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ON A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS ON FINITE GRAPHS

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Abstract

Suppose that $G = (V, E)$ is a finite graph with the vertex set V and the edge set E. Let Δ be the usual graph Laplacian. Consider the nonlinear Schrödinger equation of the form

$$-\Delta u - \alpha u = f(x, u), \quad u \in W^{1,2}(V),$$

on the graph G, where $f(x, u) : V \times \mathbb{R} \to \mathbb{R}$ is a nonlinear real-valued function and α is a parameter. We prove an integral inequality on G under the assumption that G satisfies the curvature-dimension type inequality $CD(m, \xi)$. Then by using the Poincaré–Sobolev inequality, the Trudinger–Moser inequality and the integral inequality on G, we prove that there is a nontrivial solution to the nonlinear Schrödinger equation if $\alpha < 2\lambda_1^2/m(\lambda_1 - \xi)$, where λ_1 is the first positive eigenvalue of the graph Laplacian.

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1. Introduction

During the past several decades, the nonlinear Schrödinger equation of the form

$$-\Delta u + b(x)u = f(x, u), \quad u \in W^{1,2}(\Omega), \tag{1.1}$$

has been extensively studied. In (1.1), $\Omega \sqsubseteq R^n$, $n \geq 2$, $f(x, u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function and $b(x) \in C(\Omega, \mathbb{R})$ is a given potential. This type of equation provides a good model for developing new mathematical methods and has important applications in science and engineering.

Most recently, the investigation of discrete weighted Laplacians and various equations on graphs has attracted much attention. In [6], Grigoryan, Lin and Yang proved that there exists a positive solution to

$$\begin{cases} -\Delta u - \alpha u = |u|^{p-2}u & \text{in } \Omega^\circ \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

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on graphs for any $p > 2$ if

$$\alpha < \lambda_1(\Omega), \tag{1.3}$$

where

$$\lambda_1(\Omega) = \inf_{u \not\equiv 0,\, u|_{\partial\Omega}=0} \frac{\int_\Omega |\nabla u|^2 \, d\mu}{\int_\Omega u^2 \, d\mu}.$$

In [7], the same authors obtained a Poincaré inequality and one type of Trudinger–Moser embedding on finite graphs. Furthermore, they gave various conditions such that the Kazdan–Warner equation $\Delta u = c - he^u$ has a solution on finite graphs, where c is a constant and $h : V \to \mathbb{R}$ is a function.

In this paper we consider a class of nonlinear Schrödinger equations of the form

$$-\Delta u - \alpha u = f(x, u), \quad u \in W^{1,2}(V), \tag{1.4}$$

on a finite graph G. Here $f(x, u) : V \times \mathbb{R} \to \mathbb{R}$ is a nonlinear real-valued function, α is a parameter and $W^{1,2}(V)$ is a Sobolev space. The Equation (1.4) can be viewed as one type of discrete version of Equation (1.1).

We begin with some notation and settings. Let $G = (V, E)$ be a weighted graph where V denotes the vertex set and E denotes the edge set. We write $x \sim y$ if vertex x is adjacent to vertex y. We use (x, y) to denote an edge in E connecting vertices x and y. A graph G is called connected if, for any vertices $x, y \in V$, there exists a sequence $\{x_i\}_{i=0}^n$ that satisfies $x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_n = y$. Let ω_{xy} be the edge weights with $\omega_{xy} = \omega_{yx} > 0$. The degree of vertex x is given by the measure $\mu(x) = \sum_{y\sim x} \omega_{xy}$. If $\mu(x)$ is finite for every vertex x of V, we say that G is a locally finite graph. If V contains only finitely many vertices, we say that G is a finite graph. A finite graph is certainly locally finite.

