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CYCLIC ORDERS AND GRAPHS OF GROUPS

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Abstract We examine a cyclic order on the directed edges of a tree whose vertices have cyclically ordered links. We use it to show that a graph of groups with left-cyclically ordered vertex groups and convex left-ordered edge groups is left-cyclically orderable.

Keywords: cyclic ordering; graph of groups

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1. Introduction

Dicks and Sunic gave an elegant way of totally ordering the vertex set of a directed tree [9]. They applied this to give a simple proof of Vinogradov's result that free groups, and more generally, free products of left-orderable groups are left-orderable. The purpose of this text is to describe a cyclically ordered counterpart.

Our basic observation is that:

Lemma 1.1. *Let $T = (V, E)$ be a tree. Suppose there is a cyclic order on link(v) for each $v \in V$. Then there is an induced cyclic order on the directed edges of T.*

Using this natural cyclic order, we examine graphs of groups and obtain:

Theorem 1.2. *Let G split as a graph of groups with left-cyclically ordered vertex groups and convex left-ordered edge groups. Then G is left-cyclically ordered in a manner compatible with its vertex and edge groups.*

This generalizes the result of Baik and Samperton that free products of left-cyclically ordered groups are left-cyclically ordered [2]. Another recent study probing more deeply than our own, was given by Clay and Ghaswala who characterized when an amalgam of cyclically ordered groups is cyclically ordered [5]. The approach of Clay and Ghaswala specializes to give a proof of the amalgamated product case of our result. Moreover, it was pointed out to us that Calegari suggested a similar approach to cyclically ordering the boundary of the Bass–Serre tree of an amalgamated product [4, Ex 2.116], from which one can sometimes deduce a cyclic ordering on the amalgam after some additional care and hypotheses.

There has been increased activity in the study of cyclically ordered groups, which are a bit more general than ordered groups. Some perspective on the relationship between them is given by the intriguing characterization that G is left-ordered if and only if $G\times\mathbb{Z}_n$ is left cyclically ordered for each n [1]. Finally, we refer to [10] and [3] for surveys on cyclically ordered groups.

2. Cyclic orders

Definition 2.1. *(Cyclic order) A cyclic order on a set A is a function $\Theta : A \times A \times A \to \{-1, 0, 1\}$ satisfying the following conditions:*

- ***Non-degeneracy**: $\Theta(x, y, z) = \pm 1$ if and only if x, y, z are pairwise distinct.*
- ***Cyclicity**: If $\Theta(x, y, z) = 1$, then $\Theta(z, x, y) = 1$.*
- ***Asymmetry**: $\Theta(x, y, z) = -\Theta(y, x, z)$.*
- ***Transitivity**: If $\Theta(x, y, z) = 1$ and $\Theta(x, z, w) = 1$, then $\Theta(x, y, w) = 1$.*

We write $[x, y, z]$ whenever $\Theta(x, y, z) = 1$.

Definition 2.2. *A strict total order is a binary relation $\prec$ on a set X satisfying the following conditions for all $x, y, z \in X$:*

- ***Irreflexivity**: $x \not\prec x$ for all $x \in X$.*
- ***Comparability**: if $x\neq y$ then $x \prec y$ or $y \prec x$.*
- ***Transitivity**: if $x \prec y$ and $y \prec z$ then $x \prec z$.*

The associated total order is denoted by $x \preceq y$ which means $x \prec y$ or $x = y$. We refer to $(X, \preceq)$ as a totally ordered set, and $(X, \prec)$ as a strict-totally ordered set.

Remark 2.3. For a strict-totally ordered set $(X, \prec)$, an associated cyclic order on X is defined by: $[x, y, z]$ holds provided $x \prec y \prec z$ or $y \prec z \prec x$ or $z \prec x \prec y$.

Remark 2.4. Consider $[0, 2\pi)$ with the usual total order. Identifying $[0, 2\pi)$ with S^1 using $\theta \mapsto \mathrm{e}^{\theta i}$, and applying Remark 2.3 provides a cyclic order on S^1.

