \gamma_1 + f_2 \cdot \gamma_2)(x) &:= \varphi_\alpha(x, f_1(x) \cdot t_\alpha(\gamma_1(x)) + f_2 \cdot t_\alpha(\gamma_2(x))), \\ (\gamma_1 \cdot \gamma_2)(x) &:= \varphi_\alpha(x, t_\alpha(\gamma_1(x)) \cdot t_\alpha(\gamma_2(x))) \end{aligned} \tag{2.1}$$

where $t_\alpha \equiv \varphi_{\alpha, x}$ (see (1.3)).

The elements $f \cdot \gamma$, $\gamma \cdot f$, $f_1 \cdot \gamma_1 + f_2 \cdot \gamma_2$, $\gamma_1 \cdot \gamma_2$ belong to $\Gamma(\xi)$ and, moreover, these are independent of the choice of U_α: Indeed, if $x \in U_\beta$, then by (2.1)

$$\begin{aligned} (\gamma_1 \cdot \gamma_2)(x) &= \varphi_\alpha(x, t_\alpha(\gamma_1(x)) \cdot t_\alpha(\gamma_2(x))) \\ &= \varphi_\alpha\big(x, \lambda_{\alpha\beta}(x)\big(t_\beta(\gamma_1(x)) \cdot t_\beta(\gamma_2(x))\big)\big) \\ &= \varphi_\beta\big(x, t_\beta(\gamma_1(x)) \cdot t_\beta(\gamma_2(x))\big) \quad \text{(by (1.5))}. \end{aligned}$$

We also define the zero section $\mathbf{0}(x) := \varphi_\alpha(x, 0)$ and moreover, if the algebras M_α have identities 1_α, we still define $\mathbb{1}(x) := \varphi_\alpha(x, 1_\alpha)$, $x \in U_\alpha$, these sections being also independent of the choice of U_α; furthermore, for all $\gamma \in \Gamma(\xi)$, one gets

$$\gamma \cdot \mathbb{1} = \mathbb{1} \cdot \gamma = \gamma, \quad 0 \cdot \gamma = \gamma \cdot 0 = 0, \quad \gamma + 0 = 0 + \gamma = \gamma.$$

Thus, $\Gamma(\xi)$ is a *$\mathcal{C}(X)$-algebra* (with identity element if the M_α are unital algebras). Before we define $\Gamma(\xi)$ as a topological algebra, some more comments are necessary.

So, let E be a topological algebra and $\mathcal{C}_u(K, E)$ the algebra of continuous E-valued functions on a compact space K endowed with the topology of uniform convergence in K. Then, by definition of the topology u, $\mathcal{C}_u(K, E)$ is a topological algebra (see [12, Lemma 2.1]). Moreover, the algebra of continuous E-valued functions on a topological space X endowed with the topology of compact convergence in X (denoted by $\mathcal{C}_c(X, E)$) is also a topological algebra, as this

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follows by

$$\mathscr{C}_c(X, E) = \varprojlim \mathscr{C}_u(K, E), \tag{2.2}$$

where the projective limit is defined over the compact subspaces K of X (see [9, Chapter VI, Section 4, (4.12)]).

Now, for any U_α, the local sections of ξ "are" M_α-valued continuous functions on U_α due to the "local triviality" of ξ; that is one has the following algebra isomorphism

$$\Gamma(U_\alpha, \xi) \underset{t}{\cong} \mathscr{C}(U_\alpha, M_\alpha). \tag{2.3}$$

Thus, if $\Gamma(U_\alpha, \xi)$ is equipped with the (compact-open) topology of $\mathscr{C}_c(U_\alpha, M_\alpha)$ then $\Gamma(U_\alpha, \xi)$ becomes a topological algebra of the same type with M_α. Moreover, if i_α: $\Gamma(\xi) \to \Gamma(U_\alpha, \xi)$ is the canonical restriction map ($X = \bigcup_\alpha U_\alpha$), then $\Gamma(\xi)$ endowed with the inverse topology defined via i_α, is a topological algebra of the same type with M_α. In particular, if M_α is a (complete) locally (m-) convex algebra, $\Gamma(\xi)$ is such an algebra too, while if M_α is a Fréchet locally (m-) convex algebra and X a second countable space, $\Gamma(\xi)$ is a Fréchet locally (m-) convex algebra as well. The seminorms which define the locally (m-) convex topology on $\Gamma(U_\alpha, \xi)$ are of the form

$$N_{K_\alpha, p_\alpha, \lambda}(\gamma) := \sup_{x \in K_\alpha \subseteq U_\alpha} \left(p_{\alpha,\lambda}(t_\alpha(\gamma(x))) \right), \tag{2.4}$$

where K_α is a compact subspace of U_α and $p_{\alpha,\lambda}$ a seminorm from the family of seminorms $\{p_{\alpha,\lambda}\}_\lambda$ defining the locally (m-) convex topology of M_α.

The fact that the local sections of ξ are (usual (!) vector-valued) continuous functions suggests now a more convenient form for the algebra $\Gamma(\xi)$.

Namely, if $\prod_\alpha \mathscr{C}_c(U_\alpha, M_\alpha)$ is the topological algebra, cartesian product of the topological algebras $\mathscr{C}_c(U_\alpha, M_\alpha)$, consider the algebra

(2.5)

$$\mathscr{B} = \left\{ \tau \equiv (\tau_\alpha) \in \prod_\alpha \mathscr{C}_c(U_\alpha, M_\alpha) \colon \tau_\alpha(x) = \left[\lambda_{\alpha\beta}(x)\right]\left(\tau_\beta(x)\right),\ x \in U_\alpha \cap U_\beta \right\}.$$

We assume $\mathscr{B}$ endowed with the relative topology from $\prod_\alpha \mathscr{C}_c(U_\alpha, M_\alpha)$, so that $\mathscr{B}$ is a topological algebra too. In particular, if M_α are unital algebras, $\mathscr{B}$ is such, since $\lambda_{\alpha\beta}(x)$, $x \in U_\alpha \cap U_\beta$, is identity preserving (see (1.3b) and also (2.5)).

The following identifies the algebra $\mathscr{B}$ as $\Gamma(\xi)$.

THEOREM 2.1. *Let* $\Gamma(\xi)$ *be the algebra of sections of a topological algebra bundle* ξ *and* $\mathscr{B}$ *the algebra defined by* (2.5). *Then,*

$$\Gamma(\xi) = \mathscr{B}, \tag{2.6}$$

within an isomorphism of topological algebras.

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PROOF. By (2.5), for any $\tau \equiv (\tau_\alpha)_\alpha$ in $\mathscr{B}$ and $x \in U_\alpha \cap U_\beta$, one has $\tau_\alpha(x) = [\lambda_{\alpha\beta}(x)](\tau_\beta(x))$ such that (1.5) implies $\varphi_\alpha(x, \tau_\alpha(x)) = \varphi_\beta(x, \tau_\beta(x))$. Thus, by setting $\tau(x) := \varphi_\alpha(x, \tau_\alpha(x))$ one has $\pi(\varphi_\alpha(x, \tau_\alpha(x))) = x$, such that the map

$$\omega\colon \mathscr{B} \to \Gamma(\xi)\colon \tau \mapsto \omega(\tau)(x) := \varphi_\alpha(x, \tau_\alpha(x)) \tag{2.7}$$

is well defined; that is, $\omega(\tau)$ is a continuous section of ξ, since φ_α is a homeomorphism and continuity is of a "local nature".

