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RIEMANN–LEBESGUE PROPERTIES OF BANACH SPACES ASSOCIATED WITH SUBSETS OF COUNTABLE DISCRETE ABELIAN GROUPS

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Abstract. For a subset Λ of the dual group of a compact metrizable abelian group, we introduce the type I-, II-, and III-Λ-Riemann–Lebesgue property of a Banach space. As an application we use these properties to characterize Rajchman sets.

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1. Introduction. Generalizations of the Radon–Nikodym property and the analytic Radon–Nikodym property of Banach spaces have been extensively studied over the past twenty years. Edgar [**7**] and later Dowling [**6**], introduced and studied Radon–Nikodym properties associated with subsets of countable discrete abelian groups (types I, II and III-Λ-RNP). Robdera and Saab [**11**] introduced and studied the concept of the analytic complete continuity property and later introduced complete continuity properties associated with subsets of countable discrete abelian groups (types I, II and III-Λ-CCP) [**12**]. Of particular interest is that $L^1[0, 1]$ has type I, II or III-Λ-RNP (or CCP) if and only if Λ is a Riesz set.

In a recent paper, Bu and Chill [**2**] introduced the notions of the Riemann–Lebesgue property and the analytic Riemann–Lebesgue property, which are weakenings of the complete continuity property and the analytic complete continuity property, respectively. In this note, we shall define and study Riemann–Lebesgue properties of Banach spaces associated with subsets of countable discrete abelian groups. In particular, we shall give conditions under which $L^1[0, 1]$ has a Riemann–Lebesgue property.

2. Preliminaries and definitions. Throughout this paper, G will denote a compact metrizable abelian group, $\mathcal{B}(G)$ is the σ-algebra of Borel subsets of G, and λ is normalized Haar measure on G. The dual group of G will be denoted by Γ. We note that Γ is a countable discrete abelian group [**13**].

Let X be a complex Banach space and let $1 \leq p \leq \infty$. For an X-valued measure, μ on $\mathcal{B}(G)$ we define

$$\mathbb{E}(\mu|\pi) = \sum_{E\in\pi} \frac{\mu(E)}{\lambda(E)}\chi_E,$$

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where π is a finite measurable partition of G, along with the convention $\frac{0}{0}=0$. The space $V^p(G;X)$ consists of all X-valued measures μ on $\mathcal{B}(G)$ with $\|\mu\|_p<\infty$, where

$$\|\mu\|_p=\sup_{\pi}\|\mathbb{E}(\mu|\pi)\|_{L^p(G;X)}$$

and the supremum is taken over all finite measurable partitions of G.

If $\mu\in V^P(G;X)$ and $\gamma\in\Gamma$, then the Fourier coefficient $\hat{\mu}(\gamma)$ is defined by

$$\hat{\mu}(\gamma)=\int_G\bar{\gamma}(x)\,d\mu(x).$$

If $f\in L^p(G;X)$ and $\gamma\in\Gamma$, then $\hat{f}(\gamma)$ is defined by

$$\hat{f}(\gamma)=\int_G f(x)\bar{\gamma}(x)\,d\lambda(x).$$

If $\Lambda\subseteq\Gamma$, we define

$$L^p_\Lambda(G;X)=\{f\in L^p(G;X):\hat{f}(\gamma)=0\text{ for all }\gamma\notin\Lambda\}$$

and

$$V^p_\Lambda(G;X)=\{\mu\in V^p(G;X):\hat{\mu}(\gamma)=0\text{ for all }\gamma\notin\Lambda\}.$$

We also need the following notation:

$$V^1_{\Lambda,ac}(G;X)=\{\mu\in V^1(G;X):\mu\text{ is }\lambda-\text{continuous and }\hat{\mu}(\gamma)=0\text{ for all }\gamma\notin\Lambda\}.$$

For more information on vector measures, we refer the reader to the monograph of Diestel and Uhl [**3**]. We are now ready to define Riemann–Lebesgue properties associated to a subset Λ of Γ. To motivate these definitions recall that a Banach space X has type-I-Λ-Radon–Nikodym property (I-Λ-RNP), (resp. type II-Λ-Radon–Nikodym property (II-Λ-RNP)) if every $\mu\in V^\infty_\Lambda(G;X)$ (resp. every $\mu\in V^1_{\Lambda,ac}(G;X)$) is differentiable [**5**], and X has the type-I-Λ-complete continuity property (I-Λ-CCP), (resp. type-II-Λ-complete continuity property (II-Λ-CCP)) if every $\mu\in V^\infty_\Lambda(G;X)$ (resp. every $\mu\in V^1_{\Lambda,ac}(G;X)$) has relatively compact range [**12**].