From [8], for any function $u : V \to \mathbb{R}$, the μ-Laplacian of u is defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}[u(y) - u(x)]. \tag{1.5}$$

The associated gradient form is

$$\Gamma(u, v)(x) = \tfrac{1}{2}\{\Delta(u(x)v(x)) - u(x)\Delta v(x) - v(x)\Delta u(x)\}. \tag{1.6}$$

The length of the gradient for u is

$$|\nabla u|(x) = \sqrt{2\Gamma(u, u)(x)} = \left(\frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) - u(x))^2\right)^{1/2}. \tag{1.7}$$

The Ricci curvature operator Γ_2 on graphs is obtained by iterating Γ:

$$\Gamma_2(u, v)(x) = \tfrac{1}{2}\{\Delta\Gamma(u, v)(x) - \Gamma(u, \Delta v)(x) - \Gamma(v, \Delta u)(x)\}. \tag{1.8}$$

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To compare with the Euclidean setting, for any function $u : V \to \mathbb{R}$, we write

$$\int_V u\,d\mu = \sum_{x \in V} \mu(x)u(x).$$

From [4], all eigenvalues of the Laplacian $-\Delta$ on $G = (V, E)$ are nonnegative and the minimum positive eigenvalue, also called the first positive eigenvalue, is given by

$$\lambda_1 = \lambda_G = \inf_{f \perp T\mathbf{1}} \frac{\sum_{x,y \in V, y\sim x}(u(y) - u(x))^2 \omega_{xy}}{\sum_{x\in V} u^2(x)\mu(x)} = \inf_{f \perp T\mathbf{1}} \frac{\int_\Omega |\nabla u|^2\,d\mu}{\int_\Omega u^2\,d\mu}, \tag{1.9}$$

where the nontrivial function u achieving (1.9) is called a harmonic eigenfunction of $-\Delta$ on G with eigenvalue λ_1 and $T\mathbf{1}$ is the vector each of whose elements is the degree of the corresponding vertex. (For more details, we refer to [4].) By [4, Lemma 1.10], if $u(x)$ is a harmonic eigenfunction achieving λ_1 in (1.9), then, for any vertex $x \in V$,

$$-\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}[u(x) - u(y)] = \lambda_1 u(x).$$

In [8], Lin and Yau introduced the curvature-dimension type inequality $CD(m, \xi)$.

Definition 1.1 (Curvature-dimension type inequality). We say that a graph G satisfies the curvature-dimension type inequality $CD(x, m, \xi)$ for some $m > 1$, $\xi \in \mathbb{R}$ and $x \in V$ if, for any function $u : V \to \mathbb{R}$,

$$\Gamma_2(u, u)(x) \geq \frac{1}{m}(\Delta u(x))^2 + \xi\Gamma(u, u)(x).$$

We call m the dimension of the operator Δ and ξ the lower bound of the Ricci curvature of the operator Δ. Furthermore, we say that $CD(m, \xi)$ is satisfied if $CD(x, m, \xi)$ is satisfied for all $x \in V$.

It is easy to see that, for $m < m'$, the operator Δ satisfies the curvature-dimension type inequality $CD(m', \xi)$ if it satisfies the curvature-dimension type inequality $CD(m, \xi)$.

Lin and Yau [8] proved that any locally finite graph satisfies either $CD(2, 2/d - 1)$ if d is finite, or $CD(2, -1)$ if d is infinite, where $d = \sup_{x\in V} \sup_{y\sim x} \mu(x)/\omega_{xy}$.

In addition to the curvature-dimension type inequality, we introduce the well-known Trudinger–Moser inequality. From [2, 9], when $p > 2$,

$$\exp(\beta|u|^{p/(p-1)}) \in L^1(\Omega)$$

and there exists a constant $C = C(p, \beta)$ which depends only on p and β such that

$$\sup_{\|u\|_{W_0^{1,p}(\Omega)} \leq 1} \int_\Omega \exp(\beta|u|^{p/(p-1)})\,dx \leq C|\Omega|, \quad \text{if } \beta \leq \beta_p, \tag{1.10}$$

where $p > 2$, $u \in W_0^{1,p}(\Omega)$, $\|u\|_{W_0^{1,p}(\Omega)} = (\int_\Omega |\nabla u|^p\,dx)^{1/p}$, $\beta_p = p(\omega_{p-1})^{1/(p-1)}$ and ω_{p-1} is the measure of the unit sphere in $\mathbb{R}^p$. In Section 2, we will give a Trudinger–Moser inequality on finite graphs in Lemma 2.6 as a discrete version of (1.10).

The next definition is motivated by the Trudinger–Moser inequality.

