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NORMAL, LOCALLY COMPACT, BOUNDEDLY METACOMPACT SPACES ARE PARACOMPACT: AN APPLICATION OF PIXLEY-ROY SPACES

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1. Introduction. Let $PR(X)$ denote the Pixley-Roy topology on the collection of all nonempty, finite subsets of a space X. For each cardinal κ, let κ^* be the cardinal κ with the co-finite topology. We use $PR(\kappa^*)$ to obtain a partial solution in ZFC to F. Tall's question whether every normal, locally compact, metacompact space is paracompact [**6**]. W.S. Watson has answered this question affirmatively assuming $V = L$ [**7**]. The question also has an affirmative answer if we assume either that the space is perfectly normal [**1**] or that it is locally connected [**4**].

A space X is said to be *boundedly metacompact* (*boundedly paracompact*) provided that for each open cover $\mathscr{U}$ of X there is a positive integer n such that $\mathscr{U}$ has a point finite (locally finite) open refinement of order n. As the main result of this paper, we show every normal, locally compact, boundedly metacompact space is paracompact. Thus, by a theorem of P. Fletcher, R.A. McCoy and R. Slover, such spaces are boundedly paracompact [**3**]. We also show that if there is a normal, zero-dimensional, locally compact, metacompact space that is not paracompact, then there is a cardinal κ, and a subspace of $PR(\kappa^*)$ with the same properties. More generally, we show that if there is a normal, locally compact, metacompact space that is not paracompact, then there is a cardinal κ and a subspace Y of $PR(\kappa^*)$ with the following two properties: (1) any two disjoint subsets of $\{\{\alpha\}: \alpha \in \kappa\}$ can be separated by disjoint open subsets of Y, and (2) $\{\{\alpha\}: \alpha \in \kappa\}$ is a discrete closed subset of Y, the points of which cannot be separated by disjoint open subsets of Y. Finally, we show every zero-dimensional, normal, locally compact, metacompact space is subparacompact.

2. Pixley-Roy spaces on κ^*. First let us recall the definition of the Pixley-Roy topology on the collection of all nonempty, finite subsets of a space X. Given a space X, let $\mathscr{P}(X)$ be the collection of all nonempty, finite subsets of X. For each $A \in \mathscr{P}(X)$ and each open set U of X, let

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$$[A, U] = \{B \in \mathscr{P}(x): A \subseteq B \subseteq U\}.$$

Then $\{ [A, U]: A \in \mathscr{P}(X)$ and U is an open set in $X\}$ forms a basis for a topology on $\mathscr{P}(X)$, called the *Pixley-Roy topology* on $\mathscr{P}(X)$. Let $PR(X)$ denote $\mathscr{P}(X)$ with this topology. It is well known that if X is a T_1-space, then each element of this basis is clopen, and hence $PR(X)$ is completely regular and zero-dimensional. Also, if X is a T_1-space, then $PR(X)$ is hereditarily metacompact [**2**].

As an aid in notation, given a cardinal κ, a set $A \in PR(\kappa^*)$, and a finite subset F of $\kappa - A$, we let $\mathscr{U}(A, F) = [A, \kappa - F]$. Also, for each positive integer n, let $PR_{\leqq n}(\kappa^*)$ denote the subspace of $PR(\kappa^*)$ consisting of all subsets of κ^* of cardinality less than or equal to n. Finally, we consider κ to be a subspace of $PR(\kappa^*)$, that is, we identify $\{\alpha\}$ with α for each $\alpha \in \kappa$. Note that since κ^* is T_1, $PR(\kappa^*)$ is completely regular, zero-dimensional, and hereditarily metacompact.

To help visualize these spaces, let us recall that $PR_{\leqq 2}(\mathbf{R})$ is homeomorphic to R. Heath's tangent-V space, where $\mathbf{R}$ is the set of real numbers with the usual topology. The tangent-V space is not locally compact, but if we extend each edge of a tangent V infinitely far and give it the topology of the one-point compactification of an infinite discrete space, we get a locally compact space. More precisely, for $p = (x, 0)$, let

$$\mathscr{U}(p) = \{p\} \cup \{ (x', y'): y' = x' - x \text{ or } y' = x - x'\}.$$

Then a basic open set containing p is $\mathscr{U}(p) - F$ where F is a finite subset of $\mathscr{U}(p) - \{p\}$. This space is locally compact and metacompact, but not collectionwise-Hausdorff (since the tangent-V space isn't collectionwise-Hausdorff), and hence, it is not paracompact. It turns out that this space also is homeomorphic to a Pixley-Roy space, namely $PR_{\leqq 2}(\mathbf{R}^*)$, where $\mathbf{R}^*$ is the set of all real numbers with the co-finite topology. In fact, the only property of the real numbers affecting the space $PR(\mathbf{R}^*)$ is their cardinality, i.e., $PR(\mathbf{R}^*)$ is homomorphic to $PR(c^*)$, where c^* is $c = 2^\omega$ with the co-finite topology. This led us to consider Pixley-Roy spaces of the form $PR(\kappa^*)$ where κ^* is some cardinal κ with the co-finite topology. We begin with a few more simple facts about $PR(\kappa^*)$.

THEOREM 1. *For each positive integer n and each cardinal* κ, $PR_{\leqq n}(\kappa^*)$ *is locally compact. If* $n \geqq 2$ *and* $\kappa > \omega$, *then* $PR_{\leqq n}(\kappa^*)$ *is not collectionwise-Hausdorff, and hence, not paracompact. In fact,* κ *is a closed discrete subset of* $PR_{\leqq n}(\kappa^*)$ *that cannot be separated in* $PR_{\leqq n}(\kappa^*)$ *by disjoint open sets.*

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Proof. Suppose κ is a cardinal and $n \in \omega$. Suppose $A \in PR_{\leqq n}(\kappa^*)$. We show that $\mathscr{U}(A, \emptyset)$ is compact. Since $\mathscr{U}(A, \emptyset)$ is metacompact, is suffices to show that $\mathscr{U}(A, \emptyset)$ is countably compact. Suppose that

$$\{X_m : m \in \omega\} \subseteq \mathscr{U}(A, \emptyset).$$

Without loss of generality, we may assume that all the elements of this set have the same cardinality, and hence form a Δ-system with root B. It is easy to check that B is a limit point of this set in $\mathscr{U}(A, \emptyset)$, and so $PR_{\leqq n}(\kappa^*)$ is locally compact.

