

## RIGIDITY OF HILBERT METRICS

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We study the groups of isometries for Hilbert metrics on bounded open convex domains in  $\mathbb{R}^n$  and show that if  $\mathcal{C}$  is such a set with a *strictly* convex boundary, the Hilbert geometry is asymptotically Riemannian at infinity. As a consequence of this result, we prove there are no Hausdorff quotients of  $\mathcal{C}$  by isometry subgroups with finite *volume* except when  $\partial\mathcal{C}$  is an ellipsoid.

### INTRODUCTION

Let  $\mathcal{C}$  be a bounded open convex domain in  $\mathbb{R}^n$  with boundary  $\partial\mathcal{C}$  and  $\|\cdot\|$  be the canonical Euclidean norm in  $\mathbb{R}^n$ . It is then possible to define a distance  $d_{\mathcal{C}}$  on  $\mathcal{C}$ , the so-called Hilbert metric (discovered by D. Hilbert in 1894), as follows. Given two distinct points  $p$  and  $q$  in  $\mathcal{C}$ , let  $a$  and  $b$  be the intersection points of the straight line defined by  $p$  and  $q$  with  $\partial\mathcal{C}$  so that  $p = ta + (1-t)b$  and  $q = sa + (1-s)b$  with  $0 < s < t < 1$ . Then  $d_{\mathcal{C}}(p, p) = 0$  and  $d_{\mathcal{C}}(p, q) = \ln[a, p, q, b]$ , where  $[a, p, q, b] = (1-t)s/((1-s)/t) > 1$  is the cross ratio of the ordered collinear points  $\{a, p, q, b\}$ . The fact  $d_{\mathcal{C}}$  is a distance comes from basic properties of the cross ratio and the metric space  $(\mathcal{C}, d_{\mathcal{C}})$  thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of  $\mathbb{R}^n$  and in which the affine open segments joining two points of the boundary are geodesics isometric to  $(\mathbb{R}, |\cdot|)$ . On the other hand, the distance  $d_{\mathcal{C}}$  is associated to the Finsler metric  $F_{\mathcal{C}}$  on  $\mathcal{C}$  given for  $p \in \mathcal{C}$  and  $v \in T_p\mathcal{C} = \mathbb{R}^n$  by  $F_{\mathcal{C}}(p, v) = \|v\| \left( (1/\|p - p^-\|) + (1/\|p - p^+\|) \right)$ , where  $p^-$  (respectively  $p^+$ ) is the intersection point of the half line  $p + \mathbb{R}^-v$  (respectively  $p + \mathbb{R}^+v$ ) with  $\partial\mathcal{C}$ . For further information, we refer to [3, 4, 6, 9] for an introduction to the subject. In the present paper, we study the subgroups  $\Gamma$  of the group  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  of isometries of  $(\mathcal{C}, d_{\mathcal{C}})$  with *proper* actions on  $\mathcal{C}$  (that is, such that the quotient topological space  $\mathcal{C}/\Gamma$  is Hausdorff) and prove the following *rigidity* result:

**THEOREM.** *Let  $\mathcal{C}$  be a bounded open convex domain in  $\mathbb{R}^n$  whose boundary  $\partial\mathcal{C}$  is a hypersurface of class  $C^3$  which is strictly convex (in the sense the Hessian is positive*

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definite). Then if  $\partial\mathcal{C}$  is not an ellipsoid, any subgroup of the isometry group  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  whose action on  $\mathcal{C}$  is proper is finite.

We then deduce an important consequence involving the volume measure associated to the Finsler metric  $F_{\mathcal{C}}$  on  $\mathcal{C}$  which is defined by  $\mu(f) = \int_{\mathcal{C}} f(p)\sigma(p)dp$  for every continuous function  $f : \mathcal{C} \rightarrow \mathbb{R}$  with compact support, where  $\sigma(p)$  is the Finsler density for the metric  $F_{\mathcal{C}}$ . Recall here that for any Finsler manifold  $(F, M)$  the density function  $\sigma$  over  $M$  is defined as follows. Given an arbitrary Riemannian metric  $g$  on  $M$ , for all  $p \in M$  the value  $\sigma(p)$  is equal to the square root of the ratio of the *Euclidean* volume of the ball  $\{v \in T_p M : g(p) \cdot (v, v) \leq 1\}$  by the *Euclidean* volume of the ball  $\{v \in T_p M : F(p, v) \leq 1\}$  in the Euclidean vector space  $(T_p M, g(p))$ . This definition does not depend on the choice of  $g$  and it generalises the well known Riemannian density (indeed, if  $F$  is Riemannian, then  $\sigma$  is nothing else than the famous  $\sqrt{\det(g_{ij})}$ ). For more information about the Finsler density and some problems related to this notion, one can have a look at [14].

**COROLLARY.** *Let  $\mathcal{C}$  be as above. Then if  $\partial\mathcal{C}$  is not an ellipsoid,  $(\mathcal{C}, d_{\mathcal{C}})$  does not allow quotients of finite volume (and thus compact quotients) by subgroups of  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  whose actions on  $\mathcal{C}$  are proper.*

Hilbert metrics are important objects that people have been interested in for many reasons. First of all, they give a range of basic and rich enough examples of geodesic metric spaces (see the investigations of Busemann in [3, 4]). They next generalise in Finsler geometry the Riemannian hyperbolic spaces which are obtained in the case  $\partial\mathcal{C}$  is an ellipsoid and correspond to Klein's model (see [3, 4]). On the other hand, Hilbert geometries are considered in affine geometry from the 'projective transformation groups' point of view (see [9] and the recent preprint of Benoist [1]). The question we are dealing with in the present paper has already been tackled in the literature for the *compact* case. Namely, the non-existence of compact quotients has been proved in the framework of affine geometry by Benzécri in [2] (see also the illuminating Lecture Notes of Goldman [9]). Using a dynamical systems approach, a proof of this result has also been given by Egloff ([5, Theorem 3.59]) in dimension two and by Foulon in higher dimensions as a consequence of a more general rigidity theorem (see [7, 8]). The situation when the boundary of  $\mathcal{C}$  is no longer *strictly* convex has been investigated in [5, Proposition 3.2], [9] and [10], and it is known that if  $\mathcal{C}$  is for example a triangle, there *exist* co-compact subgroups of  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ .

The starting point of the present paper was to know whether there are quotients of  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  with finite *volume* and the strategy we use here to prove that in general such quotients do *not* exist involves very basic tools in metric and Finsler geometry. Indeed, the key idea of the proof consists in showing that the closer to  $\partial\mathcal{C}$  we are, the less the metric  $d_{\mathcal{C}}$  on  $\mathcal{C}$  is different from a Riemannian one (this is given by Proposition 1.3 below). Therefore, if an infinite group of isometries for  $d_{\mathcal{C}}$  had a proper action on  $\mathcal{C}$ , then every point would be sent to the boundary  $\partial\mathcal{C}$  and the Hilbert metric would be as close as

desired to a Riemannian metric and thus Riemannian itself. Then we conclude  $\partial\mathcal{C}$  is an ellipsoid by combining results due to different authors ([3, 11]) in the following general statement:

**THEOREM.** *Let  $\mathcal{C}$  be any bounded open convex domain in  $\mathbb{R}^n$ . If the metric  $d_{\mathcal{C}}$  is Riemannian, then  $\partial\mathcal{C}$  is an ellipsoid.*

To be complete let us point out that in December 1999 we discussed a first version of this work (see [15]) with Édith Socié-Méthou who proposed to use an approach for this problem that is different from ours and that she later developed in her Ph.D. thesis ([12]) and in [13].

## 1. GEOMETRY AT INFINITY

In this part, we give two independent technical lemmas before proving that for a bounded open convex domain  $\mathcal{C} \subset \mathbb{R}^n$  with a strictly convex boundary of class  $C^3$  the Hilbert metric  $d_{\mathcal{C}}$  is asymptotically Riemannian at infinity (that is, when moving towards the boundary  $\partial\mathcal{C}$ ). Our approach for doing this consists in a local approximation of  $\partial\mathcal{C}$  by a parabola (Lemma 1.1) in order to show the geometry of  $(\mathcal{C}, d_{\mathcal{C}})$  behaves near  $\partial\mathcal{C}$  as if it were Klein's geometry in the Euclidean unit ball near its boundary (Lemma 1.2). We then give Proposition 1.3.

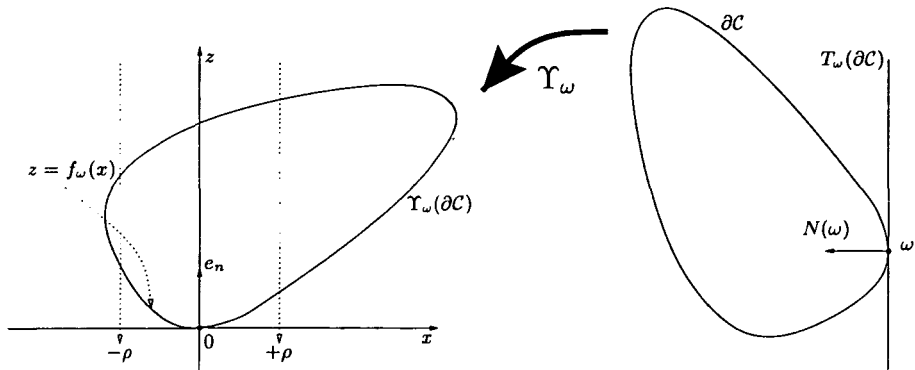
Throughout the section, the canonical Euclidean norms in  $\mathbb{R}^{n-1}$  and  $\mathbb{R}^n$  will be both denoted by  $\|\cdot\|$  and the open ball in  $(\mathbb{R}^{n-1}, \|\cdot\|)$  centred at 0 with radius  $r > 0$  by  $B(0, r)$ . On the other hand,  $(e_1, \dots, e_n)$  will be the canonical basis in  $\mathbb{R}^n$  and  $\langle \cdot | \cdot \rangle$  the canonical scalar product.

The first lemma shows that for each point  $\omega \in \partial\mathcal{C}$  there is a ball in  $(\mathbb{R}^n, \|\cdot\|)$  with centre  $\omega$  and radius *independent* of  $\omega$  in which the boundary of  $\mathcal{C}$  can be written as the graph of a function defined in  $\mathbb{R}^{n-1}$  (after an appropriate coordinate change depending on  $\omega$ ) having the property that its value at  $x$  lies between  $(1 - M\|x\|)\|x\|^2$  and  $(1 + M\|x\|)\|x\|^2$ , where  $M > 0$  is a number which does *not* depend on  $\omega$  but *only* on  $\mathcal{C}$ .

**LEMMA 1.1.** *Let  $\mathcal{C}$  be a bounded open convex domain in  $\mathbb{R}^n$  whose boundary  $\partial\mathcal{C}$  is a hypersurface of class  $C^3$  that is strictly convex (in the sense the Hessian is positive definite) and denote  $N : \mathcal{C} \rightarrow \mathbb{R}^n$  the normal vector field over  $\partial\mathcal{C}$  pointing inwards.*

*Then there are positive constants  $0 < M$  and  $0 < \rho \leq 1/M$  with a family  $(\Upsilon_{\omega})_{\omega \in \partial\mathcal{C}}$  of affine isometries in  $(\mathbb{R}^n, \|\cdot\|)$  together with a family  $(f_{\omega})_{\omega \in \partial\mathcal{C}}$  of functions defined in  $B(0, \rho) \subset \mathbb{R}^{n-1}$  such that for each  $\omega \in \partial\mathcal{C}$  we have the following:*

- (i)  $\Upsilon_{\omega}(\mathcal{C}) \subset \{(x, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid z > 0\}$ ,  $\Upsilon_{\omega}(\omega) = 0$  and  $\vec{\Upsilon}_{\omega} \cdot N(\omega) = e_n$ , where  $\vec{\Upsilon}_{\omega}$  is the linear part of  $\Upsilon_{\omega}$ .
- (ii) For all  $(x, z) \in B(0, \rho) \times \mathbb{R}$ ,  $z = f_{\omega}(x) \implies (x, z) \in \Upsilon_{\omega}(\partial\mathcal{C})$ .
- (iii) For all  $x \in B(0, \rho)$ ,  $(1 - M\|x\|)\|x\|^2 \leq f_{\omega}(x) \leq (1 + M\|x\|)\|x\|^2$ .



