MAXIMAL INVERSE SUBSEMIGROUPS OF $S(X)$

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1. Introduction. If X is a topological space then $S(X)$ will denote the semigroup, under composition, of all continuous functions from X into X. An element f in a semigroup is regular if there is an element g such that $fgf = f$. The regular elements of $S(X)$ will be denoted by $R(X)$. Elements f and g are inverses of each other if $fgf = f$ and $gfg = g$. Every regular element has an inverse [**1**]. If every element in a semigroup has a unique inverse then the semigroup is an inverse semigroup. In this paper we examine maximal inverse subsemigroups of $S(X)$.

For certain idempotents e we will define a set I_e and show that I_e is a maximal inverse subsemigroup of $S(X)$ with e as its smallest idempotent. N. R. Reilly [**5**], J. W. Nichols [**4**] and B. M. Schein [**7**] have looked at maximal inverse subsemigroups of T_X, the full transformation semigroup on the set X. By letting X have the discrete topology we can apply our theorems about 0-dimensional spaces to yield the results of Nichols and Reilly. Further results give conditions on X which ensure that $G(X)$, the group of units of $S(X)$, is a maximal inverse subsemigroup. Other theorems will give results for X a Euclidean n-cell or Euclidean n-space.

2. Preliminary results. Throughout the paper we will use the notation and basic results about semigroups from Clifford and Preston [**1**]. A retract is the range of an idempotent in $S(X), f|_A$ will denote the restriction of the map f to the set A. The juxtaposition fg will mean the composition $f \circ g$. We begin with a result of R. D. Hofer [**2**] which gives conditions for f and g to be inverses of each other.

PROPOSITION 1. *Let $f \in R(X)$. Then g is an inverse for f if and only if there exist retracts A, B of X such that B = range of f, A = range of g, $f|_A$ is a homeomorphism onto B, $g|_B$ is a homeomorphism onto A, $fg|_B = \mathrm{id}|_B$ (identity map on B) and $gf|_A = \mathrm{id}|_A$.*

Note that if $f \in R(X)$ then the set B above is uniquely determined; we will denote it by B_f. If the set A is also uniquely determined (for example, if f belongs to an inverse semigroup) then it will be denoted by A_f. If f is an idempotent then we will say $A_f = B_f$. Finally, if f belongs to an inverse semigroup J then the unique algebraic inverse of f (in J) will be denoted by f^{-1}. We will also occasionally use the symbol f^{-1} for the inverse image of the map f; no confusion should result from this.

The next lemma is concerned with composing two elements in $R(X)$.

LEMMA 2. *Suppose $f, g \in R(X)$ with inverses f', g' respectively. Let $A = g'(B_{f'} \cap B_g)$ and $B = fg(A)$.*

(1) *If range of $fg = B$ then $(fg)(g'f')(fg) = fg$, $fg \in R(X)$ and fg maps A homeomorphically onto B.*

(2) *If range of $fg = B$ and range of $g'f' = A$ then $g'f'$ is an inverse for fg.*

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Proof. (1) Suppose range of $fg = B$. If $y \in A$ then $y \in B_{g'}$, $g(y) \in B_{f'}$, and so $f'fg(y) = g(y)$ $(f'f|_{B_{f'}} = \mathrm{id}|_{B_{f'}})$ and hence $g'f'fg(y) = g'g(y) = y$. But now, if $x \in X$ then $fg(x) = fg(y)$ for some y in A. Thus

$$(fg)(g'f')(fg)(x) = (fg)(g'f')(fg)(y) = fg(y) = fg(x).$$

Thus $fg \in R(X)$ and fg maps A homeomorphically onto B $(g|_A$ and $f|_{B_{f'}}$ are both homeomorphisms).

(2) Assume range of $fg = B$ and range of $g'f' = A$. If we show that $A = g'f'(f(B_g \cap B_{f'}))$ then we can apply (1) to the element $g'f'$ to conclude that $(g'f')(fg)(g'f') = g'f'$. But this is true since $A = g'(B_{f'} \cap B_g)$ and $f'f|_{B_{f'}} = \mathrm{id}|_{B_{f'}}$.

We now introduce a new notion.

Definition. Let e be an idempotent in $S(X)$. We say that an element $f \in S(X)$ *respects* A_e if there exists an inverse f' of f with $A_e \subseteq B_{f'} \cap B_f$ and $f|_{A_e}$ is a homeomorphism onto A_e. If we wish to emphasize the role of f' we will say f respects A_e *via* f'.

Next we consider Green's relation $\mathscr{H}$. Let H_e denote the $\mathscr{H}$-class of an idempotent $e \in S(X)$. Then by using results of K. D. Magill, Jr. and S. Subbiah [**3**] we see that

$$H_e = \{f \in R(X)\text{: there exists an inverse } f' \text{ of } f \text{ such that } B_f = B_{f'} = A_e,\ e(x) = e(y) \text{ if and only if } f(x) = f(y)\}.$$

Note that every element of H_e respects A_e and that if $f \in H_e$ then $e(x) = e(y)$ if and only if $ef(x) = ef(y)$ (e is the identity on B_f). We now state a result pertaining to these notions (the proof will be omitted).

Lemma 3. *Let e be an idempotent in $S(X)$ and suppose that h respects A_e. Then $he \in H_e$ and $he|_{A_e} = h|_{A_e}$.*

Lemma 4. *Suppose e and f are idempotents in $S(X)$ which commute.*

(1) *If $A_e = A_f$ then $e = f$.*

(2) *If $e(x), f(x) \in A_e \cap A_f$ then $e(x) = f(x)$. In particular, if $A_e \subseteq A_f$ and $f(x) \in A_e$ then $e(x) = f(x)$.*

Proof. The proof is straightforward and will be omitted.