On the other hand, ω *is an algebra morphism*. Thus, for any τ, τ' in $\mathscr{B}$ and $x \in X$, one gets $(\omega(\tau) \cdot \omega(\tau'))(x) = \omega(\tau)(x) \cdot \omega(\tau')(x) = \varphi_\alpha(x, \tau_\alpha(x)) \cdot \varphi_\alpha(x, \tau'_\alpha(x)) = \varphi_\alpha(x, \tau_\alpha(x) \cdot \tau'_\alpha(x)) = \varphi_\alpha(x, (\tau_\alpha \cdot \tau'_\alpha)(x)) = \omega(\tau \cdot \tau')(x)$ (similarly for the other operations). Furthermore, if $\tau, \tau' \in \mathscr{B}$ with $\omega(\tau) = \omega(\tau')$ then, for any $x \in U_\alpha$, $\varphi_\alpha(x, \tau_\alpha(x)) = \varphi_\alpha(x, \tau'_\alpha(x))$ and hence $\tau_\alpha = \tau'_\alpha$, that is, $\tau = \tau'$, which shows that ω is 1-1. Now, each $\gamma \in \Gamma(\xi)$ is locally in $\mathscr{C}(U_\alpha, M_\alpha)$, that is, $\gamma|_{U_\alpha} \equiv \gamma_\alpha \in \Gamma(U_\alpha, \xi) \underset{l}{\cong} \mathscr{C}_c(U_\alpha, M_\alpha)$, so that $\tau_\alpha := l(\gamma_\alpha)$ (see (2.3)). Indeed, $\tau \equiv (\tau_\alpha)_\alpha \in \prod_\alpha \mathscr{C}_c(U_\alpha, M_\alpha)$ is an element of $\mathscr{B}$. Thus, for any $x \in U_\alpha \cap U_\beta$, we have $\varphi_\alpha(x, t_\alpha(\gamma(x))) = \varphi_\beta(x, t_\beta(\gamma(x))) =$ (by (1.5)) $\varphi_\alpha(x, [\lambda_{\alpha\beta}(x)](t_\beta(\gamma(x))))$, which entails $t_\alpha(\gamma(x)) = [\lambda_{\alpha\beta}(x)](t_\beta(\gamma(x)))$, that is $\tau_\alpha(x) = l(\gamma_\alpha)(x) = [\lambda_{\alpha\beta}(x)](l(\gamma_\beta))(x) = [\lambda_{\alpha\beta}(x)](\tau_\beta(x))$. Hence ω is onto. Finally, ω *is bicontinuous*. That is, since, by definition, $\Gamma(\xi)$ has the initial topology of the canonical restriction map $i_\alpha\colon \Gamma(\xi) \to \Gamma(U_\alpha, \xi)$, it suffices to prove the continuity of each one of the maps $i_\alpha \circ \omega$, which is valid since $i_\alpha \circ \omega = l \circ p_\alpha \circ j$, the second map being, of course, continuous, where $p_\alpha\colon \prod_\alpha \mathscr{C}_c(U_\alpha, M_\alpha) \to \mathscr{C}_c(U_\alpha, M_\alpha)$ is the canonical projection and $j\colon \mathscr{B} \hookrightarrow \prod_\alpha \mathscr{C}_c(U_\alpha, M_\alpha)$ the inclusion map (see also (2.3)). Moreover, by the definition of the topology on $\mathscr{B}$, $\omega^{-1}\colon \Gamma(\xi) \to \mathscr{B}$ is continuous if and only if $p_\alpha \circ j \circ \omega^{-1}$ is continuous, which is valid because $p_\alpha \circ j \circ \omega^{-1} = l \circ i_\alpha$.

3.

In the present section we establish a (category) antiequivalent between the category of topological algebra bundles $\mathscr{E}(X)$ (see Section 1) and that of (locally trivial) fibre bundles $\mathscr{F}(X)$ over a topological space X (see Corollary 3.1). An object of $\mathscr{F}(X)$ is a triplet $\xi = (E, \rho, X)$ endowed with an atlas $\mathscr{A} \equiv \{(U_\alpha, \varphi_\alpha, M_\alpha)\}$ (Definition 1.1), where now the fibre M_α is a topological space. On the other hand, a morphism $a\colon \xi = (E, \rho, X) \to \xi' = (E', \rho', X)$ in $\mathscr{F}(X)$ is a continuous map $a\colon E \to E'$ such that $\rho' \circ a = \rho$; that is, a is "fibre preserving", such that

$$a_x \equiv a|_{\rho^{-1}(x)}\colon \rho^{-1}(x) \to \rho'^{-1}(x), \qquad x \in X, \tag{3.1}$$

is a continuous map.

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Let $\xi = (E, \pi, X)$ be a semi-simple topological algebra bundle and $\mathscr{A} = \{(U_\alpha, \varphi_\alpha, M_\alpha)\}$ an atlas of E. If $u \in \operatorname{Isom}(M_\beta, M_\alpha)$, then ${}^t u \in \operatorname{Isom}(\mathfrak{M}(M_\alpha), \mathfrak{M}(M_\beta))$, with ${}^t u(\chi) := \chi \circ u$, $\chi \in \mathfrak{M}(M_\alpha)$ (numerical spectrum of M_α, [9]). The map $u \to {}^t u$ yields a bijection

$$\text{(3.2)}\qquad \operatorname{Isom}_s(M_\beta, M_\alpha) \cong {}^t\operatorname{Isom}(M_\beta, M_\alpha) \equiv \{{}^t u: u \in \operatorname{Isom}(M_\beta, M_\alpha)\} \subseteq \operatorname{Isom}(\mathfrak{M}(M_\alpha), \mathfrak{M}(M_\beta))$$

through which one endows ${}^t\operatorname{Isom}(M_\beta, M_\alpha)$ with the topology of the first member of (3.2). Now, the continuity of $\lambda_{\alpha\beta}: U_\alpha \cap U_\beta \to \operatorname{Isom}_s(M_\beta, M_\alpha)$ (see (1.10), (1.13)) implies the continuity of the (dual) maps

$$\text{(3.3)}\qquad \lambda^*_{\beta\alpha}: U_\alpha \cap U_\beta \to {}^t\operatorname{Isom}_s(M_\beta, M_\alpha): x \mapsto \lambda^*_{\beta\alpha}(x) := {}^t[\lambda_{\alpha\beta}(x)]$$

so that the system of transition functions $\Lambda \equiv \{(U_\alpha), (\lambda_{\alpha\beta}), (M_\alpha)\}_\alpha$ defines a (dual) system of transition functions $\Lambda^* \equiv \{(U_\alpha), (\lambda^*_{\beta\alpha}), (\mathfrak{M}(M_\alpha))\}_\alpha$ on X.