3. Cyclic orders on trees

A *tree* is a non-empty, connected, acyclic, simplicial graph. An edge with vertices u, v is associated to two *directed edges*: (u, v) and (v, u).

In this section, we cyclically order the directed edges of a tree. We emphasize that each edge corresponds to two directed edges. The cyclic ordering arises from the following statement, which is illustrated in Figure 1.

Lemma 3.1. *Let T be a finite tree embedded in the plane. There is an induced cyclic ordering on the directed edges of T.*

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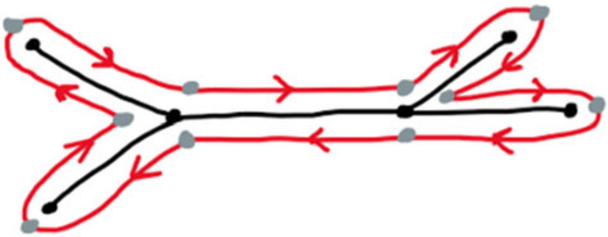

Figure 1. A clockwise boundary path cyclically orders the directed edges.

Proof. Regarding T as a disc diagram, the clockwise boundary path $\partial_{\mathsf{p}}(T)$ provides an embedding of the directed edges into S^1, hence inducing a cyclic order via Remark 2.4. Note that the boundary path traverses each edge twice: once in each direction. □

Definition 3.2. *A tree $T = (V, E)$ is a c-tree if link(v) has a cyclic order for each $v \in V$. Equivalently, there is a cyclic order on the edges adjacent to each vertex.*

We emphasize that *link*(v) has a point for each edge containing v. We are not considering directed edges here.

Definition 3.3. *An embedding $T \to \mathbb{R}^2$ of a locally finite c-tree is coordinated if for each $v \in V$ with adjacent edges $e_1 \prec e_2 \prec \cdots \prec e_n \prec e_1$, their images $\bar{e}_1 \prec \bar{e}_2 \prec \cdots \prec \bar{e}_n \prec \bar{e}_1$ are in the same clockwise order about $\bar{v} \in \mathbb{R}^2$.*

Lemma 3.4. *Each (locally) finite c-tree T has a coordinated embedding $T \to \mathbb{R}^2$.*

Proof. We produce a 'thickening' of T into 0-handles and 1-handles to obtain a disk as follows. Embed a valence n-vertex v with cyclically ordered edges $e_1, \ldots, e_n$ in a unit disk, by identifying v with 0 and identifying each edge with the segment joining 0 and $\mathrm{e}^{\frac{2\pi}{n}i}$. Join disks for adjacent vertices along neighbourhoods (consistently orientated) to form a surface S homeomorphic to the unit disk, see Figure 2. □

Remark 3.5. The embedding of Lemma 3.4 is unique up to ambient isotopy. Hence, for any finite subtrees $T_a \subset T_b$, a coordinated embedding of T_a is essentially the same as an embedding of T_a induced by a coordinated embedding of T_b. Indeed, the way Lemma 3.4 embeds T_b induces the way it embeds T_a simply by 'forgetting' $T_b - T_a$.

For any two finite subtrees, their embeddings agree with a coordinated embedding of a larger finite tree containing them.

Theorem 3.6. *Let T be a c-tree. There is an induced cyclic order on the set of directed edges of T. It is uniquely determined by the cyclic orders on vertex links.*

Proof. For a c-tree, take a coordinated embedding of a finite subtree T'. Lemma 3.1 yields a cyclic order on the directed edges of T'. This cyclic order is consistent for $T' \subset T''$ whenever T'' is a larger finite subtree. Hence, it induces a cyclic order on all directed edges of T.