Recall that in an inverse semigroup J all idempotents commute. J has a smallest idempotent e if $fe = ef = e$ for all idempotents f in J. If this is the case then $A_e \subseteq A_f$, with equality occurring only if $e = f$ (by the last lemma).

Lemma 5. *Let J be an inverse subsemigroup of $S(X)$ with smallest idempotent e and suppose $g \in J$. Then g respects A_e, $g^{-1}eg = e$, $ge = eg$ and for all $x, y \in X$, $e(x) = e(y)$ if and only if $eg(x) = eg(y)$.*

Proof. The elements geg^{-1} and $g^{-1}eg$ are idempotents in J and so $A_e \subseteq g(A_e) \subseteq B_g$ and $A_e \subseteq g^{-1}(A_e) \subseteq A_g$. But then $g|_{A_e}$ maps onto A_e and so g respects A_e. Now $A_{geg^{-1}} =$

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$A_{g^{-1}eg} = A_e$ and so, by Lemma 4, $geg^{-1} = e$ and thus $eg = geg^{-1}g = gg^{-1}ge = ge$. Now

$$\begin{aligned} e(x) = e(y) &\Leftrightarrow ge(x) = ge(y) \ (g \text{ is one-to-one on } A_e) \\ &\Leftrightarrow eg(x) = eg(y). \end{aligned}$$

The next corollary shows us that every maximal inverse subsemigroup with a smallest idempotent e must contain H_e (also proved by Reilly [**5**]).

Corollary 6. *Let J be an inverse subsemigroup of $S(X)$ with smallest idempotent e and let $g \in J$. If $f \in H_e$ then $fg, gf \in H_e$; if J is maximal then $H_e \subseteq J$.*

Proof. Suppose f respects A_e via f'. Then we apply Lemmas 2 and 5 to show that fg and gf are in $R(X)$ and that $B_{g^{-1}f'} = B_{fg} = B_{f'g^{-1}} = B_{gf} = A_e$. Now if $f \in H_e$ then $ef = fe = f$. Thus

$$\begin{aligned} fg(x) = fg(y) &\Leftrightarrow feg(x) = feg(y) \\ &\Leftrightarrow eg(x) = eg(y) \quad (f \text{ is one-to-one on } A_e) \\ &\Leftrightarrow e(x) = e(y) \qquad (\text{by the last Lemma}). \end{aligned}$$

Also,

$$\begin{aligned} gf(x) = gf(y) &\Leftrightarrow f(x) = f(y) \quad (g \text{ is one-to-one on } B_f) \\ &\Leftrightarrow e(x) = e(y) \qquad (f \in H_e). \end{aligned}$$

Thus fg and gf both belong to H_e. Now suppose J is maximal. Then $H_e \cup J$ is a subsemigroup by the above. Clearly idempotents in $H_e \cup J$ commute and so $H_e \cup J$ is an inverse subsemigroup [**1**]. Hence $H_e \subseteq J$ by maximality of J.

Later in the paper we will define several maximal inverse subsemigroups with smallest idempotent e. This last corollary then tells us that each of these maximal inverse subsemigroups contains H_e. The next results indicate when such a smallest idempotent is present.

Definition. Let J be an inverse subsemigroup of $S(X)$. Then we define $A_J = \cap\{A_f : f \in J\}$. (Note that the collection $\{A_f : f \in J\}$ satisfies the finite intersection property; if X is compact then $A_J \neq \varnothing$.)

Lemma 7. *Let J be an inverse subsemigroup of $S(X)$ and suppose $f \in J$. Then $A_J \subseteq A_f \cap B_f$ and $f|_{A_J}$ is a homeomorphism onto A_J. If there exists an idempotent $e \in J$ such that $A_e = A_J$ then e is the smallest idempotent of J.*

Proof. $A_J \subseteq A_f$ by definition and since there exists $f^{-1} \in J$ with $A_{f^{-1}} = B_f$ we have $A_J \subseteq B_f$ also. Thus $f|_{A_J}$ is a homeomorphism. If $x \in A_J$ and $f(x) \notin A_J$ then there exists $g \in J$ such that $f(x) \notin B_g$. Without loss of generality we may assume g is an idempotent and $B_g \subseteq B_f$ $(ff^{-1}gg^{-1} \in J)$. Now $f^{-1}gf$ is an idempotent in J and so $f^{-1}gf(x) = x$ $(x \in A_J \subseteq A_{f^{-1}gf})$. But then $ff^{-1}gf(x) = f(x)$. Since $B_g \subseteq B_f$ we have $ff^{-1}gf(x) = gf(x)$. Thus $gf(x) = f(x)$ but $f(x) \notin B_g$. This is a contradiction. Hence $f(x) \in A_J$. This means that f maps

A_J into A_J. Apply this result to f^{-1} to conclude that f maps A_J onto A_J. Now suppose e is an idempotent in J with $A_e = A_J$. Then if f is any other idempotent, $f(x) = x$ for all $x \in A_e = A_J$ $(A_J \subseteq A_f)$. Thus $ef = fe = e$ and so e is the smallest idempotent.

