A GENERALISATION OF A FORMULA DUE TO SCHUBERT

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1. Let there be given, on an algebraic curve C, of genus p, a linear series g_m^r and an algebraic series γ_n^1 of index ν, both without fixed points. The number of groups of $r+1$ points which are common to a set of g_m^r and a set of γ_n^1 has been shown by Schubert (**1**) to be

$$m\nu\binom{n-1}{r}-\tfrac{1}{2}\binom{n-2}{r-1}d$$

where d is the number of double points of γ_n^1.

The object of this note is to generalise the above result by seeking the number of groups of $s=\sum_{i=1}^{\rho}a_i$ points which are common to a set of g_m^r and a set of γ_n^1, these s points consisting of a batch of a_1 points of multiplicity k_1, a batch of a_2 points of multiplicity k_2, ..., and a batch of a_ρ points of multiplicity k_ρ for the set of g_m^r which contains them, and being all simple points of the set of γ_n^1 which contains them. In order that there shall be a finite number of such groups of points it is necessary that $r+1=\sum_{i=1}^{\rho}a_ik_i$. The number sought may then be denoted by

$$Z\begin{bmatrix} r \\ m \end{bmatrix};\begin{matrix} a_1 \\ k_1 \end{matrix},\begin{matrix} a_2 \\ k_2 \end{matrix},\ldots,\begin{matrix} a_\rho \\ k_\rho \end{matrix}\Bigg].$$

2. Consider an algebraic series γ_n^1 and a linear series $g_{m+l-k_1+1}^{r+l-k_1+1}$ both on C, where $l<k_1<k_2<\ldots<k_\rho$.

A general point P of C belongs to ν sets of γ_n^1. If Q be any one of the points of these sets, other than P, there is defined a correspondence (P, Q), with indices $\nu(n-1)$, whose united points are the d double points of γ_n^1.

From the $n-1$ points Q of such a set, a batch of (a_1-1) points Q_1, a batch of a_2 points Q_2, ..., and a batch of a_ρ points Q_ρ may be chosen in

$$\binom{n-1}{a_1-1,\ a_2,\ \ldots,\ a_\rho}=\frac{(n-1)!}{(a_1-1)!\ a_2!\ \ldots\ a_\rho!\ (n-s)!}$$

ways. Each such choice defines a set of $g_{m+l-k_1+1}^{r+l-k_1+1}$ for which P is an l-ple point and which has each point Q_i as a k_i-ple point $(i=1,\ 2,\ \ldots,\ \rho)$. Let this be done for each of the ν sets of γ_n^1 defined by P. Then if R be any one of the further points of $g_{m+l-k_1+1}^{r+l-k_1+1}$ thus defined, there is established a correspondence (P, R) whose second index is

$$\nu\binom{n-1}{a_1-1,\ a_2,\ \ldots,\ a_\rho}(m-r)$$

and whose first index is N_l where

$$N_l = Z\begin{bmatrix} r+l-k_1 \\ m+l-k_1 \end{bmatrix}; \begin{matrix} 1 \\ l \end{matrix}, \begin{matrix} \alpha_1-1 \\ k_1 \end{matrix}, \begin{matrix} \alpha_2 \\ k_2 \end{matrix}, \ldots, \begin{matrix} \alpha_\rho \\ k_\rho \end{matrix}\Bigg] \text{ if } l \neq 0,$$

while

$$N_0 = (n-s+1)Z\begin{bmatrix} r-k_1 & \alpha_1-1 & \alpha_2 & & \alpha_\rho \\ m-k_1 ; & k_1 , & k_2 , & \ldots, & k_\rho \end{bmatrix}.$$

The united points of (P, R) are U_l in number where

$$U_l = Z\begin{bmatrix} r+l-k_1+1 & 1 & \alpha_1-1 & \alpha_2 & & \alpha_\rho \\ m+l-k_1+1 ; & l+1 , & k_1 , & k_2 , & \ldots, & k_\rho \end{bmatrix} \text{ if } l \neq k_1-1,$$

while

$$U_{k_1-1} = \alpha_1 Z\begin{bmatrix} r & \alpha_1 & & \alpha_\rho \\ m ; & k_1 , & \ldots, & k_\rho \end{bmatrix}.$$

3. The sets of $g_{m+l-k_1+1}^{r+l-k_1+1}$ defined above contain, in addition to the points R, the point P counted $\nu l \binom{n-1}{\alpha_1-1, \alpha_2, \ldots, \alpha_\rho}$ times and the points Q each counted

$$k_1\binom{n-2}{\alpha_1-2, \alpha_2, \ldots, \alpha_\rho} + \sum_{i=2}^{\rho} k_i \binom{n-2}{\alpha_1-1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i-1, \alpha_{i+1}, \ldots, \alpha_\rho}$$
$$= \frac{r-k_1+1}{n-1}\binom{n-1}{\alpha_1-1, \alpha_2, \ldots, \alpha_\rho}$$

times. It follows that the correspondence

$$T_l \equiv \frac{r-k_1+1}{n-1}\binom{n-1}{\alpha_1-1, \alpha_2, \ldots, \alpha_\rho}(P, Q) + (P, R)$$