DEFINITION 1. Let G be a compact abelian metrizable group and let Λ be a subset of Γ. A Banach space X is said to have the *type*-I-Λ-*Riemann–Lebesgue property* (I-Λ-RLP) if every $\mu\in V^\infty_\Lambda(G;X)$ satisfies $\lim_{\substack{\gamma\to\infty\\ \gamma\in\Lambda}}\|\hat{\mu}(\gamma)\|=0$.

REMARK. It is easily seen that I-Λ-RNP implies I-Λ-CCP and I-Λ-CCP implies I-Λ-RLP.

DEFINITION 2. Let G be a compact abelian metrizable group and let Λ be a subset of Γ. A Banach space X is said to have the *type*-II-Λ-*Riemann–Lebesgue property* (II-Λ-RLP) if every $\mu\in V^1_{\Lambda,ac}(G;X)$ satisfies $\lim_{\substack{\gamma\to\infty\\ \gamma\in\Lambda}}\|\hat{\mu}(\gamma)\|=0$.

REMARK. It is obvious that II-Λ-RLP implies I-Λ-RLP, since every element of $V^\infty_\Lambda(G;X)$ is an element of $V^1_{\Lambda,ac}(G;X)$. Also II-Λ-RNP implies II-Λ-CCP and II-Λ-CCP implies II-Λ-RLP. Furthermore, by the method of proof of [**12**, Theorem 4.3] we can see that I-Γ-RLP is equivalent to II-Γ-RLP.

REMARK. If $G=\mathbb{T}$, the circle group, then $\Gamma=\mathbb{Z}$. In [**2**], Bu and Chill defined the notion of a Banach space having the Riemann–Lebesgue property. It can be easily seen that their concept of Riemann–Lebesgue property is equivalent to both I-$\mathbb{Z}$-RLP and II-$\mathbb{Z}$-RLP. The concept of the analytic Riemann–Lebesgue property introduced by Bu and Chill is equivalent to both I-$(\mathbb{N}\cup\{0\})$-RLP and II-$(\mathbb{N}\cup\{0\})$-RLP.

REMARK. It is clear that if a Banach space has I-Λ-RLP (resp. II-Λ-RLP), then so does every subspace of X. Moreover, since G is compact and metrizable, $\mathcal{B}(G)$ is countably generated and hence every element of $V^1_{\Lambda,ac}(G, X)$ has separable range. Consequently, X has I-Λ-RLP (resp. II-Λ-RLP) if every separable subspace of X has I-Λ-RLP (resp. II-Λ-RLP).

To motivate the third type of Riemann–Lebesgue property we recall that a Banach space X has type-III-Λ-Radon–Nikodym property (III-Λ-RNP), (resp. type III-Λ-complete continuity property (III-Λ-CCP)) if every absolutely summing operator $T: C(G)\to X$ with $T\equiv 0$ on $C_{\Lambda'}(G)$ is nuclear (resp. compact).

DEFINITION 3. Let G be a compact abelian metrizable group and let Λ be a subset of Γ. A Banach space X is said to have *type*-III-Λ-*Riemann–Lebesgue property* (III-Λ-RLP) if every absolutely summing operator $T: C(G)\to X$ with $T\equiv 0$ on $C_{\Lambda'}(G)$ has the property that $\{T(\bar{\gamma}) : \gamma\in\Lambda\}$ is relatively compact in X.

REMARK. It is clear that III-Λ-RNP implies III-Λ-CCP and III-Λ-CCP implies III-Λ-RLP. It is not so obvious that III-Λ-RLP implies II-Λ-RLP. This is the result we shall now prove.