Now let us suppose that $n \geqq 2$ and $\kappa > \omega$. Suppose that

$$\{\mathscr{U}(\alpha, F_\alpha) \cap PR_{\leqq n}(\kappa^*) : \alpha < \kappa\}$$

is a collection of pairwise disjoint open subsets of $PR_{\leqq n}(\kappa^*)$. For each pair of points α and β, it must be the case that either $\beta \in F_\alpha$ or $\alpha \in F_\beta$. For each natural number m, F_m is finite, so for each $\beta \notin F_m$, $m \in F_\beta$. Since $\kappa > \omega$,

$$\kappa - \bigcup_{m \in \omega} F_m \neq \emptyset.$$

Let

$$\beta \in \kappa - \bigcup_{m \in \omega} F_m.$$

But then for each $m \in \omega$, $m \in F_\beta$, a contradiction. So κ cannot be separated in $PR_{\leqq n}(\kappa^*)$ by disjoint open sets.

Let us say that given a space X and a pairwise disjoint collection $\mathscr{A} \subseteq \mathscr{P}(X)$, *$\mathscr{A}$ can be separated in X* provided that there exists a collection $\{U_A : A \in \mathscr{A}\}$ of pairwise disjoint open sets in X such that for each $A \in \mathscr{A}$, $A \subseteq U_A$. Also, *$\mathscr{A}$ is normalized in X* provided that for each $\mathscr{B} \subseteq \mathscr{A}$ there exist disjoint open sets U and V in X such that $\cup \mathscr{B} \subseteq U$ and $\cup (\mathscr{A} - \mathscr{B}) \subseteq V$.

Now we begin to relate the study of Pixley-Roy spaces to F. Tall's question. The next theorem illustrates the close relationship between normal, locally compact, metacompact spaces and subspaces of Pixley-Roy spaces of the form $PR(\kappa^*)$ for various cardinals κ, particularly if the original spaces are zero-dimensional. This theorem is proved using techniques that are very useful in proving the main result.

THEOREM 2. *If there is a normal, locally compact, metacompact space Y that is not paracompact, then there is a cardinal κ and a subspace Z of $PR(\kappa^*)$ with the following properties:*

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(1) *κ is normalized in Z,*

(2) *κ is a closed discrete subset of Z that cannot be separated in Z. Furthermore, if Y is also zero-dimensional, then there is such a subspace Z which is a perfect image of Y, hence also normal, locally compact, metacompact; Z is not paracompact.*

Before proving Theorem 2, we state and prove a lemma useful in proving this theorem and the main result.

LEMMA 3. *Suppose X is normal, locally compact, and (boundedly) metacompact, and $D = \{d_\alpha : \alpha < \kappa\}$ is a discrete closed subset of X. Then there exists a collection $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of open sets with compact closures such that*

(1) *For each $\alpha < \kappa$, $d_\alpha \in U_\alpha$, and if $\beta \neq \alpha$, then $d_\alpha \notin \bar{U}_\beta$, and*

(2) *Each point of X belongs to only finitely many (at most n, for some integer n) elements of $\mathcal{U}$.*

Proof of Lemma 3. Suppose X is normal, locally compact, and metacompact, and $D = \{d_\alpha : \alpha < \kappa\}$ is a discrete closed subset of X. For each $x \in X$, let U_x be an open set with compact closure containing x such that $\bar{U}_x \cap D \subseteq \{x\}$. Since $\{U_x : x \in X\}$ covers X, let $\{V_x : x \in X\}$ be a precise point-finite open refinement of $\{U_x : x \in X\}$. (If X is boundedly metacompact, we may assume each point of X is in at most n elements of the refinement, for some positive integer n.) If α and β are two elements of κ, then $d_\beta \in V_{d_\beta}$, and since $\bar{V}_{d_\beta} \subseteq \bar{U}_{d_\beta}$, $d_\alpha \notin \bar{V}_{d_\beta}$. So $\{V_x : x \in X\}$ has the desired properties.

Proof of Theorem 2. Suppose that every normal, locally compact, metacompact space is collectionwise-Hausdorff. We show that a normal, locally compact, metacompact space X is collectionwise-normal with respect to compact sets and, hence, is paracompact. Take a discrete collection of compact subsets of X, say $\{H_\alpha : \alpha \in \Lambda\}$ and collapse each H_α to a point. This new quotient space, call it Y, is normal, locally compact, metacompact (using a result of J. Worrell [**8**]) and, by supposition, collectionwise-Hausdorff. The points of $\{H_\alpha : \alpha \in \Lambda\}$ can thus be separated in Y, and therefore the elements of $\{H_\alpha : \alpha \in \Lambda\}$ can be separated in X.

Suppose there is a normal, locally compact, metacompact space, say Y, that is not collectionwise-Hausdorff. Let $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete closed subset of Y that cannot be separated in Y. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be as in Lemma 3. Let X' be an open set in Y such that

$$D \subseteq X' \subseteq \bar{X}' \subseteq \bigcup_{\alpha < \kappa} U_\alpha$$

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and let $\bar{X}' = X$. Let $f:X \to PR(\kappa^*)$ be the function defined by

$$f(x) = \{\alpha : x \in U_\alpha\} \quad \text{for each } x \in X.$$

We claim that $f(X)$ is a subspace of $PR(\kappa^*)$ with the required properties.