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DEFINITION 1.2 [5]. Suppose that $f(x,t) : V \times \mathbb{R} \to \mathbb{R}$. We say the function f has subcritical growth at $+\infty$ if, for all $\beta > 0$ and $p > 2$,

$$\lim_{t\to+\infty} \frac{f(x,t)}{\exp(\beta|t|^{p/(p-1)})} = 0. \tag{1.11}$$

Suppose that $G = (V, E)$ is a connected finite graph that satisfies the curvature-dimension type inequality $CD(m, \xi)$. Firstly, we state the Poincaré–Sobolev inequality and the Trudinger–Moser inequality on G and we prove an integral inequality on G (in Theorem 1.3). Then, by using the three inequalities and the Mountain-Pass theorem, we prove that there exists a nontrivial solution to the nonlinear Schrödinger type equation (1.4) if $\alpha < 2\lambda_1^2/m(\lambda_1 - \xi)$, extending the result (1.3) for (1.2).

We give some notation before we state our main theorems. Throughout, $L^p(V)$ denotes the Banach space with the norm $\|u\|_{L^p} = (\int_V |u|^p \, d\mu)^{1/p}$. Furthermore, we define a Sobolev space and a norm on it by

$$W^{1,p}(V) = \left\{u : V \to \mathbb{R} \mid \int_V (|\nabla u|^p + |u|^p) \, d\mu < \infty\right\}$$

and

$$\|u\|_{W^{1,p}(V)} = \left(\int_V (|\nabla u|^p + |u|^p) \, d\mu\right)^{1/p}.$$

We can now state our main theorems.

THEOREM 1.3. *Suppose that $G = (V, E)$ is a finite graph that satisfies the curvature-dimension type inequality $CD(m, \xi)$ and u is a harmonic eigenfunction of $-\Delta$ with eigenvalue λ_1. Then*

$$\frac{2\lambda_1^2}{m(\lambda_1 - \xi)} \int_V u^2 \, d\mu \le \int_V |\nabla u|^2 \, d\mu.$$

REMARK 1.4. For u a harmonic eigenfunction of $-\Delta$ on G with eigenvalue λ_1, we define a function space

$$\mathcal{H} = \{u \in W^{1,2}(V) \mid -\Delta u = \lambda_1 u\}.$$

By Theorem 1.3, when $\alpha < 2\lambda_1^2/m(\lambda_1 - \xi)$, we can define a norm on $\mathcal{H}$ by

$$\|u\|_{1,\alpha} = \left(\int_V (|\nabla u|^2 - \alpha|u|^2) \, d\mu\right)^{1/2}.$$

By Remark 1.4, we obtain the following theorem.

THEOREM 1.5. *Suppose that $G = (V, E)$ is a finite graph that satisfies the curvature-dimension type inequality $CD(m, \xi)$. Assume that $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses.*

(H1) *For any $x \in V$, $f(x,t)$ is continuous in $t \in \mathbb{R}$, and $f(x,-t) = -f(x,t)$ for all $(x,t) \in V \times \mathbb{R}$.*

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(H2) *For all* $(x,t) \in V \times [0,+\infty)$, $f(x,t) \geq 0$ *and* $f(x,0) = 0$.

(H3) $f(x,t)$ *has subcritical growth at* $+\infty$ *, that is,* f *satisfies (1.11).*

(H4) *There exists some* $p > 2$ *such that* $\lim_{t\to 0+} f(x,t)/t^{p-1} = 0$ *for all* $x \in V$.

(H5) *(Ambrosetti–Rabinowitz condition) There exist constants* $q > 2$ *and* $s_0 > 0$ *such that if* $|u| \geq s_0$*, then* $0 < qF(x,u) < uf(x,u)$ *for any* $x \in V$*, where* $F(x,u) = \int_0^u f(x,t)\,dt$.

Then, for any $p > 2$ *and*

$$\alpha < \frac{2\lambda_1^2}{m(\lambda_1 - \xi)},$$

there exists a nontrivial solution $u \in \mathcal{H}$ *to (1.4).*

Remark 1.6.