PROPOSITION 4. *Let G be a compact abelian metrizable group and let $\Lambda\subseteq\Gamma$. If a Banach space X has III-Λ-RLP, then X has II-Λ-RLP.*

Proof. Suppose that X has III-Λ-RLP and let $\mu\in V^1_{\Lambda,ac}(G;X)$. Define an operator $T: C(G)\to X$ by

$$T(f)=\int_G f\,d\mu \qquad \text{for all } f\in C(G).$$

Since μ is the representing measure for T and μ is of bounded variation, T is an absolutely summing operator. Also, if $\gamma\in\Lambda'$, then $\bar{\gamma}\notin\Lambda$ and so $T(\gamma)=\int_G \gamma\, d\mu=\hat{\mu}(\bar{\gamma})=0$. Hence $T\equiv 0$. on $C_{\Lambda'}(G)$. Therefore, since X has III-Λ-RLP, $\{T(\bar{\gamma}) : \gamma\in\Lambda\}$ is relatively norm compact in X. This means that $\{\hat{\mu}(\gamma) : \gamma\in\Lambda\}$ is relatively norm compact in X. For each $x^*\in X^*$, $x^*\mu$ is a scalar measure of bounded variation that is absolutely continuous with respect to λ. Hence $\lim_{\substack{\gamma\to\infty\\ \gamma\in\Lambda}} x^*(\hat{\mu}(\gamma))=\lim_{\substack{\gamma\to\infty\\ \gamma\in\Lambda}} \widehat{(x^*\mu)}(\gamma)=0$ for all $x^*\in X^*$. Therefore $(\hat{\mu}(\gamma))_{\gamma\in\Lambda}$ is a weakly null sequence in X and thus, since $\{\hat{\mu}(\gamma) : \gamma\in\Lambda\}$ is relatively compact, $(\hat{\mu}(\gamma))_{\gamma\in\Lambda}$ is norm null. This proves that X has II-Λ-RLP. □

3. The results. We begin this section with a characterization of type I-Λ-RLP. For this result we need to recall the concept of a good approximate identity.

A sequence $(i_n)_{n\in\mathbb{N}}$ of measurable functions $i_n: G\to\mathbb{R}$ is called a *good approximate identity* on G if

1. $i_n\geq 0$ for all $n\in\mathbb{N}$,
2. $\int_G i_n(x)d\lambda(x)=1$ for all $n\in\mathbb{N}$,

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3. $\sum_{\gamma\in\Gamma} \hat{i}_n(\gamma) < \infty$ for all $n \in \mathbb{N}$,
4. $\lim_{n\to\infty} \int_U i_n(x)\, d\lambda(x) = 1$ for all neighborhoods U of 1 in G.

It is well known that a good approximate identity always exists on G if G is a compact abelian metrizable group.

Our first result is analogous to the Theorem in [**7**] and Theorem 3.3 in [**12**]. We shall omit the easy proof.

THEOREM 5. *Let G be a compact abelian metrizable group, let $\Lambda \subseteq \Gamma$ and let $(i_n)_{n\in\mathbb{N}}$ be a good approximate identity on G. For a Banach space X the following are equivalent.*

1. *X has I-Λ-RLP.*
2. *For each bounded linear operator $S : L^1(G)/L^1_{\Lambda'}(G) \to X$, the sequence $(SQ(\bar{\gamma}))_{\gamma\in\Lambda}$ is norm null in X, where $Q : L^1(G) \to L^1(G)/L^1_{\Lambda'}(G)$ is the natural quotient mapping.*
3. *If $(a_\gamma)_{\gamma\in\Lambda}$ is a sequence in X such that $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty_\Lambda(G; X)$, where $f_n = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma) a_\gamma \gamma$, then $(a_\gamma)_{\gamma\in\Lambda}$ is a norm null sequence in X.*

COROLLARY 6. *Let G be a compact abelian metrizable group and let Λ be an infinite subset of Γ. Then c_0 fails to have I-Λ-RLP.*

Proof. Since Λ is countably infinite, c_0 and $c_0(\Lambda)$ are isometric. Let $(e_\gamma)_{\gamma\in\Lambda}$ be the canonical unit vector basis of $c_0(\Lambda)$. For each $n \in \mathbb{N}$, define $f_n = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma) e_\gamma \gamma$. It is easy to see that $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty_\Lambda(G; c_0(\Lambda))$, but $(e_\gamma)_{\gamma\in\Lambda}$ is not a norm null sequence in $c_0(\Lambda)$. Thus, by Theorem 5, $c_0(\Lambda)$ (and hence c_0) fails to have I-Λ-RLP. □

COROLLARY 7. *Let G be a compact abelian metrizable group and let Λ be an infinite subset of Γ. Then $L^1(G)/L^1_{\Lambda'}(G)$ fails to have I-Λ-RLP.*

Proof. Let $Q : L^1(G) \to L^1(G)/L^1_{\Lambda'}(G)$ be the natural quotient mapping and note that $(Q(\bar{\gamma}))_{\gamma\in\Lambda}$ is a sequence of elements of $L^1(G)/L^1_{\Lambda'}(G)$, each with norm at least 1. Hence, by Theorem 5, $L^1(G)/L^1_{\Lambda'}(G)$ fails to have I-Λ-RLP. □

Before we get to our next result, recall that a subset Λ of Γ is called a *Sidon set* if $C_\Lambda(G) = \ell^1(\Lambda)$.