COROLLARY 8. *Let J be an inverse subsemigroup of $S(X)$, e an idempotent in J. Suppose the following condition is satisfied: if B is any retract of X with $B \subsetneq A_e$ then there exists $f \in J$ such that $f(B) \cap B = \varnothing$. Then $A_J = A_e$, e is the smallest idempotent in J and if $g \in J$ then g respects A_e.*

Proof. We know $A_J \subseteq A_e$. If $A_e \nsubseteq A_J$ then there exists an idempotent $g \in J$ such that $A_g \subsetneq A_e$. Then by the condition there exists an $f \in J$ such that $f(A_g) \cap A_g = \varnothing$. Then fgf^{-1} is an idempotent in J and so $g(fgf^{-1}) = (fgf^{-1})g$. But $B_{gfgf^{-1}} \subseteq A_g$, $B_{fgf^{-1}g} \subseteq f(A_g)$ and $f(A_g) \cap A_g = \varnothing$. This is a contradiction. Thus $A_e = A_J$. The rest of the corollary follows from Lemmas 7 and 5.

3. Main results. We first prove several results about maximal inverse subsemigroups of $S(X)$ where X is 0-dimensional. The symbol c_y will signify the constant map in $S(X)$ which sends everything to the point y.

THEOREM 9. *Let X be T_1 and 0-dimensional and suppose $e = c_y$ for some fixed $y \in X$. Let*

$$I_e = \{f \in R(X) : f(y) = y, \text{ there exists an inverse } f' \text{ of } f \text{ such that } \{y\} \subseteq B_{f'} \cap B_f, \text{ and if } f(x) \neq y \text{ then } |\{z : f(z) = f(x)\}| = 1\}.$$

Then I_e is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent e.

Proof. We initially note that if $f \in I_e$ with inverse f' then f respects A_e via f', if $x \in B_{f'}$ and $x \neq y$ then $f(x) \neq y$, and if $x \notin B_{f'}$ then $f(x) = y$. This, coupled with the fact that X is T_1, means that the boundaries of B_f and $B_{f'}$ are contained in $\{y\}$. We can now show that if $f \in I_e$ then f has an inverse $k \in I_e$; define k by

$$k(x) = \begin{cases} f'(x) & x \in B_f \\ y & \text{otherwise.} \end{cases}$$

Note that k is continuous by the above remarks and it is straightforward to show that $k \in I_e$. Now suppose $f, g \in I_e$ with inverses $f', g' \in I_e$. Let $h = fg$. If $A = g'(B_{f'} \cap B_g)$ and $B = h(A)$ we show that $B =$ range of h. Let $x \in X$. Then there exists z such that $g(z) = g(x)$ and $g'g(z) = z$. If $g(z) \in B_{f'}$ then $z \in A$ and $h(z) = h(x)$. If $g(z) \notin B_{f'}$ then $fg(x) = y$ and $h(y) = h(x)$ with $y \in A$. Thus range of $h = B$. Now by Lemma 2, $h \in R(X)$. Clearly h respects A_e since $h(y) = fg(y) = y$. It is also clear that if $h(x) \neq y$ then $|\{z : h(z) = h(x)\}| = 1$. Hence $h \in I_e$ and so I_e is a subsemigroup. We have already shown that I_e contains inverses. Note that if f is an idempotent in I_e then

$$f(x) = \begin{cases} x & \text{if } x \in A_f, \\ y & \text{otherwise.} \end{cases}$$

Two such idempotents commute and so I_e is an inverse subsemigroup of $S(X)$.

To show that I_e is maximal suppose $I_e \subseteq J$ where J is an inverse subsemigroup. By Corollary 8 we have that e is the smallest idempotent in J and if $f \in J$ then f respects A_e. Now suppose $f(w) \neq y$ and

$$|\{z : f(z) = f(w)\}| > 1.$$

We may assume $w \in A_f$. Then there exists $z \notin A_f$ such that $f(w) = f(z)$. Choose a clopen (closed and open) set G so that $z, y \in G$ and $w \notin G$. Define $g \in S(X)$ by

$$g(x) = \begin{cases} x & \text{if } x \in G, \\ y & \text{otherwise.} \end{cases}$$

It is easy to see that g is an idempotent in I_e, hence in J. Thus $gf^{-1}f = f^{-1}fg$. But $gf^{-1}f(z) = g(w) = y$ and $f^{-1}fg(z) = f^{-1}f(z) = w$ and $w \neq y$. This is a contradiction. Hence if $f(w) \neq y$ then $|\{z : f(z) = f(w)\}| = 1$ and so $f \in I_e$. Thus $J \subseteq I_e$ and I_e is maximal with smallest idempotent e.

If we let X be discrete then $S(X) = T_X$, the full transformation semigroup on the set X. We may then apply the last theorem to obtain the result of Nichols [**4**]. The next theorem is also concerned with 0-dimensional spaces. Recall that a space X is homogeneous if for every two points x and y there exists a homeomorphism h of X onto X such that $h(x) = y$.