So by adapting Theorem 1.2 to the present framework, one obtains a topological (fibre) bundle ξ'. Namely, one considers the quotient space

$$\text{(3.4)}\qquad L \equiv \Big(\dot{\bigcup_\alpha} (U_\alpha \times \mathfrak{M}(M_\alpha))\Big)/\sim,$$

where the equivalence relation " $\sim$ " on the disjoint union involved is given by

$$\text{(3.5)}\qquad (x_\alpha, \chi_\alpha) \sim (x_\beta, \chi_\beta) \quad \text{if and only if} \quad \begin{cases} x_\alpha = x_\beta = x \in U_\alpha \cap U_\beta, \\ \chi_\beta = [\lambda^*_{\beta\alpha}(x)](\chi_\alpha). \end{cases}$$

The obvious projection map $P: L \to X$ is, of course, continuous.

Thus, the topological algebra bundle $\xi = (E, \pi, X)$ defines a fibre bundle $\xi' = (L, P, X)$ of fibre type $\mathfrak{M}(M_\alpha)$. Namely, one gets an object of $\mathscr{F}(X)$, which we call the "*spectrum bundle*" of ξ and denote by $\mathfrak{M}(\xi)$.

On the other hand, given the topological algebra bundles $\xi = (E, \rho, X)$ and $\xi' = (E', \rho', X)$ of fibre types M_α, respectively, a bundle morphism $h: \xi \to \xi'$ defines a morphism ("*spectrum morphism*")

$$\text{(3.6)}\qquad \mathfrak{M}(h): \mathfrak{M}(\xi') \equiv (L', P', X) \to (L, P, X) \equiv \mathfrak{M}(\xi),$$

between the corresponding spectra bundles, as follows: Namely, if

$$\text{(3.7)}\qquad h_x \equiv h|_{\rho^{-1}(x)}: \rho^{-1}(x) \underset{\text{iso}}{\cong} M_\alpha \to M'_\alpha \underset{\text{iso}}{\cong} \rho'^{-1}(x),$$

$x \in U_\alpha$, is the (canonical) continuous algebra morphism defined by $h: \xi \to \xi'$ (see (3.1)), then the continuous map

$$U_\alpha \times \mathfrak{M}(M'_\alpha) \ni (x, \chi) \overset{\nu_\alpha}{\to} (x, h_x \circ \chi) \in U_\alpha \times \mathfrak{M}(M_\alpha)$$

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entails a continuous map $\nu\colon \dot{\bigcup}_\alpha(U_\alpha \times \mathfrak{M}(M'_\alpha)) \to \dot{\bigcup}_\alpha(U_\alpha \times \mathfrak{M}(M_\alpha))$, so that $\nu|_{U_\alpha \times \mathfrak{M}(M'_\alpha)} := \nu_\alpha$. Thus, the continuous map

$$(3.8)\quad \tilde{h}\colon L' \equiv \Big(\dot{\bigcup_\alpha}\big(U_\alpha \times \mathfrak{M}(M'_\alpha)\big)\Big)/\sim \;\to L \equiv \Big(\dot{\bigcup_\alpha}\big(U_\alpha \times \mathfrak{M}(M_\alpha)\big)\Big)/\sim$$

between the corresponding quotient spaces may be defined in a canonical way (see (3.4), (3.5) and also [9, Chapter II, Section 8]). Moreover, for every $x \in X$, the map

$$\tilde{h}_x \equiv h|_{P'^{-1}(x)}\colon P'^{-1}(x) \underset{\text{homeo}}{\cong} \mathfrak{M}(M'_\alpha) \to \mathfrak{M}(M_\alpha) \underset{\text{homeo}}{\cong} P^{-1}(x)$$

is continuous since $\tilde{h}_x = {}'h_x$ (see (3.7)); thus, the spectrum morphism $\mathfrak{M}(h) := \tilde{h}$ is well defined.

So the preceding discussion yields now the next

THEOREM 3.1. *If $\mathscr{E}(X)$ (resp. $\mathscr{F}(X)$) is the category of semi-simple topological algebra bundles (resp. of fibre bundles) over a topological space X, then the correspondence (see also (3.6))*

$$(3.9)\qquad \mathfrak{M}\colon \mathscr{E}(X) \to \mathscr{F}(X)\colon \xi \mapsto \mathfrak{M}(\xi)$$

yields a contravariant functor ("spectrum functor").

Let $(Y_\alpha)_{\alpha \in K}$ be a family of completely regular spaces and for any $(\alpha, \beta) \in K \times K$, let $\mathrm{Isom}(Y_\alpha, Y_\beta)$ be the set of homeomorphisms of Y_α onto Y_β. If $u \in \mathrm{Isom}(Y_\alpha, Y_\beta)$, then $u^* \in \mathrm{Isom}(\mathscr{C}_c(Y_\beta), \mathscr{C}_c(Y_\alpha))$ with $u^*(\chi) := \chi \circ u$, $\chi \in \mathscr{C}_c(Y_\beta)$ (locally m-convex algebra of $\mathbf{C}$-valued continuous functions on Y_β in the compact-open topology). Then, the map

$$(3.10)\qquad \mathrm{Isom}(Y_\alpha, Y_\beta) \to \mathrm{Isom}\big(\mathscr{C}_c(Y_\beta), \mathscr{C}_c(Y_\alpha)\big)\colon u \mapsto u^*$$

is a *bijection*: Indeed, if $u, v \in \mathrm{Isom}(Y_\alpha, Y_\beta)$, with $u^* = v^*$, then for any $\chi \in \mathscr{C}(Y_\beta)$ we have $\chi \circ u = \chi \circ v$, and hence (Urysohn's lemma), $u = v$; that is, (3.10) is 1-1. Moreover, every $g \in \mathrm{Isom}(\mathscr{C}_c(Y_\beta), \mathscr{C}_c(Y_\alpha))$ defines a ${}'g \in \mathrm{Isom}(\mathfrak{M}(\mathscr{C}_c(Y_\alpha)), \mathfrak{M}(\mathscr{C}_c(Y_\beta))) \cong \mathrm{Isom}(Y_\alpha, Y_\beta)$ with ${}'g(\chi) := \chi \circ g$, $\chi \in \mathfrak{M}(\mathscr{C}_c(Y_\alpha))$, since by hypothesis one has $\mathfrak{M}(\mathscr{C}_c(Y_\alpha)) \underset{\text{homeo}}{\cong} Y_\alpha$, $\alpha \in K$ (see [9, Chapter III, Theorem 9.2]).

Now, given a fibre bundle $\xi = (E, \rho, X)$ and an atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha, Y_\alpha)\}_\alpha$ of E, let $\Lambda = \{(U_\alpha), (\lambda_{\beta\alpha}), (Y_\alpha)\}$ be the respective system of transition functions on X (see (1.10), (1.13)). Then, Λ defines a (dual) system of transition functions $\tilde{\Lambda} = \{(U_\alpha), (\tilde{\lambda}_{\alpha\beta}), (\mathscr{C}_c(Y_\alpha))\}_\alpha$ on X, where the maps $\tilde{\lambda}_{\alpha\beta}$ are given by

$$(3.11)\quad \tilde{\lambda}_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Isom}\big(\mathscr{C}_c(Y_\beta), \mathscr{C}_c(Y_\alpha)\big)\colon x \mapsto \tilde{\lambda}_{\alpha\beta}(x) := \big[\lambda_{\beta\alpha}(x)\big]^*$$

(see (3.10)). The continuity of $\tilde{\lambda}_{\alpha\beta}$ follows by that of $\lambda_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{Isom}_s(Y_\alpha, Y_\beta)$, when $\mathrm{Isom}(\mathscr{C}_c(Y_\beta), \mathscr{C}_c(Y_\alpha))$ is endowed with the topology of $\mathrm{Isom}_s(Y_\alpha, Y_\beta)$ via the bijection (3.10).