COROLLARY 8. *Let G be a compact abelian metrizable group, let Λ be an infinite Sidon subset of Γ and let X be a Banach space. Then X has I-Λ-RLP if and only if X does not contain a subspace isomorphic to c_0.*

Proof. Since Λ is infinite, every Banach space containing c_0 will fail to have I-Λ-RLP, by Corollary 6. On the other hand, since Λ is a Sidon set, every Banach space not containing a subspace isomorphic to c_0 has the I-Λ-RNP, by [**6**], and hence has I-Λ-RLP. □

DEFINITION 9. [**9**] Let G be a compact abelian metrizable group and let Λ be a subset of Γ. Then Λ is called a *Rajchman set* if each $\nu \in V^1_\Lambda(G)$ satisfies $\lim_{\substack{\gamma\to\infty\\ \gamma\in\Lambda}} \hat{\nu}(\gamma) = 0$.

THEOREM 10. *Let Λ be a subset of Γ. Then Λ is a Rajchman set if and only if $L^1\,[0, 1]$ has I-Λ-RLP.*

Proof. Suppose that $L^1[0, 1]$ has type I-Λ-RLP. Since G is compact and metrizable, $L^1(G)$ is isomorphic to a subspace of $L^1[0, 1]$ and hence $L^1(G)$ also has type I-Λ-RLP.

Let $\mu \in V^1_\Lambda(G)$. Define $T : L^1(G) \to L^1(G)$ by

$$Th(y) = \overline{(\bar{h} * \mu)(y)} = \overline{\int_G \overline{h(yx^{-1})}\, d\mu(x)}, \quad \text{for all } h \in L^1(G).$$

The operator T is clearly linear and is bounded because $\|Th\|_1 \le \|h\|_1 \|\mu\|_1$, for all $h \in L^1(G)$. Also note that for each $\gamma \in \Gamma$, $T(\bar{\gamma}) = \overline{\hat{\mu}(\gamma)\gamma}$. Therefore, since $\mu \in V^1_\Lambda(G)$, $T(\bar{\gamma}) = 0$, for all $\gamma \notin \Lambda$. Hence $T|_{L^1_{\Lambda'}}(G) = 0$, and so there exists a bounded linear mapping $S : L^1(G)/L^1_{\Lambda'}(G) \to L^1(G)$ such that $T = SQ$, where $Q : L^1(G) \to L^1(G)/L^1_{\Lambda'}(G)$ is the natural quotient mapping. Hence, by Theorem 5, $(T(\bar{\gamma}))_{\gamma\in\Lambda}$ is norm null in $L^1(G)$; that is, $\overline{(\hat{\mu}(\gamma)\gamma)}_{\gamma\in\Lambda}$ is norm null in $L^1(G)$. This means that $(\hat{\mu}(\gamma))_{\gamma\in\Lambda}$ is a null sequence in $\mathbb{C}$. Hence Λ is a Rajchman set.

Conversely, suppose that Λ is a Rajchman set. Let $S : L^1(G)/L^1_{\Lambda'}(G) \to L^1[0, 1]$ be a bounded linear operator and define $T : L^1(G) \to L^1[0, 1]$ by $T = SQ$. By the Fakhoury–Kalton representation theorem [**8**, **10**], there is a family $\{\mu_\omega\}_{\omega\in[0,1]}$ of complex measures μ_ω on $\mathcal{B}(G)$ such that, for each $f \in L^1(G)$, we have

$$T(f)(\omega) = \int_G f(x)\, d\mu_\omega(x),$$

for m-almost all $\omega \in [0, 1]$, where m is Lebesgue measure on $[0, 1]$. The Fakhoury–Kalton representation theorem also says that $\int_{[0,1]} \|\mu_\omega\|_1 dm(\omega) \le \|T\|$ and so μ_ω is of finite variation, for m-almost all $\omega \in [0, 1]$.