First we prove that κ is normalized in $f(X)$. Suppose $H \subseteq \kappa$ and $K = \kappa - H$. Now $f^{-1}(H)$ and $f^{-1}(K)$ are disjoint closed subsets of X; so by normality, let U be an open set in X such that

$$f^{-1}(H) \subseteq U \quad \text{and} \quad \bar{U} \cap f^{-1}(K) = \emptyset.$$

We claim that $\overline{f(\bar{U})} \cap K = \emptyset$. Suppose that, on the contrary, $\alpha \in \overline{f(\bar{U})} \cap K$. Since $(\bar{U}_\alpha - U_\alpha) \cap X$ is compact, let G be a finite subset of $\kappa - \{\alpha\}$ such that $(\bar{U}_\alpha - U_\alpha) \cap X$ is covered by $\{U_\beta : \beta \in G\}$. For each finite subset F of $\kappa - \{\alpha\}$, let $u_F \in \bar{U}$ such that

$$f(u_F) \in \mathscr{U}(\alpha, F \cup G);$$

note that u_F is an element of U_α and

$$u_F \notin \bigcup_{\beta \in F \cup G} U_\beta.$$

Let

$$A = \overline{\{u_F : F \text{ is a finite subset of } \kappa - \{\alpha\}\}}.$$

A is a compact subset of $\bar{U}_\alpha$. In fact, $A \subseteq U_\alpha$: suppose

$$x \in A \cap (\bar{U}_\alpha - U_\alpha)$$

and let $\beta \in G$ such that $x \in U_\beta$. Then there is a finite subset F of $\kappa - \{\alpha\}$ such that $u_F \in U_\beta$, a contradiction. So we must have $A \subseteq U_\alpha$. Also, for each $x \in A$, there is a $\beta_x \neq \alpha$ such that $x \in U_\beta$, since otherwise $f(x) = \alpha$ and so

$$x \in f^{-1}(K) \subseteq X - \bar{U},$$

contradicting the fact that $x \in A$ implies that $x \in \bar{U}$. Since A is compact, let F be a finite subset of $\{\beta_x : x \in A\}$ such that $\{U_\beta : \beta \in F\}$ covers A. But u_F cannot be in any element of $\{U_\beta : \beta \in F\}$, a contradiction. So $\overline{f(\bar{U})} \cap K$ must be empty.

Similarly,

$$\overline{f(X - \bar{U})} \cap H = \emptyset.$$

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So

$$H \subseteq f(X) - \overline{f(X - \bar{U})} \quad \text{and} \quad K \subseteq f(X) - \overline{f(\bar{U})},$$

and since

$$f(X) = \overline{f(X - \bar{U})} \cup f(\bar{U}),$$

these sets are disjoint open sets separating H and K.

Now let us show that f is continuous on D. Suppose $\alpha < \kappa$ and U is an open set in $f(X)$ containing α. Let F be a finite subset of $\kappa - \{\alpha\}$ such that $\mathscr{U}(\alpha, F) \cap f(X)$ is contained in U. Then $U_\alpha - \cup_{\beta \in F}\bar{U}_\beta$ is an open set containing d_α whose f-image is contained in $\mathscr{U}(\alpha, F) \cap f(X)$, and therefore in U. So f is continuous on D.

Now suppose that κ can be separated in $f(X)$. By the continuity of f on D, the points of D can be separated in X, and thus in X' and in Y, which is a contradiction.

Now let us further assume that Y is zero-dimensional. Assume without loss of generality that each U_α is clopen. In this case, f is a perfect map from X onto $f(X)$. We can check that f is continuous by an argument similar to the proof that f is continuous on D. The proof that f is closed goes through like the proof that $\overline{f(\bar{U})} \cap K = \emptyset$: replace $\bar{U}$ by any closed set H of X and replace K by $f(X) - f(H)$. Finally, for each $y \in f(X)$,

$$f^{-1}(y) = \bigcap_{\alpha \in y} U_\alpha - \bigcup_{\alpha \notin y} U_\alpha,$$

a compact set. Thus f is a perfect mapping from X into $PR(\kappa^*)$. So $f(X)$ is normal, locally compact, metacompact, and zero-dimensional. Since paracompactness is invariant under perfect mappings and under their inverse images, X is paracompact if, and only if, $f(X)$ is paracompact.

We now begin with the details leading up to the proof of the main result.

LEMMA 4. *Let $n \geqq 2$, κ be a cardinal, and Y a subspace of $PR_{\leqq n}(\kappa^*)$ which contains κ, such that whenever $\alpha < \beta < \kappa$, $\{\gamma$: there is a $y \in Y$ such that $\{\alpha, \beta, \gamma\} \subseteq y\}$ is finite. If κ is normalized in Y, then κ is separated in Y.*

Proof. Let $n \geqq 2$, and suppose κ is the least cardinal for which the lemma fails. To simplify notation, if α and β are two elements of κ and $\delta \in \{\gamma$: there is a $y \in Y$ with $\{\alpha, \beta, \gamma\} \subseteq y\}$, we will say that "$\delta$ occurs with α, β in Y".

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We first prove that if $Z \subseteq \kappa$ and $|Z| = \lambda < \kappa$, then Z can be separated in Y. For each pair of points γ, β of Z, let

$$F_{\beta\gamma} = \{\delta:\delta \text{ occurs with } \beta, \gamma \text{ in } Y\}.$$

By supposition, each such $F_{\beta\gamma}$ is finite. Let

$$B = Z \cup \cup \{F_{\beta\gamma}:\beta, \gamma \in Z\}.$$

Let $Y' = \{x \in Y:x \subseteq B\}$. Since $Y' \subseteq Y$, we have that any two disjoint subsets of B can be separated in Y', and that if α and β are any two elements of B, $\{\gamma \in B: \gamma$ occurs with α, β in $Y'\}$ is finite. Note that Y' can be considered to be a subspace of $PR_{\leqq n}(\lambda^*)$, where the elements of B are identified with the elements of λ, since if X is any space of cardinality λ with the co-finite topology, $PR(X)$ is homeomorphic to $PR(\lambda^*)$. By the minimality of κ, B can be separated in Y. For each $\alpha \in B$, let F be a finite subset of $\kappa - \{\alpha\}$ such that if α and β are two elements of B, then

$$\mathscr{U}(\alpha, F_\alpha) \cap \mathscr{U}(\beta, F_\beta) \cap Y' = \emptyset;$$

now if $y \in \mathscr{U}(\alpha, F_\alpha) \cap \mathscr{U}(\beta, F_\beta)$, then $y \subseteq F_{\alpha\beta}$ and therefore, $y \in Y'$. Thus the $\mathscr{U}(\alpha, F_\alpha)$'s separate B in Y, and therefore Z can be separated in Y.

