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AN EXPLICIT CRITERION FOR THE CONVEXITY OF QUATERNIONIC NUMERICAL RANGE

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ABSTRACT. Quaternionic numerical range is not always a convex set. In this note, an explicit criterion is given for the convexity of quaternionic numerical range.

1. **Introduction.** It has been over 150 years since the discovery of quaternions by Hamilton in 1843. Various topics on quaternions were and are being studied: quaternionic equations [6, 11], quaternionic matrices [5, 9, 16], and quaternionic eigenvalues [1, 10, 15]. In this note we are interested in quaternionic numerical ranges. Kippenhahn [8] was the first one to study quaternionic numerical range as a generalization of the complex case. Unaware of Kippenhahn's paper, Jamison [7] and later Au-Yeung [2] reinitiated the study of quaternionic numerical ranges. Recently the study of quaternionic numerical ranges is revived in a series of papers [3, 4, 12, 13, 14, 17]. It seems that most of the recent papers are on the convexity of the so called upper bild instead of quaternionic numerical range.

Let **R**, **C** and **H** denote the set of real, complex and quaternionic numbers respectively. Let $q = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ be a quaternion with real a, b, c and d. Then we denote the conjugate $\overline{q} = a-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}$; the real part $\mathrm{re}(q) = a$; and the norm $|q| = \sqrt{q\overline{q}} = \sqrt{a^2+b^2+c^2+d^2}$. For an $n \times n$ quaternionic matrix A, the quaternionic numerical range of A is defined as the set

$$W(A) = \{x^*Ax : x^*x = 1\}$$

where x is an n-vector with quaternionic entries and x^* is the conjugate transpose of x. Note that $W(A)$ is a subset of **H** and can be viewed as a subset of the real 4-dimensional space. Unlike its complex analog, quaternionic numerical range is not convex in general. This fact was observed by Jamison [7] in his Ph.D. dissertation on quaternionic Hilbert space. Then it comes a natural question: when is a quaternionic numerical range convex? For $n = 1$, it is straight forward to verify that $W(A)$ is convex if and only if A is a real number. In his dissertation, Jamison also showed that if A is Hermitian then $W(A)$ is a nonempty closed interval and hence convex. Later Au-Yeung [2] extended this result and we restate it as follows.

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THEOREM 1.1. *For $n \geq 2$, let A be an $n \times n$ normal matrix with eigenvalues $h_1 + k_1\mathbf{i}, \ldots, h_n + k_n\mathbf{i}$ where $h_1 \leq \cdots \leq h_n$ and $k_t \geq 0$ for $1 \leq t \leq n$. Then $W(A)$ is convex if and only if $k_1(h_1 - h_2) = k_n(h_n - h_{n-1}) = 0$.*

Moreover Au-Yeung [2] gave an implicit characterization of matrices with convex quaternionic numerical range. To state his result, we define the set $W_{\mathbf{R}}(A) = \{\mathrm{re}(q) : q \in W(A)\}$. It is clear that $W(A) \cap \mathbf{R} \subset W_{\mathbf{R}}(A)$.

THEOREM 1.2. *The set $W(A)$ is convex if and only if $W(A) \cap \mathbf{R} = W_{\mathbf{R}}(A)$.*

However this result is not easy to use since it requires the knowledge of $W(A)$. The objective of this note is to make Theorem 1.2 more explicit. To this end, we need the notion of quasi-diagonal elements of a quaternionic matrix.

Write $A = H + K$ where $H = \frac{A+A^*}{2}$ is Hermitian and $K = \frac{A-A^*}{2}$ is skew-Hermitian. For results on quaternionic matrices used below, see [5, 9]. Let U be a unitary matrix such that U^*HU is a real diagonal matrix with elements $h_1 \leq \cdots \leq h_n$. Since U^*KU is still skew-Hermitian, the diagonal elements have zero real parts. Hence there exists a diagonal unitary matrix D such that the diagonal elements of D^*U^*KUD are of the form $k_1\mathbf{i}, \ldots, k_n\mathbf{i}$ with $k_t \geq 0$ for $1 \leq t \leq n$. Consequently A is unitarily similar to the matrix D^*U^*AUD with diagonal elements $h_1 + k_1\mathbf{i}, \ldots, h_n + k_n\mathbf{i}$ where $h_1 \leq \cdots \leq h_n$ and $k_t \geq 0$ for $1 \leq t \leq n$. The numbers $h_1 + k_1\mathbf{i}, \ldots$ and $h_n + k_n\mathbf{i}$ are called *quasi-diagonal elements* of A. Note that h_t's are uniquely determined by A but k_t's are not. Quasi-diagonal elements are useful in studying quaternionic numerical range.

PROPOSITION 1.3. *Let A be a matrix with quasi-diagonal elements $h_1 + k_1\mathbf{i}, \ldots, h_n + k_n\mathbf{i}$ where $h_1 \leq \cdots \leq h_n$ and $k_t \geq 0$ for $1 \leq t \leq n$. Then $W_{\mathbf{R}}(A) = [h_1, h_n]$.*

PROOF. Since $W(A)$ is invariant under unitary similarity, we may assume that A is of the form D^*U^*AUD. Note that $\mathrm{re}(x^*Ax) = \sum_{i=1}^n h_i|x_i|^2$ where $\sum_{i=1}^n |x_i|^2 = 1$. The result follows the fact that $h_1 \leq \cdots \leq h_n$. ∎

Although $W(A)$ is not convex in general, Jamison [7] and later Au-Yeung [2] proved that $W(A) \cap \mathbf{R}$ is convex, indeed a nonempty closed interval for $n \geq 2$. Now Theorem 1.2 can be stated explicitly as follows.