If $\gamma \notin \Lambda$, then $\bar{\gamma} \in \Lambda'$ and so $T(\bar{\gamma}) = SQ(\bar{\gamma}) = 0$. Hence for m-almost all $\omega \in [0, 1]$, we have

$$0 = T(\bar{\gamma})(\omega) = \int_G \bar{\gamma}(x)\, d\mu_\omega(x) = \widehat{\mu_\omega}(\gamma).$$

Since Λ is countable we can conclude that, for m-almost all $\omega \in [0, 1]$, we have $\widehat{\mu_\omega}(\gamma) = 0$ for all $\gamma \notin \Lambda$. Hence, for m-almost all $\omega \in [0, 1]$, $\mu_\omega \in V^1_\Lambda(G)$. Therefore, since Λ is a Rajchman set, $(\widehat{\mu_\omega}(\gamma))_{\gamma\in\Lambda}$ is a null sequence in $\mathbb{C}$, for m-almost all $\omega \in [0, 1]$. Consequently, we have by Lebesgue's dominated convergence theorem

$$\begin{aligned}
\lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \|T(\bar{\gamma})\|_1 &= \lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \int_{[0,1]} |T(\bar{\gamma})(\omega)|\, dm(\omega) \\
&= \lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \int_{[0,1]} \left| \int_G \bar{\gamma}(x)\, d\mu_\omega(x) \right| dm(\omega) \\
&= \lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \int_{[0,1]} |\widehat{\mu_\omega}(\gamma)|\, dm(\omega) \\
&= 0,
\end{aligned}$$

because the sequence of functions $\{\widehat{\mu_{(\cdot)}}(\gamma)\}_{\gamma\in\Lambda}$ converges m-almost everywhere to 0 on $[0, 1]$, $|\widehat{\mu_{(\cdot)}}(\gamma)| \le \|\widehat{\mu_{(\cdot)}}\|_1$, for all $\gamma \in \Lambda$, and $\|\widehat{\mu_{(\cdot)}}\|_1$ is a m-integrable function. Thus, by Theorem 5, $L^1[0, 1]$ has type I-Λ-RLP. □

Remark. Theorem 10 should be compared with the Proposition in [**7**], where it is proven that Λ is a Risez set if and only if $L^1[0, 1]$ has I-Λ-RNP (Λ of Γ is called a *Riesz set* if $V^1_\Lambda(G) = L^1_\Lambda(G)$). It is clear from the definitions of Rajchman and Riesz

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sets, that Riesz sets are Rajchman sets. It is unknown whether Rajchman sets are, in general Riesz sets. We mention also that a characterization of the Λ's for which $L^1[0,1]$ has I-Λ-CCP is unknown.

We shall now give a characterization of II-Λ-RLP. This characterization is analogous to Theorem 6 of [**5**] and Theorem 3.4 of [**12**]. We omit the easy proof.

THEOREM 11. *Let G be a compact abelian metrizable group, let Λ be a Riesz subset of Γ, and let $(i_n)_{n\in\mathbb{N}}$ be a good approximate identity on G. For a Banach space X the following are equivalent.*

1. *X has II-Λ-RLP.*
2. *If $(a_\gamma)_{\gamma\in\Lambda}$ is a sequence in X such that $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1_\Lambda(G;X)$, where $f_n = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma)a_\gamma\gamma$, then $(a_\gamma)_{\gamma\in\Lambda}$ is a norm null sequence in X.*

REMARK. One particularly interesting result proved in [**2**, Proposition 3.7] is that Banach spaces that are B-convex have the Riemann–Lebesgue property. The proof of [**2**, Proposition 3.7] uses the fact that the Hausdorff–Young inequality holds in B-convex Banach spaces (see [**1**] and [**4**, p. 281]). Using Theorems 5 and 11, we can easily prove that B-convex Banach spaces have II-Γ-RLP and therefore have II-Λ-RLP and I-Λ-RLP, for all $\Lambda \subseteq \Gamma$.

When Λ is a Riesz set, II-Λ-RNP and III-Λ-RNP are equivalent [**5**, Theorem 11], and II-Λ-CCP and III-Λ-CCP are equivalent [**12**, Proposition 3.8]. We now prove a corresponding result for Riemann–Lebesgue properties.