has valency

$$\nu l \binom{n-1}{\alpha_1-1, \alpha_2, \ldots, \alpha_\rho}.$$

Its indices are

$$\binom{n-1}{\alpha_1-1, \alpha_2, \ldots, \alpha_\rho}(r-k_1+1)\nu + N_l \text{ and } \binom{n-1}{\alpha_1-1, \alpha_2, \ldots, \alpha_\rho}(m-k_1+1)\nu$$

whose sum is

$$\nu\binom{n-1}{\alpha_1-1, \alpha_2, \ldots, \alpha_\rho}[m+r-2(k_1-1)] + N_l$$

and so, by the Cayley-Brill theorem (2)

$$U_l - N_l = \binom{n-1}{\alpha_1-1, \alpha_2, \ldots, \alpha_\rho}\left\{\nu[m+r+2(lp-k_1+1)] - \frac{r-k_1+1}{n-1}d\right\}.$$

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On giving l the values k_1-1, k_1-2, ..., 1, 0 in succession, and adding the resulting k_1 relations, the k_1-1 numbers

$$Z\begin{bmatrix} r+l-k_1 \\ m+l-k_1 \end{bmatrix}; \begin{matrix} 1 \\ l \end{matrix}, \begin{matrix} a_1-1 \\ k_1 \end{matrix}, \dots, \begin{matrix} a_\rho \\ k_\rho \end{matrix}\Bigg] \quad (l=k_1-1, \dots, 2, 1)$$

disappear, leaving

$$Z\left[\begin{matrix} r \\ m \end{matrix}; \begin{matrix} a_1 \\ k_1 \end{matrix}, \dots, \begin{matrix} a_\rho \\ k_\rho \end{matrix}\right] = \frac{n-s+1}{a_1} Z\left[\begin{matrix} r-k_1 \\ m-k_1 \end{matrix}; \begin{matrix} a_1-1 \\ k_1 \end{matrix}, \begin{matrix} a_2 \\ k_2 \end{matrix}, \dots, \begin{matrix} a_\rho \\ k_\rho \end{matrix}\right]$$

$$+\binom{n}{a_1, \dots, a_\rho}\left\{\frac{\nu k_1}{n}[m+r+(p-2)(k_1-1)] - \frac{(r-k_1+1)k_1 d}{n(n-1)}\right\}.$$

4. A g_m^{k-1} on C contains $k[m+(p-1)(k-1)]$ sets each possessing a point of multiplicity k (3), and since this point belongs to ν sets of γ_n^1 it follows that

$$Z\left[\begin{matrix} k-1 \\ m \end{matrix}; \begin{matrix} 1 \\ k \end{matrix}\right] = \nu k[m+(p-1)(k-1)].$$

The above recurrence relation now permits the successive calculation of

$$Z\left[\begin{matrix} a_\rho k_\rho - 1 \\ m \end{matrix}; \begin{matrix} a_\rho \\ k_\rho \end{matrix}\right], Z\left[\begin{matrix} a_{\rho-1}k_{\rho-1}+a_\rho k_\rho - 1 \\ m \end{matrix}; \begin{matrix} a_{\rho-1} \\ k_{\rho-1} \end{matrix}, \begin{matrix} a_\rho \\ k_\rho \end{matrix}\right]$$

and so on. Thus it is found that

$$Z\left[\begin{matrix} ak-1 \\ m \end{matrix}; \begin{matrix} a \\ k \end{matrix}\right] = \nu k\binom{n-1}{a-1}\{m+(k-1)(p-1)\} - \tfrac{1}{2}\binom{n-2}{a-2}k^2 d,$$

which clearly reduces to Schubert's formula when $k=1$. It may now be verified by induction that

$$Z\left[\begin{matrix} r \\ m \end{matrix}; \begin{matrix} a_1 \\ k_1 \end{matrix}, \dots, \begin{matrix} a_\rho \\ k_\rho \end{matrix}\right] = \binom{n}{a_1, \dots, a_\rho}\left[\frac{\nu}{n}\left\{(r+1)(m-p+1)+(p-1)\sum_{i=1}^{\rho} a_i k_i^2\right\}\right.$$

$$\left. - \frac{d}{2n(n-1)}\left\{(r+1)^2 - \sum_{i=1}^{\rho} a_i k_i^2\right\}\right].$$

5. By way of illustration, consider the following problem. Let C be a plane curve, of order $n \geqslant 3$, and genus p. There exists a single infinity of conics each of which osculates C at two points. What is the class of the envelope of the line of join of these points ?

The totality of conics cut a g_{2n}^5 on C, while the lines through a general point of the plane cut a g_n^1. Hence, setting

$$m=2n,\ r=5,\ \nu=1,\ d=2(n+p-1),\ \rho=1,\ a_1=2,\ k_1=3,$$

the number sought is

$$Z\left[\begin{matrix} 5 \\ 2n \end{matrix}; \begin{matrix} 2 \\ 3 \end{matrix}\right] = 3(2n-5)(n+p-1).$$

For example, if $n=3$ and $p=1$ the envelope is of class 9. It is easy to show that the envelope in this case degenerates into the nine inflexions of the cubic.

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REFERENCES

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(2) H. F. Baker, *ibid.*, pp. 8, 9.
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