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Now, we assume the local equicontinuity of $\mathrm{Isom}_s(Y_\alpha, Y_\beta)$. (The last assumption is valid if, for instance, Y_α, Y_β are metric spaces and by $\mathrm{Isom}(Y_\alpha, Y_\beta)$ one means isometries [3, Chapter X, §3.1, Examples, 2)].) Then (Theorem 1.2), one obtains a locally m-convex algebra bundle $\mathscr{C}(\xi) \equiv (\tilde{E}, \pi, X)$ of fibre type $\mathscr{C}_c(Y_\alpha)$, whose bundle space is given by $\tilde{E} = (\dot{\bigcup}_\alpha (U_\alpha \times \mathscr{C}_c(Y_\alpha)))/\sim$, where " $\sim$ " is the equivalence relation (1.14) and the projection π: $\tilde{E} \to X$ is defined by (1.15). That is, $\mathscr{C}(\xi)$ *is an object of* $\mathscr{E}(X)$ (category of locally m-convex algebra bundles).

Moreover, if h: $\xi = (E, \rho, X) \to \xi' = (E', \rho', X)$ is a bundle morphism in $\mathscr{F}(X)$, then a morphism

$$(3.12) \qquad \mathscr{C}(h)\colon \mathscr{C}(\xi') \to \mathscr{C}(\xi)$$

between the corresponding locally m-convex algebra bundles can be defined as follows: If

$$(3.13) \qquad h_x \equiv h|_{\rho^{-1}(x)}\colon \rho^{-1}(x) \underset{\text{homeo}}{\cong} Y_\alpha \to Y'_\alpha \underset{\text{homeo}}{\cong} \rho'^{-1}(x),$$

$x \in U_\alpha$, is the (canonical) continuous map defined by h: $\xi \to \xi'$ (see (3.1)), then the continuous map

$$(3.14) \qquad \mu_\alpha\colon U_\alpha \times \mathscr{C}_c(Y'_\alpha) \to U_\alpha \times \mathscr{C}_c(Y_\alpha)\colon (x, g) \mapsto (x, h_x \circ g)$$

defines a continuous map μ: $\dot{\bigcup}_\alpha (U_\alpha \times \mathscr{C}_c(Y'_\alpha)) \to \dot{\bigcup}_\alpha (U_\alpha \times \mathscr{C}_c(Y_\alpha))$, so that $\mu|_{U_\alpha \times \mathscr{C}_c(Y'_\alpha)} := \mu_\alpha$. It is directly verifiable that μ is compatible with the equivalence relation (1.14), whence μ defines a continuous map

$$(3.15) \qquad \mathscr{C}(h)\colon \tilde{E}' = \Big(\dot{\bigcup_\alpha}(U_\alpha \times \mathscr{C}_c(Y'_\alpha))\Big)/\sim\; \to \tilde{E} = \Big(\dot{\bigcup_\alpha}(U_\alpha \times \mathscr{C}_c(Y_\alpha))\Big)/\sim$$

(see Theorem 1.2). Thus, the map

$$(3.16) \qquad \mathscr{C}(h)_x \equiv \mathscr{C}(h)|_{\pi'^{-1}(x)}\colon \pi'^{-1}(x) \underset{\text{iso}}{\cong} \mathscr{C}_c(Y'_\alpha) \to \mathscr{C}_c(Y_\alpha) \underset{\text{iso}}{\cong} \pi^{-1}(x)$$

for any $x \in X$, is a continuous algebra morphism, since (3.14), (3.15) imply $\mathscr{C}(h)_x(g) = g \circ h_x$, $g \in \mathscr{C}_c(Y_\alpha)$.

So, now we get

THEOREM 3.2. *Let* $\mathscr{E}(X)$ (*resp.* $\mathscr{F}(X)$) *be the category of locally m-convex algebra bundles* (*resp. of fibre bundles*) *such that* $\mathrm{Isom}_s(Y_\alpha, Y_\beta)$ *is locally equicontinuous*, Y_α, $\alpha \in K$, *being a completely regular space the fibre of* $\xi \in \mathscr{F}(X)$. *Then, there exists a contravariant functor*

$$(3.17) \qquad \mathscr{C}\colon \mathscr{F}(X) \to \mathscr{E}(X)\colon \xi \mapsto \mathscr{C}(\xi).$$

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The next result identifies the algebra of continuous sections of the bundle $\mathscr{C}(\xi)$, defined by (3.17) for a $\xi = (E, \rho, X) \in \mathscr{F}(X)$, as the algebra of $\mathbf{C}$-valued continuous functions on E. That is we have

THEOREM 3.3. *Let $\xi = (E, \rho, X)$ be a fibre bundle (that is, an object of $\mathscr{F}(X)$) over a paracompact space X, of fibre type a locally compact space Y_α $(\alpha \in K)$ and $\mathscr{C}(\xi) \equiv \tilde{\xi}$ the corresponding locally m-convex algebra bundle defined by Theorem 3.2. Then, one has*

$$\mathscr{C}_c(E) = \Gamma(\mathscr{C}(\xi)) \tag{3.18}$$

within an isomorphism of locally m-convex algebras.

PROOF. By Theorems 2.1, 3.2 (see also (2.6), (3.11)) one gets the next isomorphism of locally m-convex algebras:

$$\Gamma(\tilde{\xi}) = \tilde{\mathscr{B}} := \left\{ \tilde{\tau} \equiv (\tilde{\tau}_\alpha) \colon \tilde{\tau}_\alpha(x) = \tilde{\lambda}_{\alpha\beta}(x)\tilde{\tau}_\beta(x),\ x \in U_\alpha \cap U_\beta \right\} \subseteq \prod_\alpha \mathscr{C}_c(U_\alpha, \mathscr{C}_c(Y_\alpha)). \tag{3.19}$$

On the other hand, by considering a chart $\varphi_\alpha \colon U_\alpha \times Y_\alpha \xrightarrow{\sim} \rho^{-1}(U_\alpha) \subseteq E$ one defines the correspondence

$$\mathscr{C}_c(E) \ni f \mapsto f_\alpha \equiv f \circ \varphi_\alpha \in \mathscr{C}_c(U_\alpha \times Y_\alpha). \tag{3.20}$$

Thus, by considering the following (canonical) topological-algebraic isomorphism

$$\omega \colon \mathscr{C}_c(U_\alpha \times Y_\alpha) \to \mathscr{C}_c(U_\alpha, \mathscr{C}_c(Y_\alpha)) \colon f_\alpha \mapsto \hat{f}_\alpha, \tag{3.21}$$

where

$$[\hat{f}_\alpha(x)](y) := f_\alpha(x, y), \tag{3.22}$$

(see [4, page 265, Theorem 5.3]), one defines the map

$$\tilde{\theta} \colon \mathscr{C}_c(E) \to \tilde{\mathscr{B}} \colon f \mapsto \tilde{\theta}(f) \equiv \hat{f} \equiv (\hat{f}_\alpha)_\alpha := (\omega(f \circ \varphi_\alpha))_\alpha. \tag{3.23}$$