PROPOSITION 12. *Let G be a compact abelian metrizable group, let Λ be a Riesz subset of Γ. Then a Banach space X has II-Λ-RLP if and only if X has III-Λ-RLP.*

Proof. If X has III-Λ-RLP, then X has II-Λ-RLP, by Proposition 4.

Conversely, suppose that X has II-Λ-RLP. Let $T: C(G)\to X$ be an absolutely summing operator with $T\equiv 0$ on $C_{\Lambda'}(G)$. Let $F:\mathcal{B}(G)\to X^{**}$ be the representing measure for T; that is, $T(f) = \int f\,dF$ for all $f\in C(G)$. Since T is absolutely summing F is an X-valued measure of bounded variation [**3**]. It is easy to see that $\hat{F}(\gamma) = T(\bar{\gamma})$, for all $\gamma\in\Gamma$. Hence, since $T\equiv 0$ on $C_{\Lambda'}(G)$, $\hat{F}(\gamma)=0$, for all $\gamma\notin\Lambda$. Thus $F\in V^1_\Lambda(G,X)$ and, since Λ is a Riesz set, $F\in V^1_{\Lambda,ac}(G,X)$. Therefore, since X has II-Λ-RLP, $\lim_{\substack{\gamma\to\infty\\ \gamma\in\Lambda}}\|\hat{F}(\gamma)\|=0$. Consequently, $\lim_{\substack{\gamma\to\infty\\ \gamma\in\Lambda}}\|T(\bar{\gamma})\|=0$ and so $\{T(\bar{\gamma}):\gamma\in\Lambda\}$ is relatively compact in X. This proves that X has III-Λ-RLP. □

REMARK. If Λ is not a Riesz set, then III-Λ-RNP is equivalent to the Radon–Nikodym property and III-Λ-CCP is equivalent to the complete continuity property. We do not know if there is a corresponding result for Riemann–Lebesgue properties.

THEOREM 13. *Let G be a compact abelian metrizable group, let Λ be a subset of Γ. If X is a Banach space such that $L^1([0,1],X)$ has I-Λ-RLP, then X has II-Λ-RLP. On the other hand, if Λ is a Riesz set, then X has II-Λ-RLP if and only if $L^1([0,1],X)$ has II-Λ-RLP.*

Proof. Suppose that $L^1([0,1],X)$ has I-Λ-RLP. Then $L^1(G,X)$ has I-Λ-RLP. Let $\mu\in V^1_{\Lambda,ac}(G,X)$. Define an operator $T: L^1(G)\to L^1(G,X)$, by $T(f)=\overline{\mu * \bar{f}}$, for all

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$f \in L^1(G)$. Note that for $\gamma \in \Gamma$ and $y \in G$,

$$\begin{aligned}(T\gamma)(y) &= \overline{\int \overline{\gamma(x^{-1}y)}\, d\mu(x)} \\ &= \overline{\int \gamma(x)\overline{\gamma(y)}\, d\mu(x)} \\ &= \gamma(y)\overline{\int \gamma(x)\, d\mu(x)} \\ &= \gamma(y)\overline{\hat{\mu}(\bar{\gamma})}.\end{aligned}$$

That is, $T(\gamma) = \overline{\hat{\mu}(\bar{\gamma})}\gamma$, for all $\gamma \in \Gamma$. In particular, if $\gamma \in \Lambda'$, then $\bar{\gamma} \notin \Lambda$ so that $T(\gamma) = \overline{\hat{\mu}(\bar{\gamma})}\gamma = 0$, since $\mu \in V^1_{\Lambda,ac}(G, X)$. Therefore $T \equiv 0$ on $L^1_{\Lambda'}(G)$ and so there exists an operator $S : L^1(G)/L^1_{\Lambda'}(G) \to X$ such that $T = SQ$, where $Q : L^1(G) \to L^1(G)/L^1_{\Lambda'}(G)$ is the natural quotient mapping. Hence, by Theorem 5, since $L^1(G, X)$ has I-Λ-RLP, $\lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \|T(\bar{\gamma})\| = 0$. That is, $\lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \|\hat{\mu}(\gamma)\| = 0$. This proves that X has II-Λ-RLP.