We now prove that κ can be separated in Y.

First suppose that κ is a regular cardinal. For each $\beta < \kappa$, let

$$A_\beta = \{\alpha < \beta: \text{there is a } y \in Y \text{ such that } \{\alpha, \beta\} \subseteq y\}.$$

Let

$$\Gamma = \{\alpha: \text{there is a } \beta \geqq \alpha \text{ such that } A_\beta \cap \alpha \text{ is infinite}\}.$$

We claim that Γ is not stationary. Suppose that it is stationary. For each $\alpha \in \Gamma$, let $\beta_\alpha \geqq \alpha$ be such that $A_{\beta_\alpha} \cap \alpha$ is infinite. Let g be the function from Γ into κ such that for each $\alpha \in \Gamma$, $g(\alpha) = \beta_\alpha$. The set of all $\alpha < \kappa$ such that $g[\alpha] \subseteq \alpha$ is a closed unbounded set K [**5**], and so $K \cap \Gamma$ is a stationary set on which g is one to one. So without loss of generality, we may assume g is one to one on Γ.

Let $\{\gamma_\beta:\beta < \kappa\}$ be an increasing enumeration of $g(\Gamma)$. We define disjoint sets A and B inductively as follows: let $\gamma_0 \in A$; suppose $\beta < \kappa$ and γ_α has been assigned to either A or B for each $\alpha < \beta$. If $A_{\gamma_\beta} \cap g^{-1}(\gamma_\beta) \cap A$ is infinite, let $\gamma_\beta \in B$; otherwise, let $\gamma_\beta \in A$.

Either $g^{-1}(A)$ or $g^{-1}(B)$ is stationary. Without loss of generality, suppose $g^{-1}(A)$ is stationary. We show that A and $\kappa - A$ cannot be separated. Let U be open in Y such that $A \subseteq U$. For each $x \in A$, let R_x be a finite subset of $\kappa - \{x\}$ such that

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$\mathscr{U}(x, R_x) \cap Y \subseteq U.$

For each $a \in g^{-1}(A)$, $A_{g(a)} \cap a \cap A$ must be finite. By definition, $A_{g(a)} \cap a$ is infinite, so $A_{g(a)} \cap a \cap (\kappa - A)$ is infinite. Let $\{\delta_m : m \in \omega\}$ be a denumerable subset of

$$A_{g(a)} \cap a \cap (\kappa - (A \cup R_{g(a)})),$$

and for each $m \in \omega$, let $x_m \in Y$ such that $\{\delta_m, g(a)\} \subseteq x_m$, since $\delta_m \in A_{g(a)}$. We wish to choose some x_m in U. We do this by showing there is an $m \in \omega$ such that

$$x_m \cap R_{g(a)} = \emptyset$$

(so $x_m \in U(g(a), R_{g(a)})$). If not, there is a $\gamma \in R_{g(a)}$ and an infinite subset J of ω such that for each $m \in J$, $\gamma \in x_m$. By property (2), $\{\phi : \phi$ occurs with $\gamma, g(a)$ in $Y\}$ is finite, but $\{\delta_m : m \in J\}$ is a subset of this set. Hence

$$x_m \in \mathscr{U}(g(a), R_{g(a)}) \subseteq U \quad \text{for some } m \in \omega.$$

So for each $a \in g^{-1}(A)$, we may let

$$\delta_a \in A_{g(a)} \cap a \cap (\kappa - A)$$

and $x_a \in Y$ such that

$$\{\delta_a, g(a)\} \subseteq x_a \quad \text{and} \quad x_a \in U.$$

Let $h : g^{-1}(A) \to \kappa$ be defined by $h(a) = \delta_a$ for each $a \in g^{-1}(A)$. Since h presses down, we may let $\delta \in \kappa$ such that $\{a \in g^{-1}(A) : \delta_a = \delta\}$ is stationary.

δ is our candidate for a point of $\kappa - A$ which is a limit point of U. Suppose that F is a finite subset of $\kappa - \{\delta\}$. Let $\beta \in \kappa$ such that if $a \in g^{-1}(A)$ and $a \geqq \beta$, then $g(a) \notin F$. By an argument similar to the one presented above, there is an $a \geqq \beta$ such that $\delta_a = \delta$ and $x_a \cap F = \emptyset$. Thus δ is a limit point of $\{x_a : a \in g^{-1}(A)$ and $\delta_a = \delta\}$ and hence of U. Thus A and $\kappa - A$ cannot be separated. So the original assumption must be false, i.e., Γ is not stationary.

By definition then, there is a closed unbounded subset of κ, call it C, which misses Γ. We will use C to partition κ into sets that can be separated from each other in Y. Let $\{c_\alpha : \alpha < \kappa\}$ be an increasing enumeration of C.

Note

$$\kappa = [0, c_0) \cup \bigcup_{\alpha < \kappa} [c_\alpha, c_{\alpha+1}).$$

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Also notice that $\beta \in [c_\alpha, c_{\alpha+1})$ implies that $A_\beta \cap c_\alpha$ is finite since $c_\alpha \notin \Gamma$ for any $\alpha < \kappa$. For each $\alpha < \kappa$ and each $\beta \in [c_\alpha, c_{\alpha+1})$, let $D_\beta = A_\beta \cap c_\alpha$.

We claim that if $\alpha < \delta < \kappa$, $\beta \in [c_\alpha, c_{\alpha+1})$, and $\gamma \in [c_\delta, c_{\delta+1})$, then

$$\mathscr{U}(\beta, D_\beta) \cap \mathscr{U}(\gamma, D_\gamma) \cap Y = \emptyset.$$

To see this, suppose we have chosen such α, β, γ, and δ, and

$$y \in \mathscr{U}(\beta, D_\beta) \cap \mathscr{U}(\gamma, D_\gamma) \cap Y.$$

Then since $\beta \notin D_\gamma$, $\beta \notin A_\gamma$. By definition of A_γ, there is no $y \in Y$ with $\{\beta, \gamma\} \subseteq y$, a contradiction. Hence

$$\mathscr{U}(\beta, D_\beta) \cap \mathscr{U}(\gamma, D_\gamma) \cap Y = \emptyset.$$

By property (1) we may also separate $[0, c_0)$ from $[c_0, \kappa)$.

