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REMARKS ON THE UNIVALENCE CRITERION OF PASCU AND PASCU

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Abstract

We consider a recent work of Pascu and Pascu [‘Neighbourhoods of univalent functions’, *Bull. Aust. Math. Soc.* **83**(2) (2011), 210–219] and rectify an error that appears in their work. In addition, we study certain analogous results for sense-preserving harmonic mappings in the unit disc $|z| < 1$. As a corollary to this result, we derive a coefficient condition for a sense-preserving harmonic mapping to be univalent in $|z| < 1$.

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1. Introduction and preliminaries

The well-known Noshiro–Warschawski–Wolff criterion (see [3, page 47]) for univalency asserts the following.

THEOREM A. *If $f : D \to \mathbb{C}$ is analytic in a convex domain D and* $\operatorname{Re} f'(z) > 0$ *for all $z \in D$, then f is univalent in D.*

As a counterpart of this result Pascu and Pascu [6] proved the following lemma.

LEMMA B [6, Proposition 2.1]. *Let $f : D \to \mathbb{C}$ be an analytic function in the domain D and define*

$$K(f, D) = \inf_{\substack{a \neq b \\ a,b \in D}} \left| \frac{f(a) - f(b)}{a - b} \right|.$$

(1) *If $K(f, D) > 0$, then f is univalent in D.*
(2) *Conversely, if f is univalent in D and $\Omega \subset \overline{\Omega} \subset D$ is a domain strictly contained in D, then $K(f, \Omega) > 0$.*

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It is worth pointing out that the converse result, namely item (2) in Lemma B, is not necessarily true. For example, consider $f(z) = e^z$ in the strip $D = \{z : -\pi < \operatorname{Im} z < \pi\}$. It is a simple exercise to see that f is univalent in D. Also let $\Omega = \{z : -\pi/2 < \operatorname{Im} z < \pi/2\}$ so that $\Omega \subset \overline{\Omega} \subset D$ and $\{-n : n \in \mathbb{N}\} \subset \Omega$. Moreover, since the sequence $\{e^{-n}\}$ converges to 0, given $\epsilon > 0$ we can find a stage $N \in \mathbb{N}$ such that

$$\left|\frac{e^{-n} - e^{-m}}{n - m}\right| \le |e^{-n} - e^{-m}| < \epsilon \quad \text{for all } n, m \ge N.$$

This observation shows that

$$K(f, \Omega) = \inf_{\substack{a \neq b \\ a,b \in \Omega}} \left|\frac{e^{-a} - e^{-b}}{a - b}\right| = 0,$$

from which we obtain that the converse part of Lemma B fails. The main mistake in the proof of part (2) of Lemma B comes from the fact that Pascu and Pascu implicitly assumed in their argument that the domain D is bounded. If this were made an explicit condition then their result would be correct.

In addition, the authors in [6] proved the following result.

Theorem C [6, Theorem 2.4]. *Let $f : D \to \mathbb{C}$ be a nonconstant analytic function in the convex domain D. If there exists an analytic function $g : D \to \mathbb{C}$ univalent in D such that*

$$|f'(z) - g'(z)| \le K(g, D), \quad z \in D,$$

then the function f is also univalent in D.

As a consequence of Theorem C, they obtained the following corollary.

Corollary D [6, Corollary 2.6]. *If $f : D \to \mathbb{C}$ is nonconstant and analytic in the convex domain D and there exists $c > 0$ such that*

$$|f'(z) - c| \le c, \quad z \in D, \tag{1.1}$$

then f is univalent in D.

Moreover, Pascu and Pascu remarked [6, Remark 2.7] that Corollary D is equivalent to Theorem A. It can easily be seen that Theorem A implies Corollary D, but again the converse is not necessarily true as the next example demonstrates.

Example 1.1. Let D be the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and consider the function $f(z) = z^2$. Then $f'(z) = 2z$ and $\operatorname{Re} f'(z) > 0$ in D. Clearly, by the Noshiro–Warschawski–Wolff univalence criterion f is univalent in D. On the other hand, univalency of f in D does not follow from Corollary D, because we cannot find a universal constant $c > 0$ satisfying (1.1). Thus the observation made by the authors in [6] about the converse of Corollary D is not true in general.

In Section 2, we extend Theorem C for sense-preserving harmonic univalent mappings and present a number of corollaries, remarks and examples.