Now suppose that Λ is a Riesz set and that X has the II-Λ-RLP property. Let $T : C(G) \to L^1([0, 1], X)$ be an absolutely summing operator with $T \equiv 0$ on $C_{\Lambda'}(G)$. By the Fakhoury–Kalton representation theorem [**8**, **10**], there is a family $\{\mu_\omega\}_{\omega\in[0,1]}$ of X-valued measures μ_ω on $\mathcal{B}(G)$, such that, for each $f \in C(G)$, we have

$$T(f)(\omega) = \int_G f(x)\, d\mu_\omega(x), \quad \text{for } m\text{-almost all } \omega \in [0, 1].$$

Also $\int_{[0,1]} \|\mu_\omega\|_1 d\lambda(\omega) \le \pi_1(T)$ and so μ_ω is of finite variation for m-almost all $\omega \in [0, 1]$.

If $\gamma \notin \Lambda$, then $\bar{\gamma} \in \Lambda'$ so $T(\bar{\gamma}) = SQ(\bar{\gamma}) = 0$. Hence for m-almost all $\omega \in G$, we have

$$0 = T(\bar{\gamma})(\omega) = \int_G \bar{\gamma}(x)\, d\mu_\omega(x) = \widehat{\mu_\omega}(\gamma).$$

Since Λ is countable we can conclude that, for m-almost all $\omega \in [0, 1]$, we have $\widehat{\mu_\omega}(\gamma) = 0$, for all $\gamma \notin \Lambda$. Hence, for m-almost all $\omega \in G$, $\mu_\omega \in V^1_\Lambda(G, X)$. Therefore, since Λ is a Riesz set, $\mu_\omega \in V^1_{\Lambda,ac}(G, X)$, for m-almost all $\omega \in G$. Since X has II-Λ-RLP, $(\widehat{\mu_\omega}(\gamma))_{\gamma\in\Lambda}$ is a null sequence in X, for m-almost all $\omega \in G$. Finally, applying Lebesgue's dominated convergence theorem, just as in Theorem 10, we get

$$\begin{aligned}\lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \|T(\bar{\gamma})\|_{L^1([0,1],X)} &= \lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \int_{[0,1]} \|T(\bar{\gamma})(\omega)\|_X\, dm(\omega) \\ &= \lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \int_{[0,1]} \left\| \int_G \bar{\gamma}(x)\, d\mu_\omega(x) \right\|_X dm(\omega) \\ &= \lim_{\substack{\gamma\to\infty \\ \gamma\in\Lambda}} \int_{[0,1]} \|\widehat{\mu_\omega}(\gamma)\|_X\, dm(\omega) \\ &= 0.\end{aligned}$$

This, in particular, says that $\{T(\bar{\gamma}) : \gamma \in \Lambda\}$ is relatively compact in $L^1([0, 1], X)$. Therefore $L^1([0, 1], X)$ has III-Λ-RLP and so it also has II-Λ-RLP. This completes the proof.

REMARK. In the proof of Theorem 13, we used the fact that Λ is a Riesz set for the sole purpose of concluding that the measures μ_ω are absolutely continuous. We can drop the Riesz set condition on Λ if we replace $L^1([0,1],X)$ by $L^p([0,1],X)$, where $1<p<\infty$. Specifically, the proof of Theorem 13 can be modified to show that, for any subset Λ of Γ, a Banach space X has III-Λ-RLP if and only if $L^p([0,1],X)$ has III-Λ-RLP, where $1<p<\infty$.

THEOREM 14. *Let Λ be a subset of Γ. Then Λ is a Rajchman set if and only if $L^1([0,1]$ has III-Λ-RLP.*

Proof. If $L^1([0,1]$ has III-Λ-RLP, it has I-Λ-RLP and so Λ is a Rajchman set, by Theorem 10.

For the converse, suppose that Λ is a Rajchman set and let $T: C(G)\to L^1([0,1]$ be an absolutely summing operator with $T\equiv 0$ on $C_{\Lambda'}(G)$. By the Fakhoury–Kalton representation theorem [**8**, **10**], there is a family $\{\mu_\omega\}_{\omega\in[0,1]}$ of complex measures μ_ω on $\mathcal{B}(G)$ such that, for each $f\in C(G)$, we have

$$T(f)(\omega)=\int_G f(x)\,d\mu_\omega(x),\quad \text{for almost all } \omega\in[0,1],$$

and $\int_{[0,1]}\|\mu_\omega\|_1 dm(\omega)\le \pi_1(T)$, so that μ_ω is of finite variation, for m-almost all $\omega\in[0,1]$. The remainder of the proof is completed just as in the proof of Theorem 10.

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