PROOF: For each  $\omega \in \partial\mathcal{C}$  let us define the ‘shape’  $\mathcal{S}_\omega(\mathcal{C})$  of  $\mathcal{C}$  with respect to  $\omega$  as the orthogonal projection of  $\mathcal{C}$  onto the affine tangent space  $T_\omega(\partial\mathcal{C})$  and denote  $e(\omega) \geq 0$  the Euclidean distance from  $\omega$  to  $\partial(\mathcal{S}_\omega(\mathcal{C}))$ . If there were a sequence of points  $(\omega_k)_{k \geq 0}$  in  $\partial\mathcal{C}$  such that  $\lim_{k \rightarrow +\infty} e(\omega_k) = 0$ , by compactness of  $\partial\mathcal{C}$  we could find  $\omega$  in this boundary with  $e(\omega) = 0$ . But this is not possible because  $\partial\mathcal{C}$  is differentiable at  $\omega$ . So there is a constant  $r > 0$  such that  $e(\omega) \geq 2r$  for all  $\omega \in \partial\mathcal{C}$ .

Moreover, if we introduce the ball  $\mathcal{B}_\omega = \{m \in T_\omega(\partial\mathcal{C}) : \|m - \omega\| < 2r\}$  in  $T_\omega(\partial\mathcal{C})$ , the intersection of the full open cylinder  $\{m + sN(\omega) : m \in \mathcal{B}_\omega, s \in \mathbb{R}\}$  with  $\partial\mathcal{C}$  has exactly two connected components one of which contains  $\omega$  and that we will denote  $\mathcal{U}_\omega$ . We then immediately get from the convexity of  $\mathcal{C}$  that for each  $m \in \mathcal{B}_\omega$  there is a unique  $s = \varphi_\omega(m) \in \mathbb{R}$  such that  $m + sN(\omega) \in \mathcal{U}_\omega$  and so we get a map  $\varphi_\omega : \mathcal{B}_\omega \rightarrow \mathbb{R}$  of class  $\mathcal{C}^3$ .

Let us now fix an open set  $\mathcal{W}$  in  $\mathbb{R}^n$  such that  $\Gamma = \mathcal{W} \cap \partial\mathcal{C}$  is non-empty and parallelisable.

We can then find a family  $(\Upsilon_\omega)_{\omega \in \Gamma}$  of affine isometries in  $(\mathbb{R}^n, \|\cdot\|)$  depending smoothly on  $\omega \in \Gamma$  such that  $\Upsilon_\omega(\omega) = 0$  and  $\vec{\Upsilon}_\omega \cdot N(\omega) = e_n$ .

Since  $\mathcal{C}$  is a star-shaped set with respect to one (arbitrary) of its point and has a boundary which is a hypersurface of class  $\mathcal{C}^3$ , there exists a function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  with the same smoothness such that  $\partial\mathcal{C} = F^{-1}(0)$  and the family  $(F_\omega)_{\omega \in \Gamma}$  defined by  $F_\omega = F \circ \Upsilon_\omega^{-1}$  thus satisfies  $\Upsilon_\omega(\partial\mathcal{C}) = F_\omega^{-1}(0)$ .

Next define for each  $\omega \in \Gamma$  the function  $g_\omega : B(0, r) \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  by  $g_\omega(u) = \varphi_\omega(\Upsilon_\omega^{-1}(u, 0))$ ; as we have  $F_\omega(u, g_\omega(u)) = 0$  for all  $u = (u_1, \dots, u_{n-1}) \in B(0, 2r)$ , we get by differentiation that all the third order partial derivatives of  $g_\omega$  with respect to  $u_1, \dots, u_{n-1}$  are rational expressions of the partial derivatives of the function  $F_\omega : (u, z) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \mapsto F_\omega(u, z)$  with respect to  $u_1, \dots, u_{n-1}, z$  computed at the point  $(u, z) = (u, g_\omega(u))$ .

Therefore, by continuity of these partial derivatives with respect to both  $u$  and  $\omega$ , if  $\mathcal{O}$  is an open set in  $\mathbb{R}^n$  whose closure is in  $\mathcal{W}$ , there is a constant  $A > 0$  such that for all

$\omega \in \mathcal{O} \cap \partial\mathcal{C}$  and all  $u \in \overline{B}(0, r)$  (closed ball) we have from the Taylor expansion

$$\sum_{i=1}^{n-1} \lambda_i(\omega) u_i^2 - A\|u\|^3 \leq g_\omega(u) \leq \sum_{i=1}^{n-1} \lambda_i(\omega) u_i^2 + A\|u\|^3,$$

where  $\lambda_1(\omega), \dots, \lambda_{n-1}(\omega)$  are the principal curvatures of  $\partial\mathcal{C}$  at the point  $\omega \in \partial\mathcal{C}$ .