(1) In (1.2), when $\Omega = V$, we consider the problem

$$-\Delta u - \alpha u = |u|^{p-2}u, \quad u \in W^{1,2}(V). \tag{1.12}$$

We can easily prove that there exists a nontrivial solution to (1.12) for any $p > 2$ if

$$\alpha < \lambda_1, \tag{1.13}$$

where λ_1 is defined as in (1.9). Taking $f(x,t) = |u|^{p-2}u$ in Theorem 1.5 and in (1.4), by Theorem 1.3, we can also prove that there exists a nontrivial solution to (1.12) for any $p > 2$ if

$$\alpha < \frac{2\lambda_1^2}{m(\lambda_1 - \xi)}.$$

(2) By Lemma 2.1 in Section 2, $\lambda_1 \geq m\xi/(m-1)$. We can easily check that when $m\xi/(m-1) \leq \lambda_1 < m\xi/(m-2)$, we have $2\lambda_1^2/m(\lambda_1 - \xi) > \lambda_1$. For example, consider a connected path with two vertices a and b. It has a nonzero eigenvalue $\lambda_1 = 2$ and satisfies $CD(2,1)$. We can check that $2\lambda_1^2/m(\lambda_1 - \xi) = 4 > \lambda_1 = m\xi/(m-1) = 2$. So Theorem 1.5 gives a significant improvement to the result (1.13) for (1.12).

The paper is organised as follows. In Section 2 we introduce some preliminary results which are useful for the proof of our main theorems. In Sections 3 and 4 we prove our main theorems.

2. Preliminaries

In this section we introduce some preliminary results which will be used to prove our main theorems.

Lemma 2.1 [3, Theorem 2.1]. *Suppose that* $G = (V,E)$ *is a finite graph that satisfies the curvature-dimension type inequality* $CD(m,\xi)$ *where* $m > 1$ *and* $\xi > 0$*, and the Ricci curvature of* G *is at least* ξ*. Then any positive eigenvalue* λ *of* $-\Delta$ *satisfies* $\lambda \geq m\xi/(m-1)$.

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REMARK 2.2. Let $m > 1$ and λ_1 be the first positive eigenvalue of $-\Delta$. From Lemma 2.1, $\lambda_1 \geq m\xi/(m-1) > \xi$ when $\xi > 0$, while $\lambda_1 > \xi$ when $\xi \leq 0$. So for $m > 1$ and $\xi \in \mathbb{R}$, we always have $\lambda_1 > \xi$.

Let $(X, \|\cdot\|)$ be a Banach space and $J : X \to \mathbb{R}$ be a functional. Following [6], we say that J satisfies the $(PS)_c$ condition for some real number c, if any sequence of functions $u_k : X \to \mathbb{R}$ such that $J(u_k) \to c$ and $J'(u_k) \to 0$ as $k \to +\infty$ has a convergent subsequence $u_{k_n} \to u$ in X.

LEMMA 2.3 (The Mountain-Pass theorem (Ambrosetti and Rabinowitz) [1] and [6, Theorem 9]). *Let $(X, \|\cdot\|)$ be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\| > r$ and $b = \inf_{\|u\|=r} J(u) > J(0) > J(e)$. If J satisfies the $(PS)_c$ condition with $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$, where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}$, then c is a critical value of J.*

Since V is a finite graph, $W^{1,p}(V)$ is a finite-dimensional space. The next lemma comes from [6, Theorem 8, together with Theorem 7] and [7, Lemma 5].

LEMMA 2.4 [6]. *Let $G = (V, E)$ be a finite graph. The Sobolev space $W^{1,s}(V)$ is pre-compact for constant $s > 1$, that is, if $\{u_k\}$ is bounded in $W^{1,s}(V)$, then there is a subsequence $\{u_{k_n}\}$ such that $u_{k_n} \to u$ in $W^{1,s}(V)$.*

The next lemma follows from Lemma 2.4 and [7, Lemma 6]. The proof is very similar to the proof of [7, Lemma 6].

LEMMA 2.5 (Poincaré–Sobolev inequality). *Let $G = (V, E)$ be a finite graph. For all $u \in W^{1,s}(V)$ with $\int_V u\, d\mu = 0$, the following Poincaré–Sobolev inequality holds for all $q \geq 1$ and $s > 1$:*

$$\left(\int_V |u|^q \, d\mu \right)^{1/q} \leq C \left(\int_V |\nabla u|^s \, d\mu \right)^{1/s}. \tag{2.1}$$

The Trudinger–Moser inequality in the next lemma follows from the Poincaré–Sobolev inequality (2.1). The proof is very similar to the proof of [7, Lemma 7].