THEOREM 10. *Let X be a homogeneous, 0-dimensional space and suppose e is an idempotent in $S(X)$ such that A_e is open. Let $I_e = \{f \in R(X) : f$ respects A_e, B_f is open, if $f(x) \notin A_e$ then $|\{y : f(y) = f(x)\}| = 1$ and for all $x, y \in X$, $e(x) = e(y)$ if and only if $ef(x) = ef(y)\}$. Then I_e is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent e.*

Proof. Note that if $f \in I_e$ respects A_e via f' and $x \notin B_{f'}$ then $f(x) \in A_e$. Thus $B_{f'} = (f^{-1}(X - A_e) \cup A_e)$ and so $B_{f'}$ is clopen. We first show that if $f \in I_e$ then there exists an inverse g of f which also belongs to I_e. Define g by

$$g(x) = \begin{cases} f'(x) & \text{if } x \in B_f, \\ f'e(x) & \text{otherwise.} \end{cases}$$

Since B_f is clopen we have that $g \in S(X)$. Clearly g is an inverse for f, B_g is open, g respects A_e and if $g(x) \notin A_e$ then $|\{y : g(y) = g(x)\}| = 1$. To show the last condition for membership in I_e we consider several cases:

(1) $x, y \in B_f : eg(x) = eg(y) \Leftrightarrow ef'(x) = ef'(y) \Leftrightarrow eff'(x) = eff'(y) \Leftrightarrow e(x) = e(y)$.

(2) $x \notin B_f, y \notin B_f : eg(x) = eg(y) \Leftrightarrow ef'e(x) = ef'e(y) \Leftrightarrow eff'e(x) = eff'e(y) \Leftrightarrow e(x) = e(y)$.

(3) $x \in B_f, y \notin B_f : eg(x) = eg(y) \Leftrightarrow ef'(x) = ef'e(y) \Leftrightarrow eff'(x) = eff'e(y) \Leftrightarrow e(x) = e(y)$.

Thus $g \in I_e$.

We now show I_e is a subsemigroup. Let $h = fg$ with $f, g \in I_e$ and inverses $f', g' \in I_e$. Let $h = fg$, let $A = g'(B_g \cap B_{f'})$ and $B = h(A)$. We show $B =$ range of h. Let $x \in X$. Then there exists y such that $g(x) = g(y)$ and $g'g(y) = y$. If $g(y) \in B_{f'}$ then $y \in A$, $g(x) = g(y)$ and hence $h(x) = h(y)$. If $g(y) \notin B_{f'}$ then $fg(x) \in A_e$ and so there exists $z \in A_e \subseteq A$ such that $h(x) = h(z)$. Now we use Lemma 2 to conclude that $h \in R(X)$. Clearly B_h is open and h respects

A_e. Now suppose $h(x)=h(y)$ where $h(x)\notin A_e$. Then $fg(x)=fg(y)$ with $fg(x)\notin A_e$. This means that $g(x)=g(y)$. Now $g(x)\notin A_e$ (otherwise $fg(x)\in A_e$) and so $x=y$. Thus if $h(x)\notin A_e$ then $|\{y:h(x)=h(y)\}|=1$. Finally, note that for any $x, y\in X$,

$$e(x)=e(y)\Leftrightarrow eg(x)=eg(y)\Leftrightarrow efg(x)=efg(y)\Leftrightarrow eh(x)=eh(y).$$

Thus $h\in I_e$.

To show that I_e is an inverse subsemigroup we need only show that idempotents in I_e commute. But note that if f is an idempotent in I_e then

$$f(x)=\begin{cases} x & \text{if } x\in A_f, \\ e(x) & \text{otherwise.}\end{cases}$$

Thus any two idempotents in I_e will commute and so I_e is an inverse subsemigroup.

For maximality suppose that $I_e\subseteq J$ where J is an inverse subsemigroup. We first show that $A_e\subseteq A_J$. If not, then there exists an idempotent $f\in J$ and $y\in X$ such that $y\in A_e-A_f$. But then $ef(y)=fe(y)\in A_f\cap A_e$ and so $fe(y)\neq y$. By the homogeneity of X choose a homeomorphism h from X onto X such that $h(y)=fe(y)$. Now choose clopen disjoint sets U, V of X so that $y\in U$, $fe(y)\in V$, $U\cup V\subseteq A_e$, $U\cap A_f=\varnothing$ and $h(U)=V$. Now define a homeomorphism k from X onto X by

$$k(x)=\begin{cases} h(x) & \text{if } x\in U, \\ h^{-1}(x) & \text{if } x\in V, \\ e(x) & \text{otherwise.}\end{cases}$$

Then $B_k=A_e$ and $ke=ek$. Thus $k\in I_e$, hence $k\in J$. Now $k^{-1}fek$ is an idempotent of J. So

$$(fe)(k^{-1}fek)=(k^{-1}fek)(fe).$$

But

$$(fe)(k^{-1}fek)(y)=(fe)(k^{-1}feh)(y)=(fe)(k^{-1}fefe)(y)=(fe)(h^{-1}fe)(y)=fe(y)$$

and $(k^{-1}fek)(fe)(y)\in k^{-1}f(U)$. Now $f(U)\cap U=\varnothing$ since $U\cap A_f=\varnothing$. Thus $k^{-1}f(U)\cap V=\varnothing$. But $fe(y)\in V$ and this is a contradiction. Thus $A_e\subseteq A_J$ and so, by Lemma 7, e is the smallest idempotent of J. Now by Lemma 5, if $g\in J$ then g respects A_e, $ge=eg$ and

$$e(x)=e(y)\Leftrightarrow eg(x)=eg(y).$$

Assume f is an idempotent in J. Suppose there exists $z\in A_f-A_e$ such that $f(z)=z=f(y)$ with $y\neq z$. Choose clopen U so that $y\in U, z\notin U$ and $U\cap A_e=\varnothing$ (note $y\notin A_e$). define $g\in S(X)$ by

$$g(x)=\begin{cases} e(x) & \text{if } x\in U, \\ x & \text{if } x\notin U.\end{cases}$$

Then g is an idempotent in I_e and so $fg = gf$. But $fg(y) = fe(y) = e(y) \in A_e$ and $gf(y) = g(z) = z$ with $z \notin A_e$. This is a contradiction. Thus if $f(z) \notin A_e$ then $|\{x : f(x) = f(z)\}| = 1$. This means that if $x \notin A_f$ then $f(x) \in A_e$. But then $X - A_f = f^{-1}(A_e) \cap (X - A_e)$ which is closed. Thus A_f (and hence B_f) is open. But then $f \in I_e$.