The strict convexity and the compactness of  $\partial\mathcal{C}$  together with the continuity of  $\lambda_1, \dots, \lambda_{n-1}$  over  $\partial\mathcal{C}$  imply there exists  $\alpha > 0$  such that  $0 < \lambda_i(\omega) \leq 1/\alpha^2$  for all  $\omega \in \partial\mathcal{C}$  and all  $i \in \{1, \dots, n-1\}$ ; so we can define  $f_\omega(x) = g_\omega(\sqrt{\lambda_1(\omega)}x_1, \dots, \sqrt{\lambda_{n-1}(\omega)}x_{n-1})$  for each  $\omega \in \mathcal{O} \cap \partial\mathcal{C}$  and all  $x = (x_1, \dots, x_{n-1})$  in  $B(0, \rho) \subset \mathbb{R}^{n-1}$ , where  $\rho = \min\{r\alpha, \alpha^3/A\} > 0$ . We then get a family of functions  $(f_\omega)_{\omega \in \mathcal{O} \cap \partial\mathcal{C}}$  that satisfies (ii) and (iii) of Lemma 1.1 with  $M = A/\alpha^3 > 0$ .

Finally, as  $\partial\mathcal{C}$  is compact, it can be recovered by a finite number of open sets like  $\mathcal{O}$  above and this proves Lemma 1.1.  $\square$

The aim of the second lemma is to give us an estimate of the value  $F_{\mathcal{C}}(m_0, v)$  of the Finsler metric  $F_{\mathcal{C}}$  at a point  $m_0 \in \mathcal{C}$  for a vector  $v \in \mathbb{R}^n$  in terms of the Euclidean distance from  $m_0$  to  $\partial\mathcal{C}$  and the direction of  $v$ . More precisely, let us consider a bounded open convex domain  $K$  in  $\mathbb{R}^2$  such that there exist positive numbers  $M > 0$ ,  $r \geq M + 2$  and a function  $\xi : (-r, r) \rightarrow \mathbb{R}$  which satisfies  $\text{Graph}(\xi) \subset \partial K$  together with

$$(1) \quad \frac{1}{4}a(r)t^2 \leq \xi(t) \leq \frac{1}{4}b(r)t^2 \text{ for all } |t| < r,$$

where  $a(r) = 1 - M/r > 0$  and  $b(r) = 1 + M/r > 0$ . For each  $c \in \mathbb{R}$  denote by  $\mathcal{P}_c$  the parabola in  $\mathbb{R}^2$  whose equation is  $y = ct^2/4$ . Then define  $H(r) \in \mathbb{R}$  so that the point  $(r, H(r) + 1)$  is in  $\mathcal{P}_{a(r)}$  and let  $h(r) \in \mathbb{R}$  so that  $h(r) + 1$  is the second component of the intersection point between  $\mathcal{P}_{b(r)}$  and the straight line passing through  $m_0 = (0, 1)$  and  $(r, H(r) + 1)$ . Hence,

$$H(r) = \frac{1}{4}a(r)r^2 - 1 \text{ and } h(r) = \frac{2H(r)}{b(r)r^2} \left( H(r) + \sqrt{H(r)^2 + b(r)r^2} \right).$$

As  $r \geq M + 2$ , we have  $H(r) > 0$  and  $h(r) > 0$ , which implies the point  $m_0$  is in  $K$  by (1) and the convexity of  $K$ .

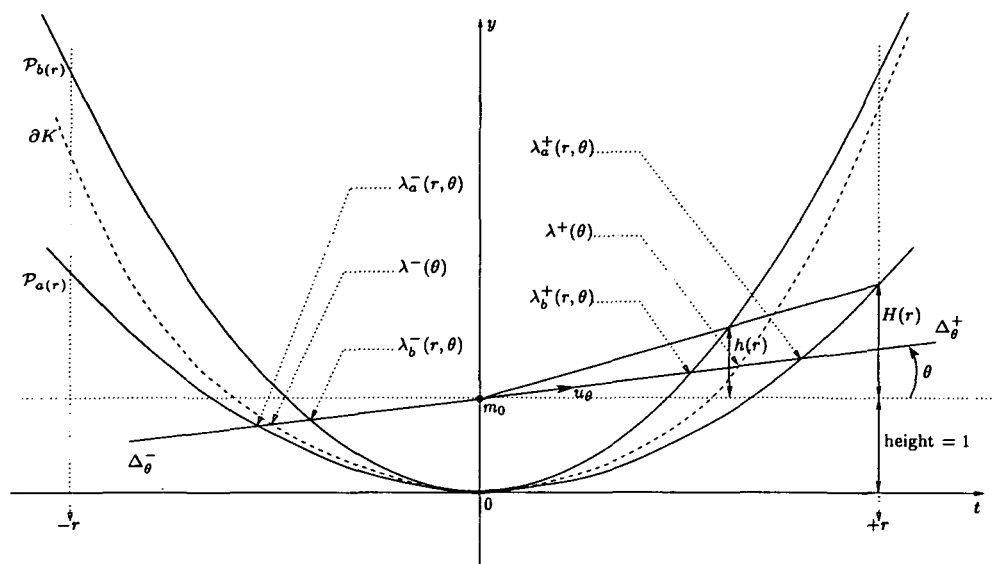
For  $\theta \in (-\pi/2, \pi/2)$  we can therefore consider  $F_K(m_0, u_\theta)$ , where  $u_\theta = (\cos \theta, \sin \theta)$ , and control this quantity in terms of  $r \in [M + 2, +\infty)$  uniformly in  $\theta$ :

**LEMMA 1.2.** *For all  $\theta \in (-\pi/2, \pi/2)$  we have*

$$\varphi(r) \leq F_K(m_0, u_\theta) \leq \psi(r),$$

where  $\varphi$  and  $\psi$  are functions with the property that  $\lim_{r \rightarrow +\infty} \varphi(r) = \lim_{r \rightarrow +\infty} \psi(r) = 1$ .

**REMARK.** This result asserts that the bigger  $r$  is (and then the closer is  $\partial K$  to the parabola with equation  $y = t^2/4$  within  $(-r, r) \times \mathbb{R}$ ), the closer to the unit canonical Euclidean sphere is the Finsler sphere  $\{v \in \mathbb{R}^2 : F_K(m_0, v) = 1\}$  at the point  $m_0$ .