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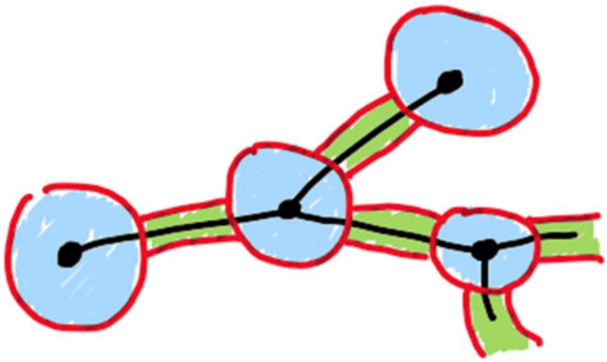

Figure 2. Handlebody decomposition of a tree in $\mathbb{R}^2$.

Uniqueness holds since the cyclic order on each $\mathrm{link}(v)$ is determined by the cyclic order of the outgoing directed edges at v. Note that in the cyclic ordering of the directed edges of $\mathrm{star}(v)$, for each edge, its two directed edges are consecutive. $\square$

Lemma 3.7. (G-invariance). *Suppose G acts on a c-tree T so that cyclic orders on vertex links are G-invariant. Then the induced cyclic order on directed edges of T is G-invariant.*

Proof. This holds by Theorem 3.6 since the induced cyclic order on directed edges of T is determined by the cyclic orderings on vertex links. $\square$

4. Cyclic orders and tree augmentation

We provide an alternate explanation of the cyclic ordering on directed edges of a c-tree given in § 3. This approach constructs a correspondence between directed edges and spurs.

Definition 4.1. *For vertices of a tree $x, y, z \in V$, the median $m(x, y, z)$ is the vertex equal to the intersection of geodesics $xy \cap yz \cap zx$.*

Lemma 4.2. *Let $T = (V, E)$ be a c-tree, there is a cyclic order on the set $L \subseteq V$ of spurs of T.*

Proof. When $x, y, z \in L$ are distinct, the median $m = m(x, y, z)$ has three distinct edges adjacent to m pointing to x, y and z. These edges e_x, e_y and e_z are cyclically ordered around m. Declare a cyclic order on L via:

$$[x, y, z] \text{ in } L \iff [e_x, e_y, e_z] \text{ in } \mathrm{link}(m).$$

Non-degeneracy, cyclicity and asymmetry all follow immediately as the link of the median is cyclically ordered. For leaves $x, y, z, w \in L$, transitivity follows if $m(x, y, z) = m(x, z, w)$. Otherwise, let S be the smallest subtree containing $\{x, y, z, w\}$. S takes the form of an 'H' with two leaves at $m_1 = m(x, y, z)$ and two leaves at $m_2 = m(x, z, w)$.

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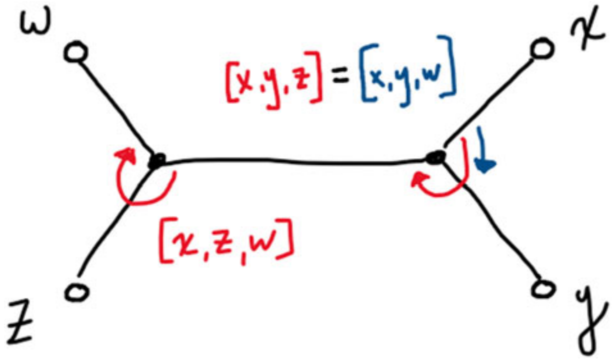

Figure 3. This explains transitivity for Lemma 4.2.

Figure 4. The direction of e determines the position of the spur.

Via Lemma 3.4, we can embed S into the plane so that links of vertices are cyclically ordered clockwise. If $[x, y, z]$ and $[x, z, w]$ hold, then $[x, y, w]$ also holds, see Figure 3. □

Definition 4.3. *(Augmented tree) Let T be a directed c-tree, the augmented tree $\overline{T}$ is obtained by adding an augmented edge e_{aug} at the barycentre of each directed edge e, see Figure 4. More precisely, for each edge $e \in E$, let b_e be its barycentre and cut e into two half edges, e_{out} and e_{in}. Orient the half edges so that e_{in} and e_{out} are incoming and outgoing at b_e. Under this construction links of vertices in the original tree T are unchanged, and the link of each barycentre vertex b_e is $\{e_{in}, e_{out}, e_{aug}\}$. Cyclically order $link(b_e)$ using the rule $[e_{in}, e_{out}, e_{aug}]$. Direct augmented edges away from barycentres, and note that the augmented tree $\overline{T}$ is now a directed c-tree.*