Indeed, for any $x \in U_\alpha \cap U_\beta$, $y_\alpha \in Y_\alpha$, one has by (1.5), $\varphi_\alpha(x, y_\alpha) = \varphi_\beta(x, \lambda_{\beta\alpha}(x)(y_\alpha))$, such that $\hat{f}_\alpha(x)(y_\alpha) = \hat{f}_\beta(x)(\lambda_{\beta\alpha}(x)(y_\alpha))$, that is $\hat{f}_\alpha(x) = \tilde{\lambda}_{\alpha\beta}(x)\hat{f}_\beta(x)$ (see (3.19)). Moreover, $\tilde{\theta}$ is an *algebra morphism*, since for any $f, g \in \mathscr{C}_c(E)$ one has $\hat{f} \cdot \hat{g} = \widehat{f \cdot g}$ by (3.20). (Similarly for the linearity of $\tilde{\theta}$.) Now, $\tilde{\theta}$ is 1-1, since for every $f, g \in \mathscr{C}_c(E)$ with $\tilde{\theta}(f) = \tilde{\theta}(g)$ one has $(\hat{f}_\alpha) = (\hat{g}_\alpha)$, that is, $\hat{f}_\alpha = \hat{g}_\alpha$ for any α. Thus, $f_\alpha = g_\alpha$ for any α (see (3.20)) and therefore $f = g$. Furthermore, $\tilde{\theta}$ *is onto*: So, if $\tilde{\tau} \equiv (\tilde{\tau}_\alpha) \in \tilde{\mathscr{B}}$ we define a continuous map $f_\alpha \colon U_\alpha \times Y_\alpha \to \mathbf{C} \colon (x, y) \mapsto f_\alpha(x, y) := (\tilde{\tau}_\alpha(x))(y)$ (see [4, page 265, Theorem 5.3]). By considering $\{\psi_\alpha\}_{\alpha \in K}$, a partition of unity subordinate to the open covering $\rho^{-1}(U_\alpha)$ of E, one has $f := \sum_\alpha \psi_\alpha f_\alpha \in \mathscr{C}(E)$ such that $\tilde{\theta}(f) = (\tilde{\tau}_\alpha) = \tilde{\tau}$

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(see (3.23)). Finally, $\tilde{\theta}$ *is bicontinuous*: Indeed, the definition of the topology of $\tilde{\mathscr{B}} \underset{i}{\subsetneq} \prod_\alpha \mathscr{C}_c(U_\alpha, \mathscr{C}_c(Y_\alpha))$ implies that $\tilde{\theta}$ is continuous if and only if $q_\alpha \circ i \circ \tilde{\theta} = \omega^{-1} \circ p'_\alpha$ is continuous, where q_α: $\prod_\alpha \mathscr{C}_c(U_\alpha, \mathscr{C}_c(Y_\alpha)) \to \mathscr{C}_c(U_\alpha, \mathscr{C}_c(Y_\alpha))$ is the canonical projection and p'_α: $\mathscr{C}_c(U_\alpha \times Y_\alpha) \to \mathscr{C}_c(E)$: $f_\alpha \mapsto p'_\alpha(f_\alpha) := f_\alpha \circ p_\alpha$; now, the last map is certainly continuous. Conversely, if $\hat{f}_\delta \to 0|_{\tilde{\mathscr{B}}}$, then $(\hat{f}_\delta)_\alpha = \omega(f_\delta \circ \varphi_\alpha) \to 0|_{\mathscr{C}_c(U_\alpha, \mathscr{C}_c(Y_\alpha))}$ (see (3.23)), for any $\alpha \in K$, such that the bicontinuity of (3.21) shows that $\omega^{-1}((\hat{f}_\delta)_\alpha) = (\hat{f}_\delta)_\alpha \to 0$ on $\mathscr{C}_c(U_\alpha \times Y_\alpha)$, for every $\alpha \in K$. Thus, by definition of the topology of compact convergence on $E = \bigcup_\alpha \rho^{-1}(U_\alpha)$, $f_\delta = \sum_\alpha \psi_\alpha (f_\delta)_\alpha$ converges to 0 on $\mathscr{C}_c(E)$, and this completes the proof.

THEOREM 3.4. *Let $\mathscr{E}(X)$ be the category of locally m-convex algebra bundles and $^{\cdot}\mathscr{F}(X)$ the category of fibre bundles having completely regular bundle spaces—locally compact fibres with X a paracompact space. Then, the functor $\mathscr{C}$: $^{\cdot}\mathscr{F}(X) \to \mathscr{E}(X)$: $\xi \mapsto \mathscr{C}(\xi)$ (see Theorem 3.2) is fully faithful* [7].

PROOF. If h_1, h_2: $\xi = (E, \rho, X) \to \xi' = (E', \rho', X)$ are two morphisms in $^{\cdot}\mathscr{F}(X)$, such that $\mathscr{C}(h_1) = \mathscr{C}(h_2)$: $\mathscr{C}(\xi') \to \mathscr{C}(\xi)$ (see (3.12)), one has the equality of continuous morphisms

$$\Gamma(\mathscr{C}(h_1)) = \Gamma(\mathscr{C}(h_2)) \colon \Gamma(\mathscr{C}(\xi')) \underset{\text{iso}}{\cong} \mathscr{C}_c(E') \to \Gamma(\mathscr{C}(\xi')) \underset{\text{iso}}{\cong} \mathscr{C}_c(E)$$

between the corresponding topological algebras of sections (see Theorem 3.3). Moreover, since E, E' are completely regular spaces, we have the homeomorphisms $\mathfrak{M}(\mathscr{C}_c(E)) = E$, $\mathfrak{M}(\mathscr{C}_c(E')) = E'$ (see [9, Chapter III, Theorem 9.2]), such that by (3.16) and the hypothesis concerning the fibres of ξ, ξ', we have $h_1 = h_2$: $E \to E'$, which, in fact, shows that $\mathscr{C}$ *is faithful.*

On the other hand, if $\xi = (E, \rho, X)$, $\xi' = (E', \rho', X)$ are objects of $^{\cdot}\mathscr{F}(X)$, let h: $\mathscr{C}(\xi') \to \mathscr{C}(\xi)$ be a morphism in the category $\mathscr{E}(X)$. Then, $\Gamma(h)$: $\Gamma(\mathscr{C}(\xi')) \underset{\text{iso}}{\cong} \mathscr{C}_c(E') \to \Gamma(\mathscr{C}(\xi)) \underset{\text{iso}}{\cong} \mathscr{C}_c(E)$ is a continuous morphism of locally m-convex algebras (see Theorem 3.3). Hence, by the hypotheses for E and E', $\mathfrak{M}(\Gamma(h))$: $E \to E'$ is a morphism in $\mathscr{E}(X)$ such that $\mathscr{C}(\mathfrak{M}(\Gamma(h))) = h$ (see (3.6), (3.7), (3.15), (3.16)), that is, $\mathscr{C}$ *is full.*

The categories $\mathscr{E}(X)$ and $^{\cdot}\mathscr{F}(X)$ as above are not in general (anti)equivalent. Under suitable conditions with respect to the (locally convex algebra) fibre of $\xi \in \mathscr{E}(X)$ we take a category antiequivalence between the previous categories, as Corollary 3.1 below, shows.