PROOF: For  $\theta \in (-\pi/2, \pi/2)$  introduce the half lines  $\Delta_\theta^+ = m_0 + \mathbb{R}^+ u_\theta$  and  $\Delta_\theta^- = m_0 + \mathbb{R}^- u_\theta$  and let  $\lambda_a^+(r, \theta)$  (respectively  $\lambda_a^-(r, \theta)$ ) be the Euclidean distance between  $m_0$  and the intersection point  $\Delta_\theta^+ \cap \mathcal{P}_{a(r)}$  (respectively  $\Delta_\theta^- \cap \mathcal{P}_{a(r)}$ ). Define  $\lambda_b^+(r, \theta)$  (respectively  $\lambda_b^-(r, \theta)$ ) in a similar way for  $\mathcal{P}_{b(r)}$ , and denote by  $\lambda^+(\theta)$  (respectively  $\lambda^-(\theta)$ ) the Euclidean distance between  $m_0$  and the intersection point  $\Delta_\theta^+ \cap \partial K$  (respectively  $\Delta_\theta^- \cap \partial K$ ).

Using these notations, we then have  $F_K(m_0, u_\theta) = (1/\lambda^-(\theta)) + (1/\lambda^+(\theta))$  from the definition of  $F_K$  (see Introduction) and a straightforward computation gives

$$(2) \quad \lambda_a^+(r, \theta) = \frac{2}{\sqrt{\sin^2 \theta + a(r) \cos^2 \theta} - \sin \theta} \quad \text{and} \quad \lambda_a^-(r, \theta) = \frac{2}{\sqrt{\sin^2 \theta + a(r) \cos^2 \theta} + \sin \theta}$$

with the analogous formulas for  $\lambda_b^+(r, \theta)$  and  $\lambda_b^-(r, \theta)$  by changing  $a(r)$  into  $b(r)$ .

There are now three cases to be considered.

FIRST CASE.  $|\tan \theta| \leq H(r)/r$ .

In this situation the half line  $\Delta_\theta^+$  (respectively  $\Delta_\theta^-$ ) cuts  $\partial K$  on the curve  $\{(t, \xi(t)) : |t| < r\}$  and we have by (1)

$$\lambda_a^-(r, \theta) \leq \lambda^-(\theta) \leq \lambda_b^-(r, \theta) \quad \text{and} \quad \lambda_b^+(r, \theta) \leq \lambda^+(\theta) \leq \lambda_a^+(r, \theta)$$

which implies

$$(3) \quad \frac{1}{\lambda_a^+(r, \theta)} + \frac{1}{\lambda_b^-(r, \theta)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq \frac{1}{\lambda_a^-(r, \theta)} + \frac{1}{\lambda_b^+(r, \theta)}.$$

Since  $0 < a(r) \leq b(r)$ , we then have by (2) the inequality

$$\begin{aligned} \frac{1}{\lambda_a^+(r, \theta)} + \frac{1}{\lambda_b^-(r, \theta)} &\geq \sqrt{\sin^2 \theta + a(r) \cos^2 \theta} \\ &= \sqrt{1 + (a(r) - 1) \cos^2 \theta} = \sqrt{1 - (M \cos^2 \theta)/r} \end{aligned}$$

and thus

$$(4) \quad \frac{1}{\lambda_a^+(r, \theta)} + \frac{1}{\lambda_b^-(r, \theta)} \geq \sqrt{a(r)}.$$

On the other hand,

$$\sqrt{\sin^2 \theta + b(r) \cos^2 \theta} = \sqrt{b(r) + (1 - b(r)) \sin^2 \theta} = \sqrt{b(r) - (M \cos^2 \theta)/r} \leq \sqrt{b(r)},$$

which leads to

$$(5) \quad \frac{1}{\lambda_a^-(r, \theta)} + \frac{1}{\lambda_b^+(r, \theta)} \leq \sqrt{b(r)}.$$

Finally, from (3), (4) and (5) we get

$$\sqrt{a(r)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq \sqrt{b(r)}.$$

SECOND CASE.  $\tan \theta > H(r)/r$ .

The convexity of  $K$  and (1) imply the intersection point  $\Delta_\theta^+ \cap \partial K$  lies in  $[0, +\infty) \times [h(r) + 1, +\infty)$  and therefore  $\lambda^+(\theta) \geq h(r)$ . As the point  $\Delta_\theta^- \cap \partial K$  is still on the curve  $\{(t, \xi(t)) : |t| < r\}$ , we can write by condition (1) that  $\lambda_a^-(r, \theta) \leq \lambda^-(\theta) \leq \lambda_b^-(r, \theta)$  in order to get

$$(6) \quad \frac{1}{\lambda_b^-(r, \theta)} \leq \frac{1}{\lambda^-(\theta)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{h(r)} \leq \frac{1}{\lambda_a^-(r, \theta)} + \frac{1}{h(r)}.$$

From  $\tan \theta > H(r)/r$  (together with  $\theta \in (-\pi/2, \pi/2)$ ), we have  $\sin \theta > (1/\sqrt{1 + H(r)^2/r^2})$  and then

$$(7) \quad \frac{1}{\lambda_b^-(r, \theta)} = \frac{1}{2} \left( \sin \theta + \sqrt{\sin^2 \theta + b(r) \cos^2 \theta} \right) \geq \sin \theta \geq \frac{1}{\sqrt{1 + H(r)^2/r^2}}.$$

On the other hand,

$$(8) \quad \begin{aligned} \frac{1}{\lambda_a^-(r, \theta)} &= \frac{1}{2} \left( \sin \theta + \sqrt{\sin^2 \theta + a(r) \cos^2 \theta} \right) \\ &\leq \frac{1}{2} \left( 1 + \sqrt{1 + (a(r) - 1) \cos^2 \theta} \right) = \frac{1}{2} \left( 1 + \sqrt{1 - (M \cos^2 \theta)/r} \right) \leq 1. \end{aligned}$$

Hence, (6), (7) and (8) imply

$$\frac{1}{\sqrt{1 + H(r)^2/r^2}} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq 1 + \frac{1}{h(r)}.$$

THIRD CASE.  $\tan \theta < -H(r)/r$ .