Now suppose $g \in J$. Then $B_g = A_{gg^{-1}}$ is open, g respects A_e and

$$e(x) = e(y) \Leftrightarrow eg(x) = eg(y).$$

If $g(x) \notin A_e$ and $g(x) = g(y)$ then $g^{-1}g(x) \notin A_e$ (g^{-1} respects A_e) and $g^{-1}g(x) = g^{-1}g(y)$. Thus $x = y (g^{-1}g \in I_e)$. But then $|\{y : g(x) = g(y)\}| = 1$. This shows $g \in I_e$. Thus $J \subseteq I_e$ and so I_e is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent e.

Corollary 11. *Let X be a homogeneous 0-dimensional space. Then $G(X)$, the group of units of $S(X)$, is a maximal inverse subsemigroup of $S(X)$.*

Proof. Let e be the identity map on X in the previous theorem.

To see that homogeneity is necessary in this corollary let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Then $G(X) \cup \{c_0\}$ is an inverse subsemigroup of $S(X)$. If X is discrete then we apply the last theorem to yield the result of Reilly [**5**]. We now consider other types of maximal inverse subsemigroups of $S(X)$. This will result in applications to $\mathbb{R}^n$ (Euclidean n-space) and I^n (Euclidean n-cell). We first make several definitions.

Definition. Suppose e is an idempotent in $S(X)$ and $\mathfrak{R}$ is a decomposition of $X - A_e$ ($\mathfrak{R}$ is an equivalence relation on $X - A_e$). We will call $\mathfrak{R}$ a *ray decomposition of $X - A_e$* if the following conditions are satisfied:

(1) for any $x \in X - A_e$, if $[x]$ denotes the $\mathfrak{R}$-equivalence class of x in $X - A_e$ then $\overline{[x]} = [x] \cup \{x_e\}$ where x_e is an element of A_e ($\overline{[x]}$ denotes the closure of the set $[x]$ in X),

(2) for any $x \in X - A_e$, $\overline{[x]}$ is homeomorphic to $[0, 1]$ or $[0, 1)$ via a homeomorphism h such that $h(x_e) = 0$.

When we write $[x]$ we shall understand that $x \in X - A_e$. If $a \in [x]$ we will use the notation $[x_e, a]$ to mean $h^{-1}[0, h(a)]$ and we will say $y > a$ ($y \geq a$) if $a, y \in \overline{[x]}$ and $h(y) > h(a)$ ($h(y) \geq h(a)$).

Definition. Suppose e is an idempotent in $S(X)$, $\mathfrak{R}$ is a ray decomposition of $X - A_e$ and for every $x \in X - A_e$, e is constant on $[x]$. A function $f \in R(X)$ is said to be *e-admissible* if the following are satisfied:

(1) there exists an inverse f' of f such that f respects A_e via f',

(2) for every $x \in X - A_e$, either f is constant on $[x]$ or $f[x] \subseteq [z]$ for some $z \in X - A_e$,

(3) for every $x \in X - A_e$, either $[x] \subseteq B_{f'}$ or there exists $x_f \in \overline{[x]}$ such that $[x_e, x_f] \subseteq B_{f'}$ (may have $x_f = x_e$) and f is constant on all $y \geq x_f$. As before, we will also say f is e-admissible *via* f'.

Note that if f is e-admissible via f', $[x] \subseteq B_{f'}$ and $f[x] \subseteq [z]$ then $f|_{\overline{[x]}}$ is a homeomorphism into $\overline{[z]}$ with $f(x_e) = z_e$; and if $[x_e, x_f] \subseteq B_{f'}$ then f is constant on all $y \geq x_f$.

Theorem 12. *Suppose X is a topological space, e is an idempotent in $S(X)$, $\mathfrak{R}$ is a ray* *decomposition of* $X-A_e$ *and the following conditions are satisfied:*

(1) *For every* $x\in X-A_e$, *e is constant on* $[x]$.

(2) *If* $a\in\overline{[x]}$ *then there exists an idempotent* $h\in R(X)$ *such that h is e-admissible,* $h|_{[x_e,a]}=\mathrm{id}|_{[x_e,a]}$ *and* $h(z)=a$ *for all* $z\geq a$. *If, in addition, there exists y such that* $[y]\neq[x]$ *then h can be chosen so that* $h|_{[y]}=\mathrm{id}|_{[y]}$.

(3) *If A is a retract of X and* $A\subsetneqq A_e$ *then there exists* $h\in R(X)$ *such that h respects* A_e *and* $h(A)\cap A=\varnothing$.