Furthermore, since $|[0, c_0)| < \kappa$ and for each $\alpha < \kappa$,

$$|[c_\alpha, c_{\alpha+1})| < \kappa,$$

the points of $[c_\alpha, c_{\alpha+1})$ can be separated in Y, and the points of $[0, c_0)$ can be separated in Y. Thus, κ can be separated in Y.

Now let us consider the case where κ is singular and cf $\kappa > \omega$. Let $\kappa = \sup\{\gamma_\beta:\beta < \alpha\}$ where cf $\kappa = \alpha$ and for each $\beta < \alpha$, γ_β is regular and $\gamma_\beta \geqq \beta$, and if $\delta < \beta$, then $\gamma_\delta < \gamma_\beta$.

By the inductive step, we may do the following: for each $\gamma < \kappa$ and each $\beta < \alpha$ such that $\gamma < \gamma_\beta$, assign a finite set $F_{\gamma\beta}$ such that if δ is another element of κ less than γ_β, then

$$\mathscr{U}(\gamma, F_{\gamma\beta}) \cap \mathscr{U}(\delta, F_{\delta\beta}) \cap Y \cap PR(\gamma_\beta{}^*) = \emptyset.$$

For each $\gamma < \kappa$, let

$$P_\gamma = \cup \{F_{\gamma\beta}:\gamma < \gamma_\beta \text{ and } \beta < \alpha\}.$$

We wish to partition κ into sets that can be easily separated from each other.

Let $B_{00} = \gamma_0$. For each $m \in \omega$, let

$$B_{0m+1} = (\{\phi\text{: there are two elements } \beta \text{ and } \gamma \text{ of } B_{0m}$$
$$\text{such that } \phi \text{ occurs with } \beta, \gamma \text{ in } Y\}$$
$$\cup \bigcup_{\gamma \in B_{0m}} (P_\gamma) \cup B_{0m}).$$

Let

$$B_0 = \bigcup_{m \in \omega} B_{0m}.$$

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Note $|B_0| \leqq \gamma_0 \cdot \alpha < \kappa$. For each $\theta < \alpha$, let $B_{\theta 0} = \gamma_\theta$. For each $m \in \omega$, let

$$B_{\theta m+1} = (\{\phi\text{: there are two elements } \beta \text{ and } \gamma \text{ of } B_{\theta m} \text{ such that } \phi \text{ occurs with } \beta, \gamma \text{ in } Y\} \cup \bigcup_{\gamma \in B_{\theta m}} (P_\gamma) \cup B_{\theta m}).$$

Let

$$B_\theta = \bigcup_{m \in \omega} B_{\theta m} - \bigcup_{\delta < \theta} B_\delta.$$

Note that $|B_\theta| \leqq \gamma_\theta \cdot \alpha < \kappa$.

Since for each $\theta < \alpha$, $|B_\theta| < \kappa$, the points of each B_θ can be separated in Y. Now we show that these sets can be separated from each other.

Suppose $\theta < \alpha$ and $\gamma \in B_\theta$. Suppose

$$\left\{\rho \in \bigcup_{\phi < \theta} B_\phi\text{: there is a } y \in Y \text{ with } \{\rho, \gamma\} \subseteq y\right\}$$

is infinite. Let $\{\rho_m : m \in \omega\}$ be a denumerable subset, and for each $m \in \omega$, let $y_m \in Y$ such that $\{\rho_m, \gamma\} \subseteq y_m$. Let $\beta < \alpha$ such that

$$\gamma_\beta > \sup(\{\rho_m : m \in \omega\} \cup \bigcup_{m \in \omega} y_m \cup \{\gamma\}).$$

For each $m \in \omega$,

$$\mathscr{U}(\gamma, F_{\gamma\beta}) \cap \mathscr{U}(\rho_m, F_{\rho_m\beta}) \cap Y \cap PR(\gamma_\beta{}^*) = \emptyset.$$

For each $m \in \omega$,

$$F_{\rho_m\beta} \subseteq P_{\rho_m} \subseteq \bigcup_{\phi < \theta} B_\phi,$$

so $\gamma \notin F_{\rho_m\beta}$. Also, there must be a $k \in \omega$ such that if $m \geqq k$, then $\rho_m \notin F_{\gamma\beta}$. For each $m \geqq k$,

$$y_m \cap F_{\rho_m\beta} = \emptyset,$$

and so

$$y_m \in \mathscr{U}(\rho_m, F_{\rho_m\beta});$$

this implies that $y_m \notin \mathscr{U}(\gamma, F_{\gamma\beta})$, and so we must have that

$$y_m \cap F_{\gamma\beta} \neq \emptyset,$$

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and neither γ nor ρ_m can be in this intersection. Now we use an argument employed before: there must be a δ and an infinite subset J of $\omega - k$ such that for each $m \in J$,

$$\delta \in y_m \cap F_{\gamma\beta};$$

for each $m \in J$, $\{\gamma, \delta, \rho_m\} \subseteq y_m$, contradicting property (2). From this we must conclude that the set

$$\{\rho \in \bigcup_{\phi<\theta} B_\phi\text{: there is a } y \in Y \text{ with } \{\rho, \gamma\} \subseteq y\}$$

is finite.

For each $\theta < \alpha$ and $\gamma \in B_\theta$, let S_γ be the finite set

$$\{\rho \in \bigcup_{\phi<\theta} B_\phi\text{: there is a } y \in Y \text{ with } \{\rho, \gamma\} \subseteq y\}.$$

These sets enable us to separate the B_θ's. This completes the proof for $\text{cf}(\kappa) > \omega$.