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2. Main results

A complex-valued function $f = u + iv$ in a simply connected domain D is said to be harmonic if the real and imaginary parts of f satisfy Laplace's equation. In D, f has the canonical decomposition $f = h + \overline{g}$, where h and g are analytic in D. The Jacobian J_f of f is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

We say that f is sense-preserving in D if $J_f(z) > 0$, for all $z \in D$. If the Jacobian of f is nonvanishing in D, then by the inverse mapping theorem it follows that f is locally univalent in $\mathbb{D}$. For harmonic functions the converse is also true as asserted by Lewy's theorem [5] (see also [4, page 20]). We refer to Clunie and Sheil-Small [2] and Duren [4] for many important results on harmonic univalent mappings.

In [7], the authors considered the class

$$C_H^1 := \{f = h + \overline{g}, f(0) = f_z(0) = 1 \text{ and } f_{\overline{z}}(0) = 0 : \operatorname{Re} h'(z) > |g'(z)|, z \in \mathbb{D}\},$$

where $\mathbb{D} = \{z : |z| < 1\}$ is the open unit disc in $\mathbb{C}$. They proved that the functions in C_H^1 are not only univalent in $\mathbb{D}$ but also close-to-convex in $\mathbb{D}$ (see [7, Lemma 1.1]). This result is regarded as a harmonic analogue of the Noshiro–Warschawski–Wolff criterion.

Theorem 2.1. *Let $f : D \to \mathbb{C}$ be a sense-preserving harmonic function in a convex domain D with the canonical decomposition $f = h + \overline{g}$. If there exists an analytic univalent function $\phi : D \to \mathbb{C}$ such that*

$$|h'(z) - \phi'(z)| + |g'(z)| \le K(\phi, D), \quad z \in D, \tag{2.1}$$

then f is univalent in D.

Proof. Assume that f is not univalent in D. Then there are points $z_1, z_2 \in D$ such that $z_1 \neq z_2$ and $f(z_1) = f(z_2)$. Since D is convex, the line segment joining z_1 and z_2 lies completely in D, that is, $\{z(t) = (1 - t)z_1 + tz_2 : 0 \le t \le 1\} \subset D$. An integration along this line segment, together with (2.1), yields

$$\begin{aligned}
|\phi(z_2) - \phi(z_1)| &= |(f(z_2) - \phi(z_2)) - (f(z_1) - \phi(z_1))| \\
&= \left| \int_0^1 \frac{d}{dt}(f(z(t)) - \phi(z(t)))\, dt \right| \\
&= \left| \int_0^1 ((h'(z(t)) - \phi'(z(t)))(z_2 - z_1) + \overline{g'(z(t))(z_2 - z_1)})\, dt \right| \\
&\le \int_0^1 (|h'(z(t)) - \phi'(z(t))| + |g'(z(t))|)|z_2 - z_1|\, dt \\
&\le K(\phi, D)|z_2 - z_1|.
\end{aligned}$$

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Since $z_1 \neq z_2$, from the above inequality and the definition of $K(\phi, D)$, as in [6],

$$K(\phi, D) = \left|\frac{\phi(z_2) - \phi(z_1)}{z_2 - z_1}\right|. \tag{2.2}$$

Again following the method of proof of [6], we consider the auxiliary function P defined on $D \setminus \{z_2\}$ by

$$P(z) = \frac{\phi(z) - \phi(z_2)}{z - z_2}, \quad z \in D \setminus \{z_2\}.$$

As ϕ is analytic in D, it follows that P is analytic in $D \setminus \{z_2\}$ and we see that the limit

$$\lim_{z \to z_2} P(z) = \lim_{z \to z_2} \frac{\phi(z) - \phi(z_2)}{z - z_2} = \phi'(z_2)$$

exists and is finite. Therefore, we can extend the function P to an analytic function in D, which we also denote by P. Since

$$\inf_{z \in D} |P(z)| = \inf_{\substack{z \neq z_2 \\ z \in D}} |P(z)| = \inf_{\substack{z \neq z_2 \\ z \in D}} \left|\frac{\phi(z) - \phi(z_2)}{z - z_2}\right| \geq \inf_{\substack{a \neq b \\ a,b \in D}} \left|\frac{\phi(a) - \phi(b)}{a - b}\right| = K(\phi, D),$$

it follows from (2.2) that

$$\inf_{z \in D} |P(z)| \geq K(\phi, D) = \left|\frac{\phi(z_2) - \phi(z_1)}{z_2 - z_1}\right| = |P(z_1)| \geq \inf_{z \in D} |P(z)|.$$

Thus, the minimum modulus value of P in D is attained at z_1.