LEMMA 2.6 (Trudinger–Moser inequality on finite graphs). *Suppose that $G = (V, E)$ is a finite graph. For all functions u with $\int_V |\nabla u|^p \, d\mu \leq 1$ and $\int_V u\, d\mu = 0$, there exists a constant C which depends only on p, β and V such that*

$$\sup_{\int_V |\nabla u|^p \, d\mu \leq 1} \int_V \exp(\beta |u|^{p/(p-1)}) \, d\mu \leq C|V|, \quad \text{for any } \beta > 1 \text{ and } p > 2,$$

where $|V| = \int_V d\mu(x) = \mathrm{Vol}\ V$ is the volume of the graph G.

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3. Proof of Theorem 1.3

In this section we prove Theorem 1.3 using the curvature-dimension type inequality $CD(m,\xi)$ in Definition 1.1.

By (1.6)–(1.8),

$$\begin{aligned}\Gamma_2(u,u)(x) &= \tfrac{1}{2}\{\Delta\Gamma(u,u)(x) - 2\Gamma(u,\Delta u)(x)\}\\ &= \tfrac{1}{4}\Delta|\nabla u|^2(x) - \Gamma(u,\Delta u)(x)\\ &= \tfrac{1}{4}\Delta|\nabla u|^2(x) - \tfrac{1}{2}\{\Delta(u(x)\Delta u(x)) - u(x)\Delta(\Delta u(x)) - (\Delta u(x))^2\}. \end{aligned}\tag{3.1}$$

On the other hand, by (1.5),

$$\begin{aligned}\Delta(u(x)\Delta u(x)) &= \frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}[u(y)\Delta u(y) - u(x)\Delta u(x)]\\ &= \frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}[u(y)\Delta u(y) - u(y)\Delta u(x) + u(y)\Delta u(x) - u(x)\Delta u(x)]\\ &= \frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}u(y)[\Delta u(y) - \Delta u(x)] + \Delta u(x)\cdot\frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}[u(y)-u(x)]\\ &= \frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}[u(y)-u(x)][\Delta u(y) - \Delta u(x)]\\ &\quad + \frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}u(x)[\Delta u(y) - \Delta u(x)] + (\Delta u(x))^2\\ &= \frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}[u(y)-u(x)][\Delta u(y) - \Delta u(x)] + u(x)\Delta(\Delta u(x)) + (\Delta u(x))^2. \end{aligned}\tag{3.2}$$

By (3.1) and (3.2),

$$\Gamma_2(u,u)(x) = \frac{1}{4}\Delta|\nabla u|^2(x) - \frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}[u(y)-u(x)][\Delta u(y)-\Delta u(x)].\tag{3.3}$$

If u is a harmonic eigenfunction that satisfies $-\Delta u(x) = \lambda_1 u(x)$ then, by (3.3),

$$\begin{aligned}\Gamma_2(u,u)(x) &= \frac{1}{4}\Delta|\nabla u|^2(x) - \frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}[u(y)-u(x)][\lambda_1 u(x) - \lambda_1 u(y)]\\ &= \frac{1}{4}\Delta|\nabla u|^2(x) + \frac{\lambda_1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}[u(y)-u(x)]^2\\ &= \frac{1}{4}\Delta|\nabla u|^2(x) + \frac{\lambda_1}{2}|\nabla u|^2(x). \end{aligned}$$