Finally, if $\text{cf}(\kappa) = \omega$, apply the fact that normal spaces are $\aleph_0$-collectionwise-normal to the space obtained from Y by isolating all points except the singletons. The proof is then complete.

Our idea now is to first prove that every normal, locally compact, boundedly metacompact space is collectionwise-Hausdorff, and then use this result to prove every normal, locally compact, boundedly metacompact space is paracompact by a method similar to that outlined at the beginning of the proof of Theorem 2.

To prove that a normal, locally compact, boundedly metacompact space X is collectionwise-Hausdorff, we take a discrete closed set $D = \{d_\alpha:\alpha < \kappa\}$ of X and a map f into $PR(\kappa^*)$ that is continuous on D, takes d_α to α, and gives us a subspace of $PR(\kappa^*)$ that has the properties mentioned in Lemma 4. Then, separating κ in the subspace allows us to separate the points of D in X.

More precisely, we start with a normal, locally compact space X, a discrete closed subset $D = \{d_\alpha:\alpha < \kappa\}$ of X, and a cover $\mathcal{U} = \{U_\alpha:\alpha < \kappa\}$ of X by open sets with compact closures, with the properties that for each $\alpha < \kappa$, $d_\alpha \in U_\alpha$, and if $\beta \neq \alpha$, then $d_\alpha \notin \bar{U}_\beta$, and that each of X belongs to at most n elements of $\mathcal{U}$ for some positive integer n. Define $f:X \to PR(\kappa^*)$ by

$$f(x) = \{\alpha:x \in U_\alpha\} \quad \text{for each } x \in X.$$

By a procedure similar to the one in the proof of Theorem 2, it can be shown that any two disjoint subsets of κ can be separated in $f(X)$. Recall

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that this is one of the properties of Lemma 4. If we can also satisfy the second property of that lemma, then we can separate κ in $f(X)$.

Since for each $\alpha < \kappa$, $f(d_\alpha) = \alpha$ and if $\beta \neq \alpha$, then $d_\alpha \notin \bar{U}_\beta$, the function f is continuous on D, so separating κ in $f(X)$ allows us to separate D in X. However, we may not be able to satisfy the second property with this function f. Consider $\alpha < \beta < \kappa$. It is not clear that $\{\gamma: \gamma$ occurs with α, β in $f(X)\}$ is finite. It is obvious, however, that if we let

$$Y = \{x \in X: x \text{ belongs to at most two elements of } \mathcal{U}\},$$

then $\{\gamma:\gamma$ occurs with α, β in $f(Y)\}$ is finite, and so κ can be separated in $f(Y)$, and D can be separated in Y. The idea in the next theorem is along the following lines: use the fact that D can be separated in Y to define a new open cover $\mathcal{V}$ of X and a new function g from X into $PR(\kappa^*)$ based on this cover so that if we let

$$Z = \{x \in X: x \text{ belongs to at most three elements of } \mathcal{V}\},$$

then $g(Z)$ witnesses the properties of Lemma 4. Then we can separate κ in $g(Z)$ and D in Z. We continue in this way, inductively generating new open covers and new functions into $PR(\kappa^*)$ that allow us to separate the points of D in more of the space X until finally we can separate D in X.

We set up the necessary machinery in the following theorem, but first we give the definition of a concept needed in the theorem. If $\{U_\alpha:\alpha < \kappa\}$ is an open cover of a space X, then an open refinement $\{V_\alpha:\alpha < \kappa\}$ is said to *shrink* $\{U_\alpha:\alpha < \kappa\}$ provided that for each $\alpha < \kappa$, $\bar{V}_\alpha \subseteq U_\alpha$. Any point-finite open cover of a normal space can be shrunk.

THEOREM 5. *If Y is normal, locally compact, and boundedly metacompact, then Y is collectionwise-Hausdorff.*

Proof. Suppose Y is normal, locally compact, and boundedly metacompact. Suppose $D = \{d_\alpha:\alpha < \kappa\}$ is a discrete closed subset of Y. Let $\{U_\alpha:\alpha < \kappa\}$ be as in Lemma 3.

Let X' be an open set such that

$$D \subseteq X' \subseteq \bar{X}' \subseteq \bigcup_{\alpha<\kappa} U_\alpha.$$

Let $\bar{X}' = X$. For each $\alpha < \kappa$, let

$$U_\alpha \cap X = U_{(n-2)\alpha},$$

and let

$$\mathcal{U}_{n-2} = \{U_{(n-2)\alpha}:\alpha < \kappa\}.$$

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For each natural number $j < n - 2$, let

$$\mathcal{U}_j = \{U_{j\alpha}:\alpha < \kappa\}$$

be a collection of open sets of X such that $\mathcal{U}_j$ shrinks $\mathcal{U}_{j+1}$. For each natural number $j \leqq n - 2$ let

$$Y_j = \{x \in X:x \text{ belongs to at most } j+2 \text{ elements of } \mathcal{U}_j\}.$$

Let P_j, $0 \leqq j \leqq n-2$, be the statement that there is an open subset Z_j' of X that contains D and a collection $\{F_{j\alpha}:\alpha < \kappa\}$ such that for each $\alpha < \kappa$, $F_{j\alpha}$ is a finite subset of $\kappa - \{\alpha\}$, and

$$\{U_{j\alpha} - \bigcup_{\gamma \in F_{j\alpha}} \overline{U_{j\gamma}}:\alpha < \kappa\}$$

is a collection of open sets such that no point of $Y_j \cap \bar{Z}_j'$ belongs to two of these sets.