Now, let ξ be a topological algebra bundle (object of $\mathscr{E}(X)$) whose fibre M_α, $\alpha \in K$, *admits a functional representation*, that is $\mathscr{C}(\mathfrak{M}(M_\alpha)) = M_\alpha$, within an isomorphism of topological algebras. This is valid, for instance, in case of a

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semi-simple, m-barreled Ptàk locally convex algebra M_α (for example, a semi-simple Fréchet locally convex algebra is of this type, cf. [10, Corollary 3.1]) for which the corresponding Gel'fand map g_α: $M_\alpha \to \mathscr{C}_c(\mathfrak{M}(M_\alpha))$, $\alpha \in K$, is an onto map (that is, g_α is an algebraic onto isomorphism; see [10, Theorem 3.1]). Then, for the corresponding spectrum bundle $\mathfrak{M}(\xi) \in \mathscr{F}(X)$ (Theorem 3.2) one has $\mathscr{C}(\mathfrak{M}(\xi)) \underset{\text{iso}}{\cong} \xi$, such that the functor (3.17) is *essentially surjective* [7], in such a way that Theorem 3.4 implies the next result.

COROLLARY 3.1. *Let* $^{\cdot}\mathscr{E}(X)$ *be the category of topological algebra bundles over a paracompact space* X *whose fibres admit functional representations. Moreover, let* $^{\cdot}\mathscr{F}(X)$ *be the category of fibre bundles having completely regular spaces and locally compact fibres. Then, one has*

$$^{\cdot}\mathscr{E}(X) = {}^{\cdot}\mathscr{F}(X), \tag{3.24}$$

within a category antiequivalence.

SCHOLIUM 3.1. Let $\mathscr{F}_\sigma(X)$ be the category of holomorphic fibre bundles over a complex manifold X with fibres (complex manifolds) Y_α, $\alpha \in K$, and the obvious morphisms. Then, there exists a contravariant functor

$$\sigma\colon \mathscr{F}_\sigma(X) \to \mathscr{E}_\sigma(X)\colon \xi \mapsto \sigma(\xi), \tag{3.25}$$

where $\mathscr{E}_\sigma(X)$ is the category, whose objects are topological algebra bundles over X of fribre type the locally m-convex algebras $\sigma(Y_\alpha)_{\alpha \in K}$, of $\mathbb{C}$-valued holomorphic functions on Y_α (see [9, Chapter II, Section 10, Example 10.3]). Moreover, if $\Gamma(\sigma(\xi))$ is the algebra of holomorphic sections of $\xi = (E, \rho, X) \in \mathscr{F}_\sigma(X)$, then

$$\sigma(E) = \Gamma(\sigma(\xi)), \tag{3.26}$$

within an isomorphism of locally m-convex algebras; this can be shown by adapting Theorems 2.1, 3.3 to the present framework. Moreover, let $^{\cdot}\mathscr{F}_\sigma(X)$ denote the category of holomorphic fibre bundles $\xi \in \mathscr{F}_\sigma(X)$ having Stein manifolds as bundle spaces. Then, analogously to Theorem 3.4, we also have that the functor σ: $\mathscr{F}_\sigma(X) \to \mathscr{E}_\sigma(X)$ (see (3.25)) is *fully faithful*.

Furthermore, let $^{\cdot}\mathscr{E}_\sigma(X)$ be the category of locally m-convex algebra bundles $\xi \in \mathscr{E}_\sigma(X)$, whose fibres M_α, $\alpha \in K$, admit "*analytic representations*", in the sense that $\sigma(\mathfrak{M}(M_\alpha)) = M_\alpha$, within an isomorphism of locally m-convex algebras (see the analogous remarks before Corollary 3.1). This happens, for example, if M_α is a Stein algebra; that is, a topological algebra isomorphic to the Fréchet algebra of holomorphic functions on a Stein space (see [9, Chapter III, Section 9, Example 9.3, (9.39)]. Then, via analogous considerations to those of Corollary 3.1 (see also (3.26) and [9, Chapter III, Section 9, Example 9.3, Theorem 9.3]), one gets

$$^{\cdot}\mathscr{F}_\sigma(X) = {}^{\cdot}\mathscr{E}_\sigma(X), \tag{3.27}$$

within a category antiequivalence.

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Finally, we note that the relations (3.18), (3.26) specialize to the facts that $\mathscr{C}_c(X \times Y) = \mathscr{C}_c(X, \mathscr{C}_c(Y))$ and $\sigma(X \times Y) = \sigma(X, \sigma(Y))$ within isomorphisms (see [9, Chapter VI, Section 4]).

4.

APPENDIX. In Section 3 we defined the spectrum bundle $\mathfrak{M}(\xi)$ of a given topological algebra bundle $\xi = (L, P, X)$ (see (3.4), (3.5)). In this section we supply another realization of $\mathfrak{M}(\xi)$, via the spectrum of the topological algebra of sections $\Gamma(\xi)$.

Thus, we come first to the *identification of L as the spectrum of the topological algebra* $\sigma(\xi)$ (see Theorem 4.1). So, for $U_\alpha \in \mathscr{U}$ (: open covering of X) we consider the map

$$\eta_\alpha: U_\alpha \times \Gamma(\xi) \to M_\alpha: (x, \gamma) \mapsto \eta_\alpha(x, \gamma) := t_\alpha(\gamma(x)) \tag{4.1}$$

such that the partial map

$$\eta_{\alpha,x}: \Gamma(\xi) \to M_\alpha: \gamma \mapsto \eta_{\alpha,x}(\gamma) := \eta_\alpha(x, \gamma), \tag{4.2}$$

with $x \in U_\alpha$, is a continuous algebra morphism, as composition of the continuous morphisms

$$\Gamma(\xi) \underset{i_\alpha}{\to} \Gamma(U_\alpha, \xi) \underset{l}{\tilde{\to}} \mathscr{C}_c(U_\alpha, M_\alpha) \underset{\varepsilon_{\alpha,x}}{\to} M_\alpha \tag{4.3}$$

where i_α is the canonical restriction map, l the isomorphism (2.3) and $\varepsilon_{\alpha,x}$ the (canonical) evaluation map, that is,

$$\mathscr{C}_c(U_\alpha, M_\alpha) \ni \tau_\alpha \mapsto \varepsilon_{\alpha,x}(\tau_\alpha) := \tau_\alpha(x) \in M_\alpha.$$

LEMMA 4.1. *If X is a compact space and $M_\alpha \equiv M$, for every $\alpha \in K$, then* (4.2) *is a continuous epimorphism.*

PROOF. If $a \in M$ and $\{\psi_\alpha\}_\alpha$ is a partition of unity subordinated to $\mathscr{U}$ (finite covering of X), we define $\gamma_\alpha(x) := \varphi_\alpha(x, \psi_\alpha(x)a)$, $x \in U_\alpha$ and $\gamma_\alpha(x) = O(x)$, $x \in U_\alpha$. It is clear that the above maps are continuous such that $\gamma := \Sigma_\alpha \gamma_\alpha$ is a section of ξ. Thus, one gets $\eta_{\alpha,x}(\gamma) = t_\alpha(\gamma(x)) = t_\alpha(\Sigma_\alpha \gamma_\alpha) = a$.