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Now consider

$$\begin{aligned}
\sum_{x\in V}\mu(x)\Gamma_2(u,u) &= \frac{1}{4}\sum_{x\in V}\mu(x)\Delta|\nabla u|^2(x) + \frac{\lambda_1}{2}\sum_{x\in V}\mu(x)|\nabla u|^2(x)\\
&= \frac{1}{4}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}[|\nabla u|^2(y) - |\nabla u|^2(x)] + \frac{\lambda_1}{2}\sum_{x\in V}\mu(x)|\nabla u|^2(x)\\
&= \frac{1}{4}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}|\nabla u|^2(y) - \frac{1}{4}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}|\nabla u|^2(x) + \frac{\lambda_1}{2}\sum_{x\in V}\mu(x)|\nabla u|^2(x)\\
&= \frac{1}{2}\sum_{(x,y)\in E}\omega_{xy}|\nabla u|^2(y) - \frac{1}{2}\sum_{(x,y)\in E}\omega_{xy}|\nabla u|^2(x) + \frac{\lambda_1}{2}\sum_{x\in V}\mu(x)|\nabla u|^2(x)\\
&= \frac{1}{2}\sum_{(x,y)\in E}\omega_{xy}|\nabla u|^2(y) - \frac{1}{2}\sum_{(y,x)\in E}\omega_{yx}|\nabla u|^2(y) + \frac{\lambda_1}{2}\sum_{x\in V}\mu(x)|\nabla u|^2(x)\\
&= \frac{\lambda_1}{2}\sum_{x\in V}\mu(x)|\nabla u|^2(x) = \frac{\lambda_1}{2}\int_V |\nabla u|^2\,d\mu. \qquad (3.4)
\end{aligned}$$

Since G satisfies the curvature-dimension type inequality $CD(m,\xi)$, that is,

$$\Gamma_2(u,u)(x) \geq \frac{1}{m}(\Delta u(x))^2 + \xi\Gamma(u,u)(x),$$

it follows that

$$\begin{aligned}
\sum_{x\in V}\mu(x)\Gamma_2(u,u) &\geq \frac{1}{m}\sum_{x\in V}\mu(x)(\Delta u(x))^2 + \xi\sum_{x\in V}\mu(x)\Gamma(u,u)(x)\\
&= \frac{1}{m}\sum_{x\in V}\mu(x)\lambda_1^2(u(x))^2 + \frac{\xi}{2}\sum_{x\in V}\mu(x)\Gamma(u,u)(x)\\
&= \frac{\lambda_1^2}{m}\int_V u^2\,d\mu + \frac{\xi}{2}\int_V |\nabla u|^2\,d\mu. \qquad (3.5)
\end{aligned}$$

By (3.4) and (3.5),

$$\frac{\lambda_1}{2}\int_V |\nabla u|^2\,d\mu \geq \frac{\lambda_1^2}{m}\int_V u^2\,d\mu + \frac{\xi}{2}\int_V |\nabla u|^2\,d\mu. \qquad (3.6)$$

By (3.6) and Remark 2.2, since $\lambda_1 > \xi$,

$$\frac{2\lambda_1^2}{m(\lambda_1-\xi)}\int_V u^2\,d\mu \leq \int_V |\nabla u|^2\,d\mu,$$

which completes the proof.

4. Proof of Theorem 1.5

In this section we prove Theorem 1.5 using Remark 1.4 and some of the lemmas from Section 2.

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Let $p > 2$ and $\alpha < 2\lambda_1^2/m(\lambda_1 - \xi)$ be fixed. Throughout this section, u is a harmonic eigenfunction of $-\Delta$ on G with eigenvalue λ_1. By Remark 1.4, the function space

$$\mathcal{H} = \{u \in W^{1,2}(V) | -\Delta u = \lambda_1 u\}$$

has the norm

$$\|u\|_{1,\alpha} = \Big(\int_V (|\nabla u|^2 - \alpha |u|^2)\, d\mu\Big)^{1/2}.$$

Furthermore, from [8, page 353)], $\int_V u\, d\mu = 0$ for all $u \in \mathcal{H}$.