Now let $I_e=\{f\in R(X)$: *there exists an inverse* f' *of f such that f is e-admissible via* f' *and* f' *is e-admissible via* $f\}$. *Then* I_e *is a maximal inverse subsemigroup of* $S(X)$ *with smallest idempotent e.*

Proof. We first show I_e is a subsemigroup. Let $f, g\in I_e$ with inverses $f', g'\in I_e$, $h=fg$, $A=g'(B_{f'}\cap B_g)$ and $B=h(A)$. We show simultaneously that range of $h=B$ and that h satisfies conditions (2) and (3) of the definition of e-admissibility. We can then apply these results to inverses f' and g' of f and g and use Lemma 2 to conclude that $h\in R(X)$, $h'=g'f'$ is an inverse for h and both h and h' are e-admissible (clearly h and h' respect A_e). This will then show that I_e is a subsemigroup. So consider $x\in X$. If $x\in A_e$ then $x\in A$ and $h(x)\in B$. If $x\notin A_e$ and g is constant on $[x]$ then h is constant on $[x]$ and $h[x]\subseteq A_e\subseteq B$. Now suppose $g[x]\subseteq[y]$. If f is constant on $[y]$ then h is constant on $[x]$ and $h[x]\subseteq A_e\subseteq B$. Now suppose $f[y]\subseteq[z]$. Then $h[x]\subseteq[z]$. If $[x]\subseteq B_{g'}$ and $[y]\subseteq B_{f'}$ then $[x]\subseteq A$, $h[x]\subseteq B$ and h is a homeomorphism on $[x]$. If $[x]\subseteq B_{g'}$ and there exists y_f such that $[y_e, y_f]\subseteq B_{f'}$ with f constant on all $w\geq y_f$, then let $x_h=g'(y_f)$. Then $[x_e, x_h]\subseteq A$ and h is constant on all $w\geq x_h$. Thus $h[x]\subseteq B$. Now suppose there exists x_g such that $[x_e, x_g]\subseteq B_{g'}$ and g is constant on all $w\geq x_g$. If $[y]\subseteq B_{f'}$ or if there exists $y_f\geq g(x_g)$ such that $[y_e, y_f]\subseteq B_{f'}$ then $[x_e, x_g]\subseteq A$, h is constant on all $w\geq x_g$, and $h[x]\subseteq B$. If there exists $y_f<g(x_g)$ such that $[y_e, y_f]\subseteq B_{f'}$ and f is constant on all $w\geq y_f$ then let $x_h=g'(y_f)$. Then $[x_e, x_h]\subseteq A$, and h is constant on all $w\geq x_h$, and again $h[x]\subseteq B$. This completes the proof that I_e is a subsemigroup.

To show that I_e is an inverse subsemigroup we need only prove that idempotents commute. Let f, g be idempotents in I_e and suppose $x\in X$. If $x\in A_e$ then $f(x)=x=g(x)$ and so $fg(x)=gf(x)$. If $x\in X-A_e$ then either $f|_{[x_e,x]}=\mathrm{id}|_{[x_e,x]}$ or $f(x)=x_f$ with $x_f<x$. If $f|_{[x_e,x]}=\mathrm{id}|_{[x_e,x]}$ then since $g(x)\in[x]$ with $g(x)\leq x$ we have $gf(x)=g(x)=fg(x)$. If $f(x)=x_f$ with $x_f<x$ and $g(x)=x$ then $gf(x)=g(x_f)=x_f=f(x)=fg(x)$. If $g(x)=x_g$ with $x_g<x$ and $x_g\geq x_f$ then $gf(x)=g(x_f)=x_f=f(x_g)=fg(x)$. If $g(x)=x_g$ with $x_g<x_f$ then $gf(x)=g(x_f)=x_g=f(x_g)=fg(x)$. In any case, $gf(x)=fg(x)$ and so I_e is an inverse subsemigroup of $S(X)$.

To show I_e is a maximal inverse subsemigroup suppose that $I_e\subseteq J$ where J is an inverse subsemigroup. Note first that we can use condition (3) of the theorem, Lemma 3, and Corollary 8 to conclude that e is the smallest idempotent in J and if $g\in J$ then g respects A_e. We now show that if f is an idempotent in J then f is an idempotent in I_e. We already have that f respects A_e and so let $x\in X-A_e$ and suppose $[x]\nsubseteq A_f$. We will show $f[x]\subseteq\overline{[x]}$ and condition (3) of e-admissibility is satisfied. Choose

$$a=\max\{z : z\in\overline{[x]},\ f(z)=z\}$$

(we may have $a=x_e$). Consider $y>a$. By condition (2) of the hypothesis choose g an idempotent such that g is e-admissible, $g|_{[x_e,a]}=\mathrm{id}|_{[x_e,a]}$ and $g(z)=a$ for $z>a$. If $f(y)\notin\overline{[x]}$ then $f(y)\notin A_e$ (otherwise $f(y)=e(y)=x_e$ by Lemma 4) and so we can also choose g so that $g|_{[f(y)]}=\mathrm{id}|_{[f(y)]}$. Then g is in I_e, hence in J and so $fg=gf$. If $f(y)\notin\overline{[x]}$ then $gf(y)=f(y)\notin\overline{[x]}$ but $fg(y)=f(a)=a\in\overline{[x]}$, which is a contradiction. Hence $f(y)\in\overline{[x]}$. Note that this means that $f(y)\leq a(f[x]\subseteq\overline{[x]}$ and so $A_f\cap\overline{[x]}$ must be an interval). Now $a=f(a)=fg(y)=gf(y)$. Thus $f(y)\geq a$. Hence $f(y)=a$ and this shows that $f\in I_e$.