Applying the same arguments as previously (where the roles of  $\Delta_\theta^-$  and  $\Delta_\theta^+$  have been exchanged), we also get here

$$\frac{1}{\sqrt{1+H(r)^2/r^2}} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq 1 + \frac{1}{h(r)}.$$

CONCLUSION. If we define  $\varphi(r) = \min\left\{1/\sqrt{1+H(r)^2/r^2}, \sqrt{a(r)}\right\}$  and  $\psi(r) = \max\left\{1+(1/h(r)), \sqrt{b(r)}\right\}$ , the three cases above say that for all  $\theta \in (-\pi/2, \pi/2)$  we have

$$\varphi(r) \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} = F_K(m_0, u_\theta) \leq \psi(r),$$

which proves Lemma 1.2 since  $\lim_{r \rightarrow +\infty} \varphi(r) = \lim_{r \rightarrow +\infty} \psi(r) = 1$ .  $\square$

Considering a convex set  $\mathcal{C}$  as described at the beginning of the present section, we are now able to prove the key idea of this paper which states that the closer to  $\partial\mathcal{C}$  we are, the less the metric  $d_{\mathcal{C}}$  on  $\mathcal{C}$  is different from a Riemannian one. In other words, we show the closer to  $\partial\mathcal{C}$  a point  $p \in \mathcal{C}$  is, the closer to an ellipsoid centred at  $p$  is the unit sphere  $\{v \in \mathbb{R}^n : F_{\mathcal{C}}(p, v) = 1\}$  of the norm  $F_{\mathcal{C}}(p, \cdot)$  in  $T_p(\mathcal{C}) \simeq \mathbb{R}^n$ .

**PROPOSITION 1.3.** *Let  $\mathcal{C}$  be a bounded open convex domain in  $\mathbb{R}^n$  whose boundary  $\partial\mathcal{C}$  is a hypersurface of class  $C^3$  that is strictly convex. For any  $p \in \mathcal{C}$  let  $\delta(p) > 0$  be the Euclidean distance from  $p$  to  $\partial\mathcal{C}$ . Then there exists a family  $(\tilde{\ell}_p)_{p \in \mathcal{C}}$  of linear transformations in  $\mathbb{R}^n$  such that*

$$\lim_{\delta(p) \rightarrow 0} \frac{F_{\mathcal{C}}(p, v)}{\|\tilde{\ell}_p(v)\|} = 1 \text{ uniformly in } v \in \mathbb{R}^n \setminus \{0\}.$$

REMARK. The proposition means that the unit sphere of the norm  $F_{\mathcal{C}}(p, \cdot)$  approaches the ellipsoid defined by the unit sphere of the Euclidean norm  $\|\cdot\| \circ \tilde{\ell}_p$  in  $\mathbb{R}^n$  as  $\delta(p)$  goes to zero.

PROOF: Let  $p \in \mathcal{C}$  sufficiently close to  $\partial\mathcal{C}$  such that  $\delta = \delta(p) < \rho$  (see Lemma 1.1) and that there is a unique  $\omega \in \partial\mathcal{C}$  satisfying  $\|p - \omega\| = \delta$ .

Define  $\Phi_p \in \text{GL}(\mathbb{R}^n)$  by  $\Phi_p(x, z) = (X, Z) = (2x/\sqrt{\delta}, z/\delta)$  from  $\mathbb{R}^{n-1} \times \mathbb{R}$  to  $\mathbb{R}^{n-1} \times \mathbb{R}$ ; it sends  $\Upsilon_\omega(p) = \delta e_n$  to  $m_0 = (0, \dots, 0, 1)$  and changes inequality (iii) in Lemma 1.1 applied to  $\omega$  into

$$\begin{aligned} \frac{1}{4}(1 - M\sqrt{\delta}\|X\|/2)\|X\|^2 &\leq f_\omega(\sqrt{\delta}X)/\delta \\ &\leq \frac{1}{4}(1 + M\sqrt{\delta}\|X\|/2)\|X\|^2 \text{ for all } X \in B(0, 1/\delta^{1/4}). \end{aligned}$$

Hence we deduce

$$(9) \quad \frac{1}{4}(1 - M\delta^{1/4})\|X\|^2 \leq f_\omega(\sqrt{\delta}X)/\delta \leq \frac{1}{4}(1 + M\delta^{1/4})\|X\|^2 \text{ for all } X \in B(0, 1/\delta^{1/4}).$$

Now introduce  $r = 1/\delta^{1/4}$ , fix  $k \in \{1, \dots, n-1\}$  and consider the function  $\xi(t) = f_\omega(\sqrt{\delta}te_k)/\delta$  for  $t \in (-r, r)$ ; in restriction to the 2-plane  $\Pi_k = \{te_k + Ze_n : (t, Z) \in \mathbb{R}^2\}$ , condition (9) means

$$(10) \quad \frac{1}{4}a(r)t^2 \leq \xi(t) \leq \frac{1}{4}b(r)t^2 \text{ for all } |t| < r,$$

where  $a(r) = 1 - M/r$  and  $b(r) = 1 + M/r$ .

If we then define  $\ell_p \in \text{Aff}(\mathbb{R}^n)$  by  $\ell_p = \Phi_p \circ \Upsilon_\omega$  and focus on the convex set  $K = \partial(\ell_p(C)) \cap \Pi_k$ , we are exactly in the situation of Lemma 1.2 (where condition (1) is given by (10) above) from which we get

$$\varphi(r) \leq \mathcal{F}_p(m_0, \cos \theta e_k + \sin \theta e_n) \leq \psi(r) \text{ for all } \theta \in (-\pi/2, \pi/2),$$

where  $\mathcal{F}_p$  is the Finsler metric associated to  $\ell_p(C)$  and  $\varphi, \psi$  are functions such that  $\lim_{r \rightarrow +\infty} \varphi(r) = \lim_{r \rightarrow +\infty} \psi(r) = 1$ .