PROPOSITION 1.4. *For $n \geq 2$, let A be an $n \times n$ matrix with $W(A) \cap \mathbf{R} = [l, r]$ and have quasi-diagonal elements $h_1 + k_1\mathbf{i}, \ldots, h_n + k_n\mathbf{i}$ where $h_1 \leq \cdots \leq h_n$ and $k_t \geq 0$ for $1 \leq t \leq n$. Then $W(A)$ is convex if and only if $l = h_1$ and $r = h_n$.*

Proposition 1.4 is not good since r and l do not have simple formulas. Nonetheless, in next section, we will give an explicit criterion in terms of quasi-diagonal elements for the convexity of quaternionic numerical range.

2. **Main result.** In this section we start with a lemma and then conclude with the main theorem.

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LEMMA 2.1. *Let* $A = \begin{bmatrix} h_1 + k_1\mathbf{i} & q \\ -\bar{q} & h_2 + k_2\mathbf{i} \end{bmatrix}$ *where* $h_1 \le h_2$ *and* $k_t \ge 0$ *for* $1 \le t \le 2$. *Denote* $W(A) \cap \mathbf{R} = [l, r]$.

1. *If* $k_1(h_2 - h_1) = 0$ *then* $h_1 = l$.
2. *If* $k_2(h_2 - h_1) = 0$ *then* $h_2 = r$.

PROOF. Note that

$$W(A) = \{h_1|x|^2 + h_2|y|^2 + k_1\bar{x}\mathbf{i}x + k_2\bar{y}\mathbf{i}y + \bar{x}qy - \overline{yq}x : x, y \in \mathbf{H}, |x|^2 + |y|^2 = 1\}.$$

Hence

$$l = \min\{h_1|x|^2 + h_2|y|^2 : x, y \in \mathbf{H}, |x|^2 + |y|^2 = 1, k_1\bar{x}\mathbf{i}x + k_2\bar{y}\mathbf{i}y + \bar{x}qy - \overline{yq}x = 0\},$$

and

$$r = \max\{h_1|x|^2 + h_2|y|^2 : x, y \in \mathbf{H}, |x|^2 + |y|^2 = 1, k_1\bar{x}\mathbf{i}x + k_2\bar{y}\mathbf{i}y + \bar{x}qy - \overline{yq}x = 0\}.$$

1. If $k_1 = 0$ then $l = h_1$ by taking $x = 1$ and $y = 0$. If $k_1 \neq 0$ then $h_1 = h_2$ from hypothesis. Consequently, $l = h_1$.
2. If $k_2 = 0$ then $h_2 = r$ by taking $x = 0$ and $y = 1$. If $k_2 \neq 0$ then $h_1 = h_2$ from hypothesis. Consequently, $r = h_2$.

■

THEOREM 2.2. *For* $n \ge 2$, *let* A *be an* $n \times n$ *matrix with quasi-diagonal elements* $h_1 + k_1\mathbf{i}, \ldots, h_n + k_n\mathbf{i}$ *where* $h_1 \le \cdots \le h_n$ *and* $k_t \ge 0$ *for* $1 \le t \le n$. *Then* $W(A)$ *is convex if and only if* $k_1(h_1 - h_2) = k_n(h_n - h_{n-1}) = 0$.

PROOF. Note that A is unitarily similar to $H + K$ where H is a diagonal matrix with real entries $h_1 \le \cdots \le h_n$ and K is a skew Hermitian matrix with diagonal elements $k_1\mathbf{i}, \ldots, k_n\mathbf{i}$, where $k_t \ge 0$ for $1 \le t \le n$. Since quaternionic numerical range is invariant under unitary similarity, we have $W(A) = W(H + K)$. Let $W(A) \cap \mathbf{R} = [l, r]$.

NECESSITY. Since $W(A)$ is convex, by Proposition 1.4, $[l, r] = [h_1, h_n]$. Hence $h_1 = l \in W(A) = W(H + K)$, *i.e.*, there exists x such that $x^*x = 1$ and $h_1 = x^*Ax = \sum_{i=1}^n h_i|x_i|^2 + x^*Kx$. Since K is skew-Hermitian, x^*Kx has zero real part. It follows that $h_1 = \sum_{t=1}^n h_t|x_t|^2$. Hence there exists an integer $1 \le s \le n$ such that $h_1 = \cdots = h_s$ and $x_{s+1} = \cdots = x_n = 0$. If $s = 1$ then $x = [x_1 0 \cdots 0]^T$ and so $0 = x^*Kx = \overline{x_1}\mathbf{i}k_1x_1$. It follows that $k_1 = 0$. If $s > 1$ then $h_1 = h_2$. Consequently $k_1(h_2 - h_1) = 0$. Similarly, $h_n = r \in W(A)$ implies that $k_n(h_n - h_{n-1}) = 0$.

SUFFICIENCY. Consider the