Now let $g\in J$. We know that g respects A_e. Let $x\in X-A_e$. Note that if $g(x)\in A_e$ then since $eg=ge$ by Lemma 5 we have $g(x)=eg(x)=ge(x)=g(x_e)$. Consider $[x]$. If $g^{-1}g$ is constant on $[x]$ and $y\in[x]$ then $g^{-1}g(y)=g^{-1}g(x_e)=x_e$. But then $g(y)=g(x_e)$ and so g is constant on $[x]$. Now suppose there exists $a>x_e$ such that $[x_e,a]\subseteq A_{g^{-1}g}$ and let $x_e<y\leq a$. Then $g(y)\notin A_e$ $(y\in A_g-A_e)$. If $g(y)\notin[g(a)]$ then choose an idempotent $f\in I_e$ so that f is the identity on $[g(a)]$ and constant on $[g(y)]$. Then $g^{-1}fg$ is an idempotent in J, hence in I_e. Now $g^{-1}fg(a)=a$ and so $g^{-1}fg(y)=y$ also $(y\leq a)$. But $g^{-1}fg(y)\in A_e$. This is a contradiction. Thus $g(y)\in[g(a)]$ for all y with $x_e<y\leq a$. Thus if $[x]\subseteq A_{g^{-1}g}$ then $[x]\subseteq A_g$ and $g[x]\subseteq[g(x)]$. Now suppose $g^{-1}g$ is such that there exists $a\in[x]$ such that $g^{-1}g$ is the identity on $[x_e,a]$ and constant thereafter. Then $[x_e,a]\subseteq A_g$ and $g[x_e,a]\subseteq[g(a)]$ by the above. Now let $y>a$. Then $g^{-1}g(y)=g^{-1}g(a)=a$ and hence $g(y)=gg^{-1}g(y)=g(a)$. Thus $g\in I_e$, $J\subseteq I_e$ and so I_e is a maximal inverse subsemigroup.

We have several corollaries.

Corollary 13. *Let $X=I$ (the unit interval) or $\mathbb{R}$ (the reals) and let e be defined by*

$$e(x)=\begin{cases} x & \text{if } a\leq x\leq b,\\ a & \text{if } x\leq a,\\ b & \text{if } x\geq b,\end{cases}$$

where $0\leq a\leq b\leq 1$ if $X=I$ and $a\leq b$ if $X=\mathbb{R}$. Then e is an idempotent and if $I_e=\{f\in R(X)$: there exists an inverse f' of f such that f respects A_e via f', if $B_{f'}=[c,d]$ then $f(x)=f(c)$ for all $x\leq c$ and $f(x)=f(d)$ for all $x\geq d\}$ we have that I_e is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent e.

Corollary 14. *Let $X=\mathbb{R}^n$ or I^n and let D be an n-dimensional disk in $\mathbb{R}^n$ (or I^n) with centre y. Define an idempotent e as follows: if $x\in D$, $e(x)=x$; if $x\in\mathbb{R}^n-D$, $e(x)=x_b$, where x_b is the unique element on the boundary of D which intersects the line segment from y to x.*

If x, $z\in X-A_e$ then we say x is $\mathfrak{R}$-equivalent to z if x and z lie on the same line segment beginning at y. Then this gives a ray decomposition of $X-A_e$ and if $I_e=\{f\in R(X)$: there exists an inverse f' of f such that f is e-admissible via f' and f' is e-admissible via $f\}$ then I_e is a maximal inverse subsemigroup of $S(X)$.

Proof. It is straightforward to see that conditions (1) and (2) of the theorem are satisfied. To see condition (3) note that if A is a retract of X and $A\subsetneq D$ then there exists a point x in the boundary of D but not in A. Since A is closed there exists an open neighborhood U of x such that U is homeomorphic to $\mathbb{R}^n$ and $\bar{U}\cap A=\varnothing$. Now there eixsts $h\in R(X)$ with inverse h' such that $B_{h'}=B_h=D$ and $h(A)\subseteq\bar{U}\cap D$.

Corollary 15. *Let $X=\mathbb{R}^n$ or I^n, $e=c_y$ for fixed $y\in X$ and let the ray decomposition of $X-\{y\}$ be defined by $z\in[x]$ if and only if z, x and y all lie on a line segment beginning at y. Then $I_e=\{f\in R(X)$: there exists an inverse f' of f such that f is e-admissible via f' and f' is e-admissible via $f\}$ is a maximal inverse subsemigroup of $S(X)$ with smallest idempotent c_y.*

Note that for the above corollary we could have chosen a different ray decomposition of $X-\{y\}$ and this would have resulted in a different maximal inverse subsemigroup, still with the same smallest idempotent c_y.

Corollary 16. *Let $X=I^n$. Then $G(X)$, the group of units of $S(I^n)$, is a maximal inverse subsemigroup of $S(X)$.*

Proof. Let e be the identity on X in Theorem 12.

Corollaries 11 and 16 give situations where $G(X)$, the group of units of $S(X)$, forms a maximal inverse subsemigroup. This is not always the case. For instance, if X is a triod then every homeomorphism of X will fix the same point y and so $G(X)\cup\{c_y\}$ is an inverse subsemigroup which properly contains $G(X)$. However, we do have the following result (also proved by Reilly [**6**]):

Proposition 17. *Suppose X is a homogeneous, compact space. Then $G(X)$, the group of units of $S(X)$, is a maximal inverse subsemigroup of $S(X)$.*

Proof. Clearly $G(X)$ is an inverse subsemigroup. Suppose $G(X)\subseteq J$ where J is an inverse subsemigroup. Then $A_J\neq\emptyset$ since X is compact. Suppose $A_J\neq X$. Then by the homogeneity of X choose $f\in G(X)$ and $x\in X$ so that $x\in A_J$ and $f(x)\notin A_J$. Then $f\in J$ but $f(A_J)\nsubseteq A_J$. This contradicts Lemma 7. Thus $A_J=X$ and so $J=G(X)$ and $G(X)$ is maximal.