We will show that P_{n-2} is true by induction. Let $f_0:X \to PR_{\leqq n}(\kappa^*)$ be the function such that

$$f_0(x) = \{\alpha:x \in U_{0\alpha}\} \quad \text{for each } x \in X.$$

Y_0 is closed in X, and so is a normal, locally compact space. For each $\alpha < \kappa$, let $V_\alpha = U_{0\alpha} \cap Y_0$. Then $\{V_\alpha:\alpha < \kappa\}$ is a point-finite open cover of Y_0 by sets with compact closures. Let $f: Y_0 \to PR(\kappa^*)$ be the function defined by

$$f(x) = \{\alpha:x \in V_\alpha\} \quad \text{for each } x \in Y_0.$$

Note that for each $x \in Y_0$, $f(x) = f_0(x)$. As previously noted, any two disjoint subsets of κ can be separated in $f(Y_0)$, and thus in $f_0(Y_0)$. Also, for each $\alpha < \beta < \kappa$, $\{\gamma:\gamma$ occurs with α, β in $f_0(Y_0)\}$ is finite. Thus, since $f_0(Y_0)$ satisfies the two properties of the hypothesis of Lemma 4, κ can be separated in $f_0(Y_0)$, i.e., for each $\alpha < \kappa$, we may assign a finite subset $F_{0\alpha}$ of $\kappa - \{\alpha\}$ such that for each $\beta < \kappa$ with $\beta \neq \alpha$,

$$\mathcal{U}(\alpha, F_{0\alpha}) \cap \mathcal{U}(\beta, F_{0\beta}) \cap f_0(Y_0) = \emptyset.$$

Let $Z_0' = X$. With the collection

$$\{U_{0\alpha} - \bigcup_{\gamma \in F_{0\alpha}} \bar{U}_{0\gamma}:\alpha < \kappa\}$$

we have shown P_0 is true.

Now suppose that P_j is true for some j, $0 \leqq j < n-2$. For each $\alpha < \kappa$, let $H_{(j+1)\alpha}$ be a finite subset of $\kappa - \{\alpha\}$ containing $F_{j\alpha}$ such that

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$$\bar{U}_{(j+1)\alpha} - U_{j\alpha} \subseteq \bigcup_{\gamma \in H_{(j+1)\alpha}} U_{j\gamma},$$

and let $Z'_{(j+1)}$ be an open set such that

$$D \subseteq Z'_{j+1} \subseteq \overline{Z'_{j+1}} \subseteq \left[\bigcup_{\alpha<\kappa} (U_{(j+1)\alpha} - \bigcup_{\gamma \in H_{(j+1)\alpha}} \overline{U_{(j+1)\gamma}}) \right] \cap Z'_j.$$

Let $\overline{Z'_{j+1}} = Z_{j+1}$ and $\bar{Z}'_j = Z_j$, and let $f_{j+1}:Z_{j+1} \to PR_{\leqq n}(\kappa^*)$ be defined by

$$f_{j+1}(x) = \{\alpha : x \in U_{(j+1)\alpha} - \bigcup_{\gamma \in H_{(j+1)\alpha}} \overline{U_{(j+1)\gamma}}\}.$$

Any two disjoint subsets of κ can be separated in $f_{j+1}(Z_{j+1} \cap Y_{j+1})$.

We now establish that $f_{j+1}(Z_{j+1} \cap Y_{j+1})$ satisfies property (2) of Lemma 4. Suppose $\alpha < \beta < \kappa$ and $\{\gamma: \gamma$ occurs with α, β in $f_{j+1}(Z_{j+1} \cap Y_{j+1})\}$ is infinite. Let $\{\gamma_m : m \in \omega\}$ be a denumerable subset of this set, and for each $m \in \omega$, let $z_m \in Z_{j+1} \cap Y_{j+1}$ such that

$$\{\alpha, \beta, \gamma_m\} \subseteq f_{j+1}(z_m).$$

Since each z_m is in $U_{(j+1)\alpha}$, let z be a limit point of $\{z_m : m \in \omega\}$. Note that since for each $m \in \omega$, z_m belongs to at most $(j + 3)$ elements of $\mathcal{U}_{j+1}$ and since $\mathcal{U}_j$ shrinks $\mathcal{U}_{j+1}$, for each $m \in \omega$, z_m belongs to at most $(j + 3)$ elements of $\mathcal{U}_j$. So z must belong to at most $j + 2$ elements of $\mathcal{U}_j$, that is, $z \in Y_j$. Recall that $Z_{j+1} \subseteq Z_j$, so $z \in Z_j$. We show that

$$z \in (U_{j\alpha} - \bigcup_{\gamma \in F_{j\alpha}} \overline{U_{j\gamma}}) \cap (U_{j\beta} - \bigcup_{\gamma \in F_{j\beta}} \overline{U_{j\gamma}})$$

which contradicts our assumptions.

Suppose that

$$z \notin U_{j\alpha} - \bigcup_{\gamma \in F_{j\alpha}} \bar{U}_{j\gamma}.$$

First suppose $z \notin U_{j\alpha}$. Then, since for each $m \in \omega$, $z_m \in U_{(j+1)\alpha}$, we have

$$z \in \overline{U_{(j+1)\alpha}} - U_{j\alpha}.$$

So there is some element of $H_{(j+1)\alpha}$, say γ, such that $z \in U_{j\gamma}$. Let $m \in \gamma$ such that $z_m \in U_{j\gamma}$, and so $z_m \in U_{(j+1)\gamma}$. This gives us a contradiction, since $\alpha \in f_{j+1}(z_m)$ means that

$$z_m \in U_{(j+1)\alpha} - \bigcup_{\delta \in H_{(j+1)\alpha}} \bar{U}_{(j+1)\delta}.$$

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Suppose that on the other hand,

$$z \in \bigcup_{\gamma \in F_{j\alpha}} \bar{U}_{j\gamma}$$

and let $\gamma \in F_{j\alpha}$ such that $z \in \bar{U}_{j\gamma}$. But since $F_{j\alpha} \subseteq H_{(j+1)\alpha}$, we may derive a similar contradiction. A similar argument shows that

$$z \in U_{j\beta} - \bigcup_{\gamma \in F_{j\beta}} \overline{U_{j\gamma}}.$$

This gives a contradiction, and indicates that $f_{j+1}(Z_{j+1} \cap Y_{j+1})$ does satisfy property (2) of Lemma 4. Hence by Lemma 4 we conclude that the points of κ can be separated, and we may assign for each $\alpha < \kappa$ a definite subset $F_{(j+1)\alpha}$ of $\kappa - \{\alpha\}$ that contains $H_{(j+1)\alpha}$ and such that for any $\beta < \kappa$ with $\beta \neq \alpha$,

$$\mathscr{U}(\alpha, F_{(j+1)\alpha}) \cap \mathscr{U}(\beta, F_{(j+1)\beta}) \cap f_{j+1}(Z_{j+1} \cap Y_{j+1}) = \emptyset.$$

With the collection

$$\{u_{(j+1)\alpha} - \bigcup_{\gamma \in F_{(j+1)\alpha}} \overline{U_{(j+1)\gamma}} : \alpha < \kappa\}$$

we have shown P_{j+1} is true. Therefore, P_{n-2} is true.