Theorem 4.4. *There is an induced cyclic order on the set of directed edges of a c-tree T.*

Proof. Construct the augmented tree $\overline{T}$ and note that each directed edge of T is associated to a spur of $\overline{T}$. Apply Lemma 4.2 to cyclically order these spurs. □

5. Ordered and cyclically ordered groups

Definition 5.1. (Left-ordered group). *A group G is left-ordered if there is a strict total order $(G, \prec)$ such that for all $x, y, g \in G$ we have:*

$$x \prec y \implies gx \prec gy.$$

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G is left-ordered if and only if $G = P \sqcup \{1_G\} \sqcup N$ with $PP \subset P$ and $NN \subset N$ where $P = \{g \in G \ : \ 1_G \prec g\}$ and $N = \{g \in G \ : \ g \prec 1_G\}$. Then $g \prec h \iff g^{-1}h \in P$. The subset P is referred to as the positive cone.

Definition 5.2. (Left-cyclically ordered group). *A group G is left-cyclically ordered if there is a cyclic order on G that is left-invariant in the sense that:*

$$[a, b, c] \quad \Longrightarrow \quad [ga, gb, gc].$$

Remark 5.3. Let G act freely on a cyclically ordered set X. Cyclically order G via:

$$[a, b, c] \text{ in } G \iff [ax, bx, cx] \text{ in } X.$$

The following well-known statements can be found in [8, § 1.1.3] and [11, Ex 2.116] respectively.

Lemma 5.4. *Let G act faithfully and order-preservingly on a strict-totally ordered set $(X, <)$. Then G has an induced left-order.*

Proof. Choose a well-ordering $\prec_w$ on X. For $g \neq h \in G$, let p be $\prec_w$-minimal with $gp \neq hp$. Declare $g \prec h$ if $gp < hp$.

This relation is irreflexive as $gp \not< gp$. Since G acts faithfully on X, for $g \neq h \in G$ there exists $x \in X$ with $gx \neq hx$, so comparability holds. G-invariance holds since $kgp < khp \iff gp < hp$. Let p_1 and p_2 be $\prec_w$-minimal with $xp_1 \neq yp_1$ and $yp_2 \neq zp_2$. If $p_1 = p_2$ we are done. If $p_1 \prec_w p_2$ then $yp_1 = zp_1$ and $xp_1 < zp_1$. If $p_2 \prec_w p_1$ then $xp_2 = yp_2$ and $xp_2 < zp_2$. Thus transitivity holds for $(G, \prec)$. □

Theorem 5.5. *Let G act faithfully and order-preservingly on a cyclically ordered set X. Then G has an induced left-cyclic order.*

Proof. Let $p \in X$ and $\dot{X} = X - \{p\}$. Observe that $\dot{X}$ is totally ordered and $H = stab(p)$ acts faithfully on $\dot{X}$. Via Lemma 5.4, H is left-ordered. There is a strict total order $(gH, \prec)$ for each left coset, by declaring $g\alpha \prec g\beta \iff \alpha \prec \beta$. This is independent of the choice g of representative, since $(H, \prec)$ is left H-invariant.

Our ordering on each coset provides a partial ordering on $G = \cup gH$. This partial ordering is G-invariant by definition. This partial ordering on G extends to a G-invariant cyclic ordering by cyclically ordering the left cosets using their bijection with Gp. Specifically $[a, b, c]$ holds if either:

(1) $a \prec b \prec c$ and $ap = bp = cp$,
(2) $a \prec b$ with $ap = bp \neq cp$, or $b \prec c$ with $bp = cp \neq ap$, or $c \prec a$ with $cp = ap \neq bp$,
(3) $[ap, bp, cp]$ in X. □