Now, by considering the continuous map

$$\nu_\alpha: U_\alpha \times \mathfrak{M}(M_\alpha) \to \mathfrak{M}(\Gamma(\xi)): (x_\alpha, \chi_\alpha) \mapsto \chi_\alpha \circ \eta_{\alpha, x_\alpha} \tag{4.4}$$

the map ν: $T \equiv \bigcup_\alpha (U_\alpha \times \mathfrak{M}(M_\alpha)) \to \mathfrak{M}(\Gamma((\xi))$ such that $\nu|_{U_\alpha \times \mathfrak{M}(M_\alpha)} := \nu_\alpha$, is a well defined continuous map compatible with the equivalence relation "$\sim$" defined in (3.5), in the sense that $t_1, t_2 \in T$ and $t_1 \sim t_2$, implies $\nu(t_1) = \nu(t_2)$.

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Hence, ν yields a continuous map

$$\theta\colon L \equiv T/\sim \to \mathfrak{M}(\Gamma(\xi)) \tag{4.5}$$

which is, in particular, a homeomorphism under suitable conditions for the base space X and the fibre M_α of the topological algebra bundle ξ (Theorem 4.1). To get the desired identification some additional comments are necessary. Thus, the following lemmas are analogous to those of [5, Lemmas 1.1, 2.1] within the present framework.

LEMMA 4.2. *Let $\xi = (E, \pi, X)$ be a topological algebra bundle of fibre type a topological algebra M and $\mathscr{U} = \{U_\alpha\}_\alpha$ an open covering of X. For $U_\alpha \in \mathscr{U}$ and $f \in \mathscr{C}_c(X, M)$ whose support is contained in U_α, we define*

$$\begin{aligned} \gamma(x) &:= \varphi_\alpha(x, f(x)), \quad x \in U_\alpha, \\ &:= O(x), \qquad\qquad x \notin U_\alpha. \end{aligned}$$

Then, $\gamma \in \Gamma(\xi)$ such that $t_\alpha \circ \gamma = f|_{U_\alpha}$.

LEMMA 4.3. *If N is a topological $\mathbf{A}$-algebra ($\mathbf{A}$ is a commutative topological algebra*; [8]), *then every regular ideal of N is also an $\mathbf{A}$-ideal.*

In the sequel by a *Waelbroeck algebra* we mean a unital (topological) Q-algebra (: its group of units is an open set, [9, Chapter I, Definition 5.2]) with a continuous inversion.

LEMMA 4.4. *Let ξ be a topological algebra bundle over a compact space X with fibre a Waelbroeck algebra M, and let $\mathscr{U} = \{U_\alpha\}_{\alpha \in K}$ be an open covering of X. Then, a 2-sided ideal I of $\Gamma(\xi)$ is proper if and only if for some $x_0 \in X$, every $U_\alpha \in \mathscr{U}$ with $x_0 \in U_\alpha$ and every $\gamma \in I$, $t_\alpha(\gamma(x_0)) \neq 1$ (: identity of M).*

PROOF. We suppose that for every $x \in X$ there is some $U_\alpha \ni x$ and $\gamma_x \in I$ such that $t_\alpha(\gamma(x)) = 1 \in M$. Then $(t_\alpha(\gamma_x(y)))^{-1}$ exists for all y in a neighborhood $N_x \subseteq U_\alpha$. We shrink $\{U_\alpha\}$ to get neighborhoods V_x, W_x satisfying $V_x \subseteq \overline{V}_x \subseteq W_x \subseteq \overline{W}_x \subseteq N_x$. If x_i, $i = 1, \ldots, n$, are such that $\{V_{x_i}\}$ to define a finite covering of X, then we can consider $f_i \in \mathscr{C}(X)$ in such a way that $f_i = 1$ on $\overline{V}_{x_i}$, $f_i = 0$ off W_{x_i}, $0 \leqslant f_i(x) \leqslant 1$, for all $x \in X$. Thus, the relation

$$\begin{aligned} g_i(x) &= f_i(x) \cdot \left(t_{\alpha_i}\left(\gamma_{x_i}(x)\right)\right)^{-1}, \qquad x \in W_{x_i}, \\ &= 0, \quad \text{otherwise}, \end{aligned}$$

defines a continuous map $g_i\colon X \to M$, since $x \mapsto (t_{\alpha_i}(\gamma_{x_i}(x)))^{-1}$ is continuous, and moreover $\mathit{support}(g_i) \subseteq U_{\alpha_i}$. So, there is a section $\gamma_i \in \Gamma(\xi)$ such that $t_{\alpha_i}(\gamma_i(x)) = g_i(x)$ (Lemma 4.2) and, moreover, $t_{\alpha_i}((\gamma_i \cdot \gamma_{x_i})(x)) = 1$, for $x \in V_{x_i}$, while $\overline{\gamma}_i := \gamma_i \cdot \gamma_{x_i} \in I$. If $\{\psi_i\}_{i=1}^n$ is a partition of unity subordinate to the

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covering $\{V_{x_i}\}$, then $\psi_i \cdot \bar{\gamma}_i \in I$ (Lemma 4.3, for $N = \Gamma(\xi)$ and $\mathbf{A} = \mathscr{C}(X)$), so that $\bar{\gamma} := \sum_{i=1}^{n} \psi_i \cdot \bar{\gamma}_i \in I$. But, for any $x \in X = \bigcup_{i=1}^{n} V_{x_i}$ there is some $V_{x_i} \ni x$ such that $t_{\alpha_i}(\bar{\gamma}(x)) = \sum_{j=1}^{n} \psi_j(x) t_{\alpha_i}(\bar{\gamma}_j(x)) = \sum_{j=1}^{n} \psi_j(x) \cdot 1 = 1$, that is, I cannot be a proper ideal of $\Gamma(\xi)$. Conversely, we suppose that I is a proper ideal. Then, by considering for every $x \in X$ the existence of a $U_\alpha \ni x$ with $t_\alpha(\gamma(x)) = 1 \in M$, for a $\gamma \in I$, one has $\gamma(x) = 1$, a contradiction, since I is a proper ideal.

Now, by a *Gel'fand-Mazur* (*topological*) *algebra* we mean a topological algebra E such that for every (2-sided proper) maximal regular closed ideal $M \subseteq E$ one has $E/M \cong \mathbf{C}$, within a topological algebraic isomorphism [9, Chapter IV, Definition 9.5]. This definition is equivalent with saying that every (2-sided) closed regular maximal ideal of E is the kernel on a continuous character of E (see [9, Chapter IV, Section 9(5)]).

Thus, we are in the position to prove the following basic result.