As  $\mathcal{F}_p(m_0, \cdot)$  is a symmetric, continuous and positive homogeneous function over  $\mathbb{R}^n$ , the last inequalities mean

$$\varphi(r)\|v\| \leq \mathcal{F}_p(m_0, v) \leq \psi(r)\|v\| \text{ for all } v \in \Pi_k.$$

But  $k \in \{1, \dots, n-1\}$  has been chosen arbitrarily and so we actually have

$$(11) \quad \varphi(r)\|v\| \leq \mathcal{F}_p(m_0, v) \leq \psi(r)\|v\| \text{ for all } v \in \mathbb{R}^n$$

because  $\varphi(r)$  and  $\psi(r)$  do not depend on  $k$ .

Since  $\ell_p$  is an affine transformation, it is an isometry from  $(C, d_C)$  to  $(\ell_p(C), d_{\ell_p(C)})$  or equivalently an isometry from  $(C, F_C)$  to  $(\ell_p(C), \mathcal{F}_p)$ , that is  $\mathcal{F}_p(\ell_p(m), \tilde{\ell}_p(v)) = F_C(m, v)$  for all  $m \in C$  and all  $v \in \mathbb{R}^n$ , where  $\tilde{\ell}_p$  is the linear part of  $\ell_p$ .

Therefore, as  $\ell_p(p) = m_0$ , we finally obtain

$$\varphi(r)\|\tilde{\ell}_p(v)\| \leq F_C(p, v) \leq \psi(r)\|\tilde{\ell}_p(v)\| \text{ for all } v \in \mathbb{R}^n$$

and Proposition 1.3 follows since  $r = 1/\delta^{1/4} \rightarrow +\infty$  as  $\delta \rightarrow 0$ . □

## 2. PROOF OF THE THEOREM

We give in this section the proof of the central *rigidity* result announced in the Introduction as a consequence of Proposition 1.3 for a bounded open convex domain  $C$  in  $\mathbb{R}^n$  with a strictly convex boundary of class  $C^3$ .

**THEOREM 2.1.** *If  $\partial C$  is not an ellipsoid, then any subgroup  $\Gamma$  of the isometry group  $\text{Isom}(C, d_C)$  with a proper action on  $C$  is finite.*

Before showing this result, we establish the following lemma:

**LEMMA 2.2.** *Let  $\mathcal{C}$  be any bounded open strictly convex domain in  $\mathbb{R}^n$  and  $\Gamma$  any subgroup of  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  with a proper action on  $\mathcal{C}$ . Then if there is a point in  $\mathcal{C}$  with a finite orbit,  $\Gamma$  has to be finite too.*

**PROOF OF LEMMA 2.2:** We shall consider  $\mathcal{C}$  in  $P(\mathbb{R}^{n+1})$  via the classical imbedding  $\mathbb{R}^n \hookrightarrow P(\mathbb{R}^{n+1})$  (see for example [9, Section 2]). Then the strict convexity of  $\mathcal{C}$  implies that  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}}) < \text{PGL}(\mathbb{R}^{n+1})$  according to [10, Proposition 3]. Let  $p_0 \in \mathcal{C}$  with a finite orbit  $\Gamma \cdot p_0$  and pick  $n+1$  points  $p_1, \dots, p_{n+1}$  in  $\mathcal{C}$  such that  $(p_0, p_1, \dots, p_{n+1})$  is a projective frame. Next define the sequence  $(\Gamma_j)_{0 \leq j \leq n+1}$  of subgroups in  $\Gamma$  by

$$\Gamma_0 = \text{Stab}_{\Gamma}(p_0) \text{ and } \Gamma_j = \text{Stab}_{\Gamma_{j-1}}(p_j) \text{ for all } j \in \{1, \dots, n+1\},$$

where  $\text{Stab}_G(x)$  denotes the stabiliser of the point  $x$  under the action of the group  $G$ . We therefore have  $\Gamma_{n+1} < \dots < \Gamma_1 < \Gamma_0 < \Gamma$  with  $\Gamma_{n+1} = \{Id_{\mathcal{C}}\}$  because  $\Gamma_{n+1} < \text{PGL}(\mathbb{R}^{n+1})$  fixes the projective frame  $(p_0, p_1, \dots, p_{n+1})$ .

As for each  $j \in \{0, \dots, n\}$  we have  $d_{\mathcal{C}}(p_j, \gamma \cdot p_{j+1}) = d_{\mathcal{C}}(p_j, p_{j+1})$  for all  $\gamma \in \Gamma_j$ , the orbit  $\Gamma_j \cdot p_{j+1}$  lies in a compact set and thus if it were infinite, it would have an accumulation point; but this is not possible because the action is proper. Hence,  $\Gamma_j \cdot p_{j+1}$  is finite for all  $j \in \{0, \dots, n\}$ .