6. Ordering collections of cosets

Definition 6.1. (Convex subgroup). *A subgroup H of a left-ordered group $(G, \prec)$ is convex if for all $h_1, h_2 \in H$ and $g \in G$, if $h_1 \prec g \prec h_2$ then $g \in H$.*

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Definition 6.2. (c-convex subgroup). *A proper subgroup H of a left-cyclically ordered group G is c-convex if there is a G-invariant cyclic order on its cosets G/H.*

Two ways of defining c-convexity for subgroups of left-cyclically ordered groups appear in [5] and [7]. We refer to these as c'-convexity and c''-convexity and show they are both equivalent to c-convexity.

Definition 6.3. (c'-convex subgroup). *Let G be a left-cyclically ordered group and $H \subset G$ a proper subgroup. We say H is c'-convex if for every $g \notin H$ and $f \in G$ and $h_1, h_2 \in H$, if $[h_1, f, h_2]$ and $[h_1, h_2, g]$ then $f \in H$.*

The definition of c''-convexity requires the following preliminary notion.

Definition 6.4. *Let G be a left-cyclically ordered group. A proper subgroup $H \subset G$ is left-ordered by restriction if for each $h_1, h_2 \in H$, if $[h_1^{-1}, 1, h_1]$ and $[h_2^{-1}, 1, h_2]$ then $[h_1^{-1}h_2^{-1}, 1, h_2h_1]$. When $H \subset G$ is left-ordered by restriction, there is an induced left-order on H given by the following positive cone:*

$$P \ = \ \{\ h \in H \ : \ [h^{-1}, 1, h] \text{ holds in } G\ \}.$$

Definition 6.5. (c''-convex subgroup). *Let G be a left-cyclically ordered group. A proper subgroup $H \subset G$ is c''-convex if:*

(1) *Whenever $h_1, h_2 \in H$ and $f \in G$, if $[h_1, 1, h_2]$ and $[h_1, f, h_2]$ then $f \in H$.*
(2) *H is left-ordered by restriction.*

Theorem 6.6. *For a proper subgroup H of a left-cyclically ordered group G, the following are equivalent:*

(1) *c-convexity.*
(2) *c'-convexity.*
(3) *Property (1) of c''-convexity.*
(4) *c''-convexity.*

Proof of (1) $\implies$ (2). See [5, Lemma 5.1]. □

Proof of (2) $\implies$ (3). We argue by contradiction. If Property (1) fails, there exists $h_1, h_2 \in H$ and $g \notin H$ with $[h_2, 1, h_1]$ and $[h_2, g, h_1]$. Suppose $[1, g, h_1]$ and left-multiply by h_1^{-1} to get $[h_1^{-1}, h_1^{-1}g, 1]$. Since $[h_1, h_1^{-1}g, 1]$ and $[h_1, 1, g]$, c'-convexity implies that $h_1^{-1}g \in H$, a contradiction. The case $[h_2, g, 1]$ is analogous. □

Proof of (3) $\implies$ (1). See [7, Proposition 2.4]. We note that Property (2) of c''-convexity is not used in that proof. □

Proof of (2) $\implies$ (4). The proof that c'-convexity implies Property (2) of c''-convexity is shown in [5, Lemma 5.2]. □

Proof of (4) $\implies$ (3). This is immediate. □

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For subsets U, V of an ordered set $(X, \prec)$, declare $U \ll V$ if there exists $v \in V$ with $u \prec v$ for all $u \in U$. Note that within a left-ordered group $(G, \prec)$ we have $U \ll V \iff gU \ll gV$ for all $g \in G$.

The following property is well known.