THEOREM 4.1. *Let $\xi = (E, \pi, X)$ be a topological algebra bundle over a compact space X of fibre type a commutative Gel'fand-Mazur Waelbroeck algebra M, and let $\mathscr{U} = \{U_\alpha\}_\alpha$ be an open covering of X. Then,*

$$L \equiv \bigcup_{\alpha} (U_\alpha \times \mathfrak{M}(M))/\sim \; = \mathfrak{M}(\Gamma(\xi)), \tag{4.6}$$

within a homeomorphism (*see* (4.5)).

PROOF. If p_α is the composition $U_\alpha \times \mathfrak{M}(M) \underset{j_\alpha}{\hookrightarrow} T \underset{p}{\to} L$, then by (4.4) one has $\theta \circ p_\alpha = \nu_\alpha$, such that θ *is injective*. Indeed, if $\theta(y) = \theta(y')$, with $y = p_\alpha(x_\alpha, \chi_\alpha)$, $y' = p_\beta(x_\beta, \chi_\beta)$, in L, to show $y = y'$, it suffices to prove $(x_\alpha, \chi_\alpha) \sim (x_\beta, \chi_\beta)$ (see (3.5)). So, $\theta(y) = \theta(y')$ entails

$$\nu_\alpha(x_\alpha, \chi_\alpha) = \nu_\beta(x_\beta, \chi_\beta), \tag{4.7}$$

and hence $x_\alpha = x_\beta \in U_\alpha \cap U_\beta$. For if $x_\alpha \neq x_\beta$, since X is Hausdorff, there exist open sets W_α, W_β with $x_\alpha \in W_\alpha \subseteq U_\alpha$ and $x_\beta \in W_\beta \subseteq U_\beta$ and $W_\alpha \cap W_\beta = \varnothing$. Thus, one gets $f \in \mathscr{C}(X)$ such that $f(x_\beta) = 1$, $f = 0$ off W_β and $0 \leqslant f(x) \leqslant 1$ for every $x \in X$. Then, for $\tilde{f} := f \cdot 1 \in \mathscr{C}(X, M)$ ($(f \cdot 1)(x) := f(x) \cdot 1$; 1 is the identity of M), we can find a $\gamma \in \Gamma(\xi)$ (Lemma 4.2) such that $\nu_\alpha(x_\alpha, \chi_\alpha)(\gamma) = \chi_\alpha(f(x_\alpha) \cdot 1) = f(x_\alpha) = 0$ and $\nu_\beta(x_\beta, \chi_\beta)(\gamma) = \chi_\beta(f(x_\beta) \cdot 1) = f(x_\beta) = 1$, that is, $\nu_\alpha(x_\alpha, \chi_\alpha) \neq \nu_\beta(x_\beta, \chi_\beta)$ a contradiction to (4.7). Now, for $x \equiv x_\alpha = x_\beta \in U_\alpha \cap U_\beta$ one has $\eta_{\alpha,x} = \lambda_{\alpha\beta}(x)\eta_{\beta,x}$ such that by (4.2), (4.7) $\chi_\alpha \circ \eta_{\alpha,x} = \chi_\beta \circ \eta_{\beta,x}$, that is, $(\chi_\alpha \circ \lambda_{\alpha\beta}(x)) \circ \eta_{\beta,x} = \chi_\beta \circ \eta_{\beta,x}$. But $\eta_{\beta,x}$ is an epimorphism (Lemma 4.1) so that $\lambda^*_{\beta\alpha}(x)(\chi_\alpha) = \chi_\beta$ (see (3.3)).

We prove next that θ *is surjective*: Indeed, if $\chi \in \mathfrak{M}(\Gamma(\xi))$, then $\ker(\chi) \equiv I$ is a (2-sided) closed maximal ideal of $\Gamma(\xi)$, such that for some x_0, every $U_{\alpha,x_0} \in \mathscr{U}$ and any $\gamma \in I$ we have $t_\alpha(\gamma(x_0)) \neq 1$ (Lemma 4.4), that is, $\eta_{\alpha,x_0}(I) \neq M$. We

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prove that $\eta_{\alpha,x_0}(I)$ is a maximal ideal. Thus, by Lemma 4.1, $\eta_{\alpha,x_0}(I)$ is an ideal of M and let $\tilde{I}$ be a maximal ideal such that $\eta_{\alpha,x_0}(I) \subsetneqq \tilde{I}$. But $\eta^{-1}_{\alpha,x_0}(\tilde{I})$ is an ideal in $\Gamma(\xi)$ (Lemma 4.1) so that the relation $I \subseteq \eta^{-1}_{\alpha,x_0}(\tilde{I})$ implies $I = \eta^{-1}_{\alpha,x_0}(\tilde{I})$ and therefore $\tilde{I} = \eta_{\alpha,x_0}(I)$, a contradiction. Thus, $\eta_{\alpha,x_0}(I) \equiv I_0$ is a (2-sided) closed (M is a Q-algebra) maximal ideal, so that let $\chi_\alpha \in \mathfrak{M}(M)$ with $I_0 = \ker(\chi_\alpha)$. Consequently, one gets an element $(x_0, \chi_\alpha) \in U_\alpha \times \mathfrak{M}(M)$ such that $\nu_\alpha(x_0, \chi_\alpha) = \chi$ since $\ker(\chi) = \ker(\chi_\alpha \circ \eta_{\alpha,x_0})$; therefore $\theta(p_\alpha(x_0, \chi_\alpha)) = \chi$, by the very definitions. In particular, one can prove that $\chi = \chi_\alpha \circ \eta_{\alpha,x_0}$, where $x_0 \in X$ is uniquely defined (the proof is a similar one as that for the injectivity of θ).

Finally, since L is compact [4] and $\mathfrak{M}(\Gamma(\xi))$ Hausdorff [9, Chapter III, Section 1], one has immediately the *continuity of* θ^{-1}, which completes the proof.

SCHOLIUM 4.1. One might ask to have an analogous situation to that of Theorem 3.4 concerning the ("spectrum") functor $\mathfrak{M}\colon \mathscr{E}(X) \to \mathscr{F}(X)$. Thus, we notice that $\mathfrak{M}$ *is full and faithful, if for every* $\xi = (E, \pi, X) \in \mathscr{E}(X)$, *the base space X is compact and its fibre M a unital commutative locally m-convex Waelbroeck algebra admitting a functional representation*, *such that* $\mathscr{A}ut(M)$ (self-homeomorphisms of M) *is an equicontinuous subset of* $\mathrm{Hom}(M, M)$. In this respect, the hypothesis that M admits a functional representation entails that the algebra of sections $\Gamma(\xi)$ admits also a functional representation and this leads to fullness and faithfulness of $\mathfrak{M}$; see Theorem 4.1.

Moreover, suppose that for every $\tilde{\xi} = (\tilde{E}, \tilde{\rho}, X) \in \mathscr{F}(X)$, the bundle space $\tilde{E}$ is completely regular and its fibre Y a compact space; besides we consider the structure group $\mathscr{A}ut(Y)$ endowed with the topology of compact convergence. Thus, one proves that the "spectrum functor" $\mathfrak{M}$ is *essentially surjective*, so that the above categories $\mathscr{E}(X)$ and $\mathscr{F}(X)$ are *antiequivalent*. However, we remark that the last conclusion about $\mathfrak{M}$ is attained under a stronger hypothesis than that for the similar result concerning the functor $\mathscr{C}$ (see Theorem 3.4, Corollary 3.1).

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