Corollary 18. *Let $X=S^n$ (the n-dimensional sphere). Then $G(X)$ is a maximal inverse subsemigroup of $S(X)$.*

We now consider one last type of maximal inverse subsemigroup of $S(I)$.

Theorem 19. *Let e be an idempotent in $S(I)$ such that if $A_e=[a,b]$ (where possibly $a=0$ or $b=1$) then e is a homeomorphism on $[0,a]$ and e is a homeomorphism on $[b,1]$. Define $I_e=\{f\in R(I)$: there exists an inverse f' of f such that $B_f=[0,b]$, $[0,1]$, $[a,b]$ or $[a,1]$, $B_{f'}$ is also one of these sets, f respects A_e via f', and $e(x)=e(y)$ if and only if $ef(x)=ef(y)\}$. Then I_e is a maximal inverse subsemigroup of $S(I)$ with smallest idempotent e.*

Proof. Suppose $f\in I_e$ with inverse f'. We define an inverse g for f by

$$g(x)=\begin{cases} f'(x) & \text{if} \quad x\in B_f,\\ f'e(x) & \text{if} \quad x\notin B_f.\end{cases}$$

It is straightforward to check that g is continuous. Clearly g is an inverse for f, g respects A_e and satisfies the conditions on B_f and B_g. The proof for the last condition follows the corresponding proof in Theorem 10. Now suppose $f, g \in I_e$ with inverses $f', g' \in I_e$ and let $h = fg$. Then $h \in R(X)$, h respects A_e and B_h, $B_{g'f'}$ are of the desired form. Now

$$e(x) = e(y) \Leftrightarrow eg(x) = eg(y) \Leftrightarrow efg(x) = efg(y) \Leftrightarrow eh(x) = eh(y).$$

So $h \in I_e$. We now show idempotents commute. Suppose f is an idempotent in I_e, $f \neq e$ and f is not the identity on I. Without loss of generality assume $[0, a) \cap A_f = \varnothing$. Then f is one-to-one on $[0, a]$ (if $f(x) = f(y)$ then $ef(x) = ef(y)$ and hence $e(x) = e(y)$, but e is one-to-one on $[0, a]$). Furthermore, if $x \in [0, a]$ then $f(x) = e(x)$ (if $f(x) \in A_e$ then $f(x) = f(y)$ for some $y \in A_e$, hence $e(x) = e(y) = f(y) = f(x)$; if $f(x) = e(x) = b$ then $x = 0$). This means that if f is an idempotent in I_e then

$$f(x) = \begin{cases} x & \text{if } x \in A_f, \\ e(x) & \text{if } x \notin A_f. \end{cases}$$

Clearly two such idempotents commute. Thus I_e is an inverse subsemigroup of $S(I)$.

To show that I_e is maximal suppose $I_e \subseteq J$ where J is an inverse subsemigroup and $g \in J$. It is straightforward to show that A_e and J satisfy the conditions of Corollary 8 and hence e is the smallest idempotent for J. Now apply Lemma 5 to conclude that g respects A_e and $e(x) = e(y)$ if and only if $eg(x) = eg(y)$. To show the remaining conditions we may assume, without loss of generality, that g is an idempotent and $A_g = [c, d]$ with $0 < c < a$. But then $g(x) = g(y)$ for some $x, y \in [0, a]$ where $x \neq y$. Thus $eg(x) = eg(y)$ and hence $e(x) = e(y)$, which is a contradiction. Thus $g \in I_e$ and so I_e is a maximal inverse subsemigroup of $S(I)$ with smallest idempotent e.

Note that it is possible to make slight modifications and prove a similar theorem if X is the reals.

As an example of this last theorem let $X = [-1, 1]$ and suppose $e(x) = |x|$. Then $I_e = \{f \in S(X) : f$ maps $[0, 1]$ homeomorphically onto $[0, 1]$ and either f is an odd function ($f(-x) = -f(x)$ for all x) or f is an even function ($f(x) = f(-x)$ for all x)$\}$ is a maximal inverse subsemigroup of $S(X)$. Or, let X be the reals and again let $e(x) = |x|$. Then $I_e = \{f \in S(X) : f$ is a homeomorphism from $[0, \infty)$ onto $[0, \infty)$ and f is either an odd or even function$\}$ is a maximal inverse subsemigroup of $S(X)$.

All of the maximal inverse subsemigroups we have considered thus far have contained a smallest idempotent e. As Reilly **[5]** remarks, this is not always the case for $S(X)$, where X is discrete. Since every inverse subsemigroup is contained in a maximal inverse subsemigroup, to produce examples of inverse subsemigroups with no smallest idempotent one needs to find subsemigroups J of $S(X)$ of commuting idempotents such that $A_J = \varnothing$. For instance, if X is the reals, define f_n for $n = 1, 2, \ldots$ as follows:

$$f_n(x) = \begin{cases} n & \text{if } x \leq n, \\ x & \text{if } x > n. \end{cases}$$

Then $J = \{f_n : n = 1, 2, \ldots\}$ is a subsemigroup of commuting idempotents but $\bigcap_{n=1}^{\infty} A_{f_n} = \varnothing$ and so $A_J = \varnothing$.

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