Lemma 6.7. *Let $(G, \prec)$ be an ordered group and H a convex subgroup. The relation $\ll$ restricts to a G-invariant strict total order on the collection G/H of left cosets.*

Proof. Comparability of $(G/H, \ll)$ holds as cosets are disjoint and H is convex. Transitivity follows since $(G, \prec)$ is left-ordered. If $U \ll U$ for some $U \in G/H$, then there exists $v \in U$ with $u \prec v$ for all $u \in U$, so $v \prec v$ which is impossible. □

Lemma 6.8. *Let G be a left-cyclically ordered group. Let $H \subsetneq K \subsetneq G$ be convex subgroups. Then $H \ll K$ (in the induced order on K).*

Proof. Let $(K, \prec)$ be the induced left-order of Definition 6.4. Consider a coset $kH \neq H$. If $H \not\ll K$, there exists $h \in H$ with $k \prec h$. By Lemma 6.7, $k' \prec h'$ for all $k' \in kH$ and $h' \in H$. In particular, $k \prec 1$. Left multiplying gives $1 \prec k^{-1}$. Since $H \not\ll K$, we have $k^{-1} \prec h''$ for some $h'' \in H$. Finally, $1 \prec k^{-1} \prec h''$ implies $k^{-1} \in H$ by convexity, a contradiction as $k \notin H$. □

Lemma 6.9. *Suppose H and K are convex subgroups of the left-cyclically ordered group G. Either $H \subset K$ or $K \subset H$.*

Proof. If $H \not\subset K$ and $K \not\subset H$ then $H \cap K \subsetneq H$ and $H \cap K \subsetneq K$. Thus, $H \cap K \ll H$ and $H \cap K \ll K$ by Lemma 6.8. Thus there exists $h \in H$ and $k \in K$ with $\alpha \prec h$ and $\alpha \prec k$ for all $\alpha \in H \cap K$. Note that $h \neq k$, as otherwise $k \in H \cap K$ hence $k \prec k$. Without loss of generality, assume $[1, h, k]$. By convexity, $h \in K$. Thus, $h \in H \cap K$ so $h \prec h$, a contradiction. □

Corollary 6.10. *Suppose K and H are convex subgroups of a left-cyclically ordered group G. Let $x, y \in G$. If $xK \cap yH \neq \emptyset$ then either $xH \subset yK$ or $yK \subset xH$.*

Proof. This follows from Lemma 6.9. □

Definition 6.11. *It will be convenient to consider indexed collections of subsets $\{H_i\}_{i \in I}$ allowing 'repeats' in the sense that $H_i = H_j$ though $i \neq j$.*

Although we will not use it, it seems worth articulating the following special case of our preliminary goal, Theorem 6.13.

Lemma 6.12. *Let $(G, \prec)$ be a left-ordered group and $\{H_i\}_{i \in I}$ an indexed collection of convex subgroups. There is a G-invariant total order on the indexed collection of left cosets $\{gH_i \ : \ g \in G, i \in I\}$.*

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Proof. Choose a strict total order $\prec_I$ on I. Let $\ll_I$ denote the relation defined by:

$$g_1H_i \ll_I g_2H_j \iff \begin{cases} g_1H_i \neq g_2H_j \text{ and } g_1H_i \ll g_2H_j \\ g_1H_i = g_2H_j \text{ and } i \prec_I j. \end{cases}$$

Transitivity and comparability of $\ll_I$ hold since $(G, \prec)$ and $(I, \prec_I)$ are strict total orders. It is impossible for $g_1H_i \ll_I g_1H_i$, as this would imply $i \prec_I i$. Thus $\ll_I$ is irreflexive, and therefore a strict total order.

Let $g_1H_i \ll_I g_2H_j$. If $g_1H_i \neq g_2H_j$, then G-invariance of $(G, \prec)$ ensures $\alpha g_1H_i \ll_I \alpha g_2H_j$ for all $\alpha \in G$. If $g_1H_i = g_2H_j$, the order depends only on $(I, \prec_I)$, and $\alpha g_1H_i \ll_I \alpha g_2H_j$ for all $\alpha \in G$. Thus, $\ll_I$ is G-invariant. □

Theorem 6.13. *Let G be a left-cyclically ordered group and let $\{H_i\}_{i\in I}$ be an indexed collection of c-convex subgroups. There is a G-invariant cyclic order on the indexed collection of left cosets $\{gH_i : g \in G, i \in I\}$.*