So we may let Z be an open subset of X that contains D and $\{F_\alpha : \alpha < \kappa\}$ be a collection such that for each $\alpha < \kappa$, F_α is a finite subset of $\kappa - \{\alpha\}$ and

$$\{(U_\alpha \cap X) - \bigcup_{\gamma \in F_\alpha} \overline{U_\gamma \cap X} : \alpha < \kappa\}$$

is a collection of open sets in X such that no point of $\bar{Z}$ belongs to two of these sets. Thus we can separate D in X. It follows that we can separate the points of D in Y.

We are now able to state and prove the main result.

THEOREM 6. *Every normal, locally compact, boundedly metacompact space is paracompact.*

Proof. Suppose X is normal, locally compact, and boundedly metacompact. To show X is paracompact, it suffices to show X is collectionwise-normal with respect to compact sets.

Suppose $\mathscr{H} = \{H_\alpha : \alpha < \kappa\}$ is a discrete collection of compact sets. Let $q : X \to X/\mathscr{H}$ be the natural quotient map. Let $Y = X/\mathscr{H}$. Then Y is locally compact and normal. We show that Y is boundedly metacompact, and hence collectionwise-Hausdorff by Theorem 5.

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Suppose $\mathscr{U}$ is an open cover of Y. Note that $\{H_\alpha:\alpha < \kappa\}$ is a discrete closed subset of Y, and denote it by H. Similar to the proof of Lemma 3, for each $y \in Y$ we choose an open set V_y with compact closure that contains y and is contained in some set of $\mathscr{U}$ and such that $\bar{V}_y \cap H \subseteq \{y\}$. Then $\{q^{-1}(V_y):y \in Y\}$ is an open cover of X. Since X is boundedly metacompact, let n be a positive integer and $\mathscr{W}$ an open refinement of $\{q^{-1}(V_y):y \in Y\}$ such that each point of X is at most n elements of $\mathscr{W}$.

For each $\alpha < \kappa$, let

$$W_\alpha = \cup \{W \in \mathscr{W}:W \cap H_\alpha \neq \emptyset\}.$$

Let R be an open set in X such that

$$\bigcup_{\alpha<\kappa} H_\alpha \subseteq R \subseteq \bar{R} \subseteq \bigcup_{\alpha<\kappa} W_\alpha$$

(by the normality of X).

$$\text{Let } \mathscr{W}' = \{W_\alpha:\alpha < \kappa\} \cup \{W \cap (X - \bar{R}): W \in \mathscr{W} \text{ and for each } \alpha < \kappa,\ W \cap H_\alpha = \emptyset\}.$$

Since for each $y \in Y$, $q^{-1}(V_y)$ meets at most one element of $\mathscr{H}$, $\mathscr{W}'$ also has the property that each point of X is in at most n elements of $\mathscr{W}'$.

Now $\{q(W):W \in \mathscr{W}'\}$ is an open cover of Y, since for each $W \in \mathscr{W}'$, $q^{-1}(q(W)) = W$. Also, each point of Y is in at most n elements of $\{q(W):W \in \mathscr{W}'\}$, and this collection is a refinement of $\{V_y:y \in Y\}$ and hence of $\mathscr{U}$.

Since we have shown that for any open cover $\mathscr{U}$ of Y, there is a positive integer n and a refinement of $\mathscr{U}$ such that each point of Y is in at most n elements of this refinement, Y is boundedly metacompact.

By Theorem 5, Y is collectionwise-Hausdorff. So we may let $\{S_\alpha:\alpha < \kappa\}$ be a collection of pairwise disjoint open sets such that $H_\alpha \in S_\alpha$ for each $\alpha < \kappa$. Then $\{q^{-1}(S_\alpha):\alpha < \kappa\}$ is a collection of pairwise disjoint open sets in X such that for each $\alpha < \kappa$, $H_\alpha \subseteq S_\alpha$.

Therefore, X is collectionwise-normal with respect to compact sets. We conclude that X is paracompact.

It is interesting to note it can be shown that for any positive integer n and cardinal κ, $PR_{\leqq n}(\kappa^*)$ is subparacompact if, and only if, $\kappa \leqq \omega_1$. (Recall that a space X is subparacompact if, and only if, every open cover of X has a σ-discrete closed refinement.) Using the results of this paper and the fact that paracompact spaces are subparacompact, we may prove the following:

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THEOREM 7. *Every zero-dimensional, normal, locally compact, metacompact space is subparacompact.*

Proof. Suppose X is zero-dimensional, normal, locally compact, and metacompact. Let $\mathscr{U} = \{U_\alpha : \alpha < \kappa\}$ be a point-finite cover of X by clopen, compact sets. Define $f: X \to PR(\kappa^*)$ to be the function such that for each $x \in X$,

$$f(x) = \{\alpha : x \in U_\alpha\}.$$

We have seen in the proof of Theorem 3 that f is perfect, and hence $f(X)$ is a zero-dimensional, normal, locally compact subspace of $PR(\kappa^*)$. For each $n \in \omega$, $f(X) \cap PR_{\leqq n}(\kappa^*)$ is normal, locally compact, and boundedly metacompact. By Theorem 6, $f(X) \cap PR_{\leqq n}(\kappa^*)$ is paracompact, and hence, subparacompact, for each $n \in \omega$, and it easily follows that

$$f(X) = \bigcup_{n \in \omega} (f(X) \cap PR_{\leqq n}(\kappa^*))$$

is subparacompact. Since the inverse image under a perfect map of a subparacompact space is subparacompact, X is subparacompact.

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