Since ϕ is univalent in D, it follows that P is a nonvanishing analytic function in D which attains its minimum modulus value in the interior of D. Hence, by the minimum modulus principle for nonvanishing analytic functions, it follows that P must be constant in D.

Thus,

$$\phi(z) = c(z - z_2) + \phi(z_2), \quad z \in D, \tag{2.3}$$

for a certain constant $c \in \mathbb{C}$. From the definition of P, one can easily see that $c = \phi'(z_2)e^{i\theta}$ for some $\theta \in \mathbb{R}$. From (2.3) we see that ϕ is a linear function and so a simple computation shows that $K(\phi, D) = |c|$ in this case.

As a consequence of the above discussion, (2.1) becomes

$$|h'(z) - c + \overline{g'(z)}| \leq |h'(z) - c| + |g'(z)| \leq |c|, \quad z \in D. \tag{2.4}$$

We need to deal with two cases.

Case (i). Suppose that equality holds in both the inequalities in (2.4) for a particular point, say at $z_0 \in D$. Now, by the maximum modulus principle for complex-valued harmonic functions (see [1, Corollary 1.11, page 8]),

$$h'(z) = l - \overline{g'(z)}, \quad z \in D,$$

where $l \in \mathbb{C}$. Since h' is an analytic function, it follows that g' is constant and so is h'. Further, from the sense-preserving property of f, we get $f(z) = \alpha z + \beta\bar{z} + \gamma$ for some α, β and $\gamma \in \mathbb{C}$ with $|\alpha| > |\beta|$.

Case (ii). Suppose Case (i) does not happen. Now, repeating the above proof with $\phi(z) = cz$,

$$\begin{aligned}
|cz_2 - cz_1| &= |(f(z_2) - cz_2) - (f(z_1) - cz_1)| \\
&= \left|\int_0^1 \frac{d}{dt}(f(z(t)) - cz(t))\, dt\right| \\
&= \left|\int_0^1 ((h'(z(t)) - c)(z_2 - z_1) + \overline{e^{i\theta} g'(z(t))}(z_2 - z_1))\, dt\right| \quad \text{for some } \theta \in \mathbb{R}, \\
&\leq \int_0^1 |h'(z(t)) - c + \overline{e^{i\theta} g'(z(t))}|\, |z_2 - z_1|\, dt \\
&< |c|\, |z_2 - z_1|,
\end{aligned}$$

which is a contradiction, where in the above $\theta = 2\arg(z_2 - z_1)$. Indeed, if we have equality in the last inequality, then as in Case (i) it is easy to see that f is an affine mapping. This contradiction shows that the function f is univalent in D. □

Remark 2.2. The sense-preserving assumption about f cannot be removed in Theorem 2.1. For example, consider the harmonic function $f(z) = \operatorname{Re} z$, $z \in \mathbb{D}$. The Jacobian of f is zero on $\mathbb{D}$, which shows that f is not even sense-preserving. Now take $\phi(z) = z/2$; then (2.1) is satisfied with $K(\phi, \mathbb{D}) = 1/2$ but f is not univalent in $\mathbb{D}$.

Remark 2.3. The right-hand side in (2.1) cannot be replaced by a larger quantity, as can be seen by the function $f(z) = z + a\overline{z}^2$ in the unit disc $\mathbb{D}$, where $a \in \mathbb{D}$. For if we take $\phi(z) = z$, then $K(\phi, \mathbb{D}) = 1$ and hence, using Theorem 2.1, we get that f is univalent in $\mathbb{D}$ if $|2az| \leq 1$ for all $z \in \mathbb{D}$, that is, if $|2a| \leq 1$. But using a direct computation, one can see that f is univalent in $\mathbb{D}$ if and only if $|2a| \leq 1$. Hence inequality (2.1) in Theorem 2.1 is sharp. Here we note that if $|2a| \leq 1$ then $f \in C_H^1$ and hence f is close-to-convex on $\mathbb{D}$.

Corollary 2.4. *Let $f : D \to \mathbb{C}$ be a sense-preserving harmonic function in a convex domain D with the canonical decomposition $f = h + \overline{g}$. If there exists a constant $c > 0$ such that*

$$|h'(z) - c| + |g'(z)| \leq c, \quad z \in D,$$

then f is univalent in D.