Since  $\Gamma/\Gamma_0 \cong \Gamma \cdot p_0$  and  $\Gamma_j/\Gamma_{j+1} \cong \Gamma_j \cdot p_{j+1}$  (as sets) for all  $j \in \{0, \dots, n\}$ , we finally get that all the indexes  $[\Gamma : \Gamma_0], [\Gamma_0 : \Gamma_1], \dots, [\Gamma_n : \Gamma_{n+1}]$  are finite. So,  $\Gamma$  is finite.  $\square$

**PROOF OF THEOREM 2.1:** Consider an infinite subgroup  $\Gamma$  of the isometry group  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  whose action on  $\mathcal{C}$  is proper and choose a point  $p$  in  $\mathcal{C}$ . As  $\Gamma$  is not finite, the orbit  $\Gamma \cdot p$  is infinite by Lemma 2.2 and thus if it were contained in a compact set, it would have an accumulation point which is not possible since the action is proper. So the Euclidean distance between  $\Gamma \cdot p$  and  $\partial\mathcal{C}$  is zero, which means there exists a sequence  $(\gamma_k)_{k \geq 0}$  in  $\Gamma$  such that the limit of the Euclidean distance between  $p_k = \gamma_k \cdot p$  and  $\partial\mathcal{C}$  is zero as  $k$  goes to infinity. Then, from Proposition 1.3, there is a sequence  $(\vec{\ell}_k)_{k \geq 0}$  in  $\text{GL}(\mathbb{R}^n)$  which satisfies

$$\lim_{k \rightarrow +\infty} \frac{F_{\mathcal{C}}(p_k, T_{p_k} \gamma_k(v))}{\|\vec{\ell}_k(T_{p_k} \gamma_k(v))\|} = 1 \text{ uniformly in } v \in \mathbb{R}^n \setminus \{0\},$$

where  $\|\cdot\|$  still denotes the canonical Euclidean norm in  $\mathbb{R}^n$ .

But since  $\gamma_k \in \text{Isom}(\mathcal{C}, d_{\mathcal{C}})$ , we have  $F_{\mathcal{C}}(p_k, T_{p_k} \gamma_k(v)) = F_{\mathcal{C}}(p, v)$  and the equality above writes

$$\lim_{k \rightarrow +\infty} \|\mathcal{L}_k(v)\| = F_{\mathcal{C}}(p, v) \text{ uniformly in } v \in \mathbb{R}^n \setminus \{0\}$$

with  $\mathcal{L}_k = \vec{\ell}_k \circ T_{p_k} \gamma_k$ .

As this limit is uniform in  $v$ , the sequence  $(\mathcal{L}_k)_{k \geq 0}$  is bounded in  $L(\mathbb{R}^n)$  (the space of linear endomorphisms of  $\mathbb{R}^n$  endowed with the operator norm associated to  $\|\cdot\|$ ) and we can therefore assume it converges to a limit  $\mathcal{L} \in L(\mathbb{R}^n)$ . We then obtain  $F_{\mathcal{C}}(p, v) = \|\mathcal{L}(v)\|$

for all  $v \in \mathbb{R}^n$  (hence  $\mathcal{L} \in \text{GL}(\mathbb{R}^n)$ ), which means the Finsler metric  $F_{\mathcal{C}}$  is Riemannian at the point  $p \in \mathcal{C}$  (the norm  $\|\cdot\| \circ \mathcal{L}$  is indeed Euclidean in  $\mathbb{R}^n$ ) and since this is true for every choice of  $p$ , we get that  $F_{\mathcal{C}}$  (or equivalently the corresponding Hilbert metric  $d_{\mathcal{C}}$ ) is a Riemannian metric on  $\mathcal{C}$ .  $\square$

At this stage, recall the following fact (see for example [3, page 85]):

**THEOREM 2.3** (E. Beltrami, 1866). *Let a connected open set  $X$  of the projective space  $P(\mathbb{R}^{n+1})$  be metrised so that the metric is Riemannian and the geodesics lie on projective lines. Then the sectional curvature of this Riemannian metric is constant.*

Using this, we deduce the very useful result:

**THEOREM 2.4.** *Let  $\mathcal{C}$  be any bounded open convex domain in  $\mathbb{R}^n$ . If the metric  $d_{\mathcal{C}}$  is Riemannian, then  $\partial\mathcal{C}$  is an ellipsoid.*

PROOF OF THEOREM 2.4: From Theorem 2.3 with  $X = \mathcal{C}$ , the sectional curvature of the Riemannian metric  $d_{\mathcal{C}}$  is constant and thus non-positive since the space  $(\mathcal{C}, d_{\mathcal{C}})$  is not compact. Then, by a theorem of Busemann ([3, page 269, Theorem 41.6]), the metric space  $(\mathcal{C}, d_{\mathcal{C}})$  has non-positive curvature in the sense of Busemann (see [3, page 237 for the definition]) and this finally implies  $\partial\mathcal{C}$  is an ellipsoid from a result due to Kelly and Straus ([11]).  $\square$

This ends the proof of Theorem 2.1.  $\square$

REMARK. Although Theorem 2.4 seems generally to have been accepted, we are unaware of any proof in the literature.

We now determine whether there are quotients of  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  with finite volume:

**COROLLARY 2.5.** *Let  $\mathcal{C}$  be as in Theorem 2.1. Then if  $\partial\mathcal{C}$  is not an ellipsoid,  $(\mathcal{C}, d_{\mathcal{C}})$  does not allow quotients of finite volume by subgroups of  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  whose actions on  $\mathcal{C}$  are proper.*

PROOF: The volume here is the one  $\mu$  associated to the Finsler metric  $F_{\mathcal{C}}$  on  $\mathcal{C}$  and described in the Introduction. Now if a subgroup  $\Gamma$  of  $\text{Isom}(\mathcal{C}, d_{\mathcal{C}})$  has a proper action on  $\mathcal{C}$ , it is finite since  $\partial\mathcal{C}$  is not an ellipsoid according to Theorem 2.1. So, if there were a fundamental domain  $D \subset \mathcal{C}$  with  $\mu(D)$  finite, we would get that  $\mu(\mathcal{C}) = \sum_{\gamma \in \Gamma} \mu(\gamma \cdot D)$  is finite too; but this is not true. (If we indeed use Lemma 1.1 and Lemma 1.2 over the intersection  $U$  of  $\mathcal{C}$  with a small enough ball in  $(\mathbb{R}^n, \|\cdot\|)$  centred at any given point in  $\partial\mathcal{C}$ , we can get an estimate of the Euclidean volume of the ball  $\{v \in \mathbb{R}^n : F_{\mathcal{C}}(p, v) \leq 1\}$  and compute that  $\mu(U) = +\infty$ .)  $\square$

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