We now define a functional $J_\alpha : \mathcal{H} \to \mathbb{R}$ by

$$J_\alpha(u) = \frac{1}{2}\|u\|_{1,\alpha}^2 - \int_V F(x,u)\, d\mu.$$

By (H4), there exist two positive constants $\tau, \delta > 0$ such that if $|u| \le \delta$ then

$$|f(x,u)| \le \tau |u|^{p-1}. \tag{4.1}$$

On the other hand, by (H3), there exist two positive constants c, β such that

$$|f(x,u)| \le c \cdot \exp(\beta |u|^{p/(p-1)}), \quad \text{for all } |u| \ge \delta. \tag{4.2}$$

Then, by (4.2), for $q > p$,

$$F(x,u) \le c \cdot \exp(\beta |u|^{p/(p-1)})|u|^q, \quad \text{for all } |u| \ge \delta. \tag{4.3}$$

Combining (4.1) and (4.3),

$$F(x,u) \le \tau \frac{|u|^p}{p} + c \cdot \exp(\beta |u|^{p/(p-1)})|u|^q.$$

By the Hölder inequality,

$$J(u) \ge \frac{1}{2}\|u\|_{1,\alpha}^2 - \frac{\tau}{p}\int_V |u|^p\, d\mu - c\Big(\int_V \exp(\beta p |u|^{p/(p-1)})\, d\mu\Big)^{1/p}\Big(\int_V |u|^{qp'}\, d\mu\Big)^{1/p'}, \tag{4.4}$$

where $1/p + 1/p' = 1$. By the Trudinger–Moser inequality in Lemma 2.6,

$$\int_V \exp(\beta p |u|^{p/(p-1)})\, d\mu = \int_V \exp\Big(\beta p \|u\|_{L^p}^{p/(p-1)}\Big(\frac{|u|}{\|u\|_{L^p}}\Big)^{p/(p-1)}\Big)\, d\mu < c|V|. \tag{4.5}$$

By Lemma 2.5, there exists some constant C that depends only on p and V such that

$$\Big(\int_V u^p\, d\mu\Big)^{1/p} \le C\Big(\int_V |\nabla u|^2\, d\mu\Big)^{1/2}. \tag{4.6}$$

Since $q > p > 2$, by (4.4)–(4.6), we can find some sufficiently small $r > 0$ such that if $\|u\|_{1,\alpha} = r$ then

$$J(u) \ge \frac{1}{2}\|u\|_{1,\alpha}^2 - C^p\Big(\frac{\tau}{p} + c|V|\Big)\|u\|_{1,\alpha}^p.$$

Therefore,

$$\inf_{\|u\|_{1,\alpha}=r} J(u) > 0. \tag{4.7}$$

By (H5) and [10, pages 141–143], there exist two positive constants c_1 and c_2 such that

$$F(x,u) \geq c_1|u|^q - c_2.$$

For any $t > 0$,

$$J_\alpha(tu) \leq \frac{t^2}{2}\|u\|_{1,\alpha}^2 - c_1 t^q \int_V |u|^q \, d\mu - c_2|V|.$$

Since $q > p > 2$, there exists some $u_1 \in \mathcal{H}$ satisfying

$$J_\alpha(u_1) < 0 \quad \text{for } \|u_1\|_{1,\alpha} > r. \tag{4.8}$$

We now prove that $J_\alpha(u)$ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. To see this, suppose $\{u_k\} \subset \mathcal{H}$ is such that $J(u_k) \to c$ and $J'(u_k) \to 0$ as $k \to \infty$, that is,

$$\frac{1}{2}\int_V (|\nabla u_k|^2 - \alpha u_k^2)\, d\mu - \int_V F(x,u_k)\, d\mu = c + o_k(1). \tag{4.9}$$

$$\int_V (|\nabla u_k|^2 - \alpha u_k^2)\, d\mu - \int_V u_k f(x,u_k)\, d\mu = o_k(1)\|u_k\|_{1,\alpha}. \tag{4.10}$$

In view of (H5), it follows from (4.9) and (4.10) that $\{u_k\}$ is bounded in $\mathcal{H}$, and the $(PS)_c$ condition follows by Lemma 2.4. Combining (4.7), (4.8) and the obvious fact that $J(0) = 0$, we conclude by Lemma 2.3 that there exists a function $u \in \mathcal{H}$ such that $J(u) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) > 0$ and $J'(u) = 0$, where

$$\Gamma = \{\gamma \in C([0,1],\mathcal{H}) : \gamma(0) = 0, \gamma(1) = u_1\}.$$

Hence there exists a nontrivial solution $u \in \mathcal{H}$ to the equation

$$-\Delta u - \alpha u = f(x,u), \quad u \in W^{1,2}(V).$$

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