Proof. The proof follows from Theorem 2.1 by taking $\phi(z) = cz$ with $c > 0$. □

Corollary 2.5. *Let $\phi : \mathbb{D} \to \mathbb{C}$ be an analytic univalent function with Taylor series expansion*

$$\phi(z) = \sum_{n=0}^{\infty} k_n z^n, \quad z \in \mathbb{D}.$$

Let f be a sense-preserving harmonic mapping with the canonical decomposition

$$f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n, \quad z \in \mathbb{D}. \tag{2.5}$$

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If the coefficients in (2.5) satisfy

$$\sum_{n=1}^{\infty} n|a_n - k_n| + \sum_{n=1}^{\infty} n|b_n| \leq K(\phi, \mathbb{D}), \tag{2.6}$$

then f is univalent in $\mathbb{D}$.

Proof. Let $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Then $f = h + \overline{g}$. Now

$$\begin{aligned}|h'(z) - \phi'(z)| + |g'(z)| &= \left|\sum_{n=1}^{\infty} na_n z^{n-1} - \sum_{n=1}^{\infty} nk_n z^{n-1}\right| + \left|\sum_{n=1}^{\infty} nb_n z^{n-1}\right| \\ &\leq \sum_{n=1}^{\infty} n|a_n - k_n|\,|z|^{n-1} + \sum_{n=1}^{\infty} n|b_n|\,|z|^{n-1} \\ &< \sum_{n=1}^{\infty} n|a_n - k_n| + \sum_{n=1}^{\infty} n|b_n| \\ &\leq K(\phi, \mathbb{D}),\end{aligned}$$

for all $z \in \mathbb{D}$. Thus, by Theorem 2.1, we conclude that f is univalent in $\mathbb{D}$. □

Example 2.6. If we take $\phi(z) = z$ in Corollary 2.5, then it follows easily that the harmonic function $f(z) = z + a\overline{z}^n$ $(n \geq 2)$ is univalent in $\mathbb{D}$ whenever $|a| \leq 1/n$ (as pointed out in Remark 2.3).

Example 2.7. Let α be such that $\alpha \in (0, 1)$ and consider the function

$$\varphi(z) = \frac{z - \alpha}{1 - \alpha z}, \quad z \in \mathbb{D}.$$

It is well known that φ is an analytic automorphism of the unit disc and

$$K(\varphi, \mathbb{D}) = \inf_{\substack{a \neq b \\ a,b \in \mathbb{D}}} \left|\frac{\varphi(a) - \varphi(b)}{a - b}\right| = \inf_{\substack{a \neq b \\ a,b \in \mathbb{D}}} \left|\frac{1 - \alpha^2}{(1 - \alpha a)(1 - \alpha b)}\right| = \frac{1 - \alpha}{1 + \alpha}.$$

Now we consider the harmonic function $f(z) = \varphi(z) + \overline{g(z)}$, where $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and the coefficients of g satisfy the condition

$$\sum_{n=1}^{\infty} n|b_n| \leq \frac{1 - \alpha}{1 + \alpha}. \tag{2.7}$$

We can easily see that (2.7) implies f is sense-preserving in $\mathbb{D}$. For

$$|g'(z)| = \left|\sum_{n=1}^{\infty} nb_n z^{n-1}\right| \leq \sum_{n=1}^{\infty} n|b_n| \leq \frac{1 - \alpha}{1 + \alpha} < \frac{1 - \alpha^2}{|1 - \alpha z|^2} = |\varphi'(z)|.$$

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By Corollary 2.5, it follows that f is univalent in $\mathbb{D}$. We observe that φ is a convex function and, by (2.7), f is sense-preserving. Thus, by a result of Clunie and Sheil-Small [2, Theorem 5.17], we conclude that the function f in this case is close-to-convex in $\mathbb{D}$.

EXAMPLE 2.8. For $0 < \alpha < 1$, consider the harmonic function

$$f_{a,\alpha}(z) = \frac{z-\alpha}{1-\alpha z} + \overline{ae^{i\beta}z + \left(\frac{1-\alpha}{1+\alpha} - a\right)e^{i\gamma}\frac{z^2}{2}}, \quad z \in \mathbb{D}$$

where β, γ are real, and $0 < a < (1-\alpha)/(1+\alpha)$. As in Example 2.7, it can be easily seen that $f_{a,\alpha}(z)$ is sense-preserving in the unit disc $\mathbb{D}$ and a simple computation shows that (2.6) is satisfied. Thus, by Corollary 2.5, $f_{a,\alpha}(z)$ is univalent and close-to-convex in $\mathbb{D}$.

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