Proof. Choose a strict total order $\prec_I$ on I. For any finite subcollection of c-convex subgroups $\{H_j\}_{j\in J} \subseteq \{H_i\}_{i\in I}$, by Lemma 6.9 there is a chain of inclusions. (We abuse notation and regard $J = \{0, 1, \ldots, n\}$.)

$$G = H_0 \supset H_1 \supseteq \cdots \supseteq H_n.$$

This chain of inclusions determines a graph of groups, whose underlying graph is a length-n subdivided interval. Direct all edges away from the root vertex v_0, whose vertex group is G. The edge e_i terminates at the vertex v_i, and $G_{e_i} = G_{v_i} = H_i$. As this graph of groups is telescopic its fundamental group is G.

Let $T = (V, E)$ be the Bass–Serre tree corresponding to this graph of groups. The vertex set $V = \sqcup_{i=0}^{n}\{gH_i : g \in G\}$ consists of the indexed collection of left cosets of vertex groups, see Figure 5.

There is a directed edge from g_1H_k to g_2H_{k+1} when $g_1H_k \supset g_2H_{k+1}$. Under this construction, each left coset gH_j for $j > 0$ is represented by a directed edge.

We turn T into a directed c-tree. For the root vertex G, note that $\mathrm{link}(G)$ corresponds to G/H_1 which has a G-invariant cyclic order by c-convexity. For any other vertex gH_k, there is one incoming parent edge of $\mathrm{link}(gH_k)$ and has outgoing edges representing containment of left-subcosets of H_{k+1}. By Lemma 6.7, $(H_k, \ll)$ induces a strict total order on H_k/H_{k+1}. This extends to a strict total order on $\{H_k\} \sqcup H_k/H_{k+1}$ by declaring H_k minimal. Translating by g provides a total order on gH_k and its left H_{k+1} cosets. This provides a cyclic order on $\mathrm{link}(gH_k)$ by Remark 2.3.

Theorem 3.6 provides a cyclic order on directed edges of the c-tree T. Hence, this gives a cyclic order on left cosets of $\{H_j\}_{j\in J}$. This holds for any finite collection of convex subgroups. The cyclic order is consistent for graphs of groups $\mathcal{G}' \subset \mathcal{G}''$ as defined above. Hence, this induces a cyclic order on all left cosets in $\{gH_i : g \in G, i \in I\}$. As G is the fundamental group of this graph of groups, the cyclic order on the link of each vertex is G-invariant. Hence, by Lemma 3.7 the cyclic order on left cosets is G-invariant. □

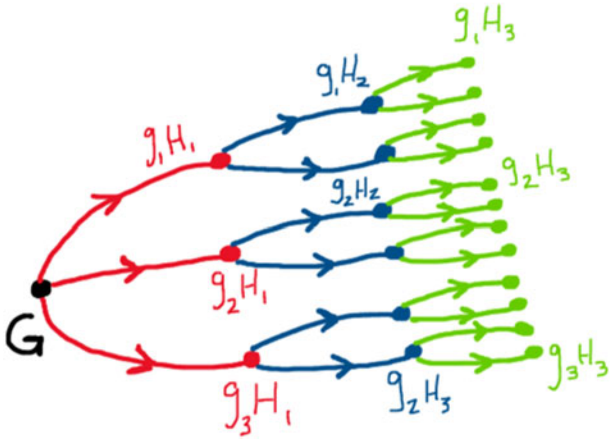

Figure 5. Part of a finite coset tree.

Remark 6.14. The referee suggests a more self-contained proof, using the fact that G has a maximal c-convex subgroup H. The left cosets within H have a left-invariant left-order by Lemma 6.12. And this can be extended to a cyclic order on the cosets in G by combining this with cyclic order on G/H.

7. Groups acting on trees

7.1. Action on tree

Definition 7.1. *An inclusion $H \to K$ of a left-ordered group into a left-cyclically ordered group is order-preserving if*

$$a \prec b \prec c \text{ in } H \quad \Longrightarrow \quad [a, b, c] \text{ in } K.$$

Theorem 7.2. *Let G act without inversions on a tree $T = (V, E)$. Suppose:*

(1) *The stabilizer G_v is left-cyclically ordered for each vertex $v \in V$.*
(2) *The stabilizer G_e is left-ordered for each edge $e \in E$.*
(3) *The inclusion $G_e \subset G_v$ is c-convex whenever v is a vertex of e.*

Then there is a c-tree $\widetilde{T} = (\widetilde{V}, \widetilde{E})$ such that:

(1) *There exists a spur $\tilde{e} \in \widetilde{E}$ such that $G\tilde{e}$ is a free orbit.*
(2) *There is a G-invariant cyclic order on the orbit $G\tilde{e}$ that induces a cyclic order on G.*
(3) *For each $e \in E$, the order on G_e is induced by the action of G_e on $\widetilde{T}$.*
(4) *For each $v \in V$, the cyclic order on G_v is induced by the action of G_v on $\widetilde{T}$.*

Proof. Build $\widetilde{T}$ from T as follows. For each $v \in V$ add a spur to v for each element of the stabilizer G_v. These spurs are in correspondence with cosets of the trivial subgroup of G_v which is a c-convex subgroup. $G_e \subset G_v$ is c-convex by hypothesis. For each $v \in V$, cyclically order link(v) via Theorem 6.13. Thus $\widetilde{T}$ is a c-tree.

Let $\tilde{e}$ be an added spur. Cyclically order the spurs and hence $G\tilde{e}$ by Theorem 4.4. By Lemma 3.7, the cyclic order on $G\tilde{e}$ is G-invariant. Finally, since G acts freely on $G\tilde{e}$, Remark 5.3 provides a left-cyclic order on G. □

7.2. Graph of groups statement

Corollary 7.3. *Let G split as a graph Γ of groups. Suppose each vertex group G_v is left-cyclically ordered, and each edge group G_e is left-ordered. Suppose each inclusion $G_e \hookrightarrow G_v$ of an edge group is c-convex. Then G has a left-cyclic order that restricts to the cyclic order of each vertex group G_v.*

Proof. Let $T = (V, E)$ be the Bass–Serre tree over Γ, which we assume to be directed. V consists of all left cosets of vertex groups of Γ in G, and E consists of left cosets of edge groups of Γ in G. That is, allowing for repeats (of edge or vertex groups):

$$V = \{gG_v \ : \ g \in G, \ v \in \text{Vertices}(\Gamma)\}$$
$$E = \{gG_e \ : \ g \in G, \ e \in \text{Edges}(\Gamma)\}.$$

Varying $g \in G$, there is an edge gG_e directed from gG_u to gG_v in T precisely when e is directed from u to v in Γ.

The stabilizer of a vertex gG_v equals gG_vg^{-1}, and similarly the stabilizer of an edge gG_e equals gG_eg^{-1}. Conjugation preserves the cyclic orders on G_v for each vertex, and similarly preserves the orderings on G_e for each edge, thus vertex and edge stabilizers are cyclically ordered. Let $\widetilde{T}$ be the c-tree obtained from T by Theorem 7.2 and note that the cyclic order on each vertex group is induced by its $\widetilde{T}$ action. $\widetilde{T}$ has a spur $\tilde{e}$ with a free G-orbit which provides a cyclic order on G by Remark 5.3. □

We note that [6] contains an analogous result to Corollary 7.3 for a graph of groups with left-orderable vertex groups and convex edge groups.

Remark 7.4. Every group acting faithfully without inversions on a c-tree arises as in Corollary 7.3. The edge stabilizers are c-convex subgroups of the vertex stabilizers. Indeed, for each edge e at a vertex v, the left cosets of $stab(e)$ in G_v correspond G_v-equivariantly to the edges in the G_v-orbit of e. The G_v-invariant cyclic order on the edges yields a G_v-invariant cyclic ordering on the cosets. Finally, every action on a tree arises as the Bass–Serre tree of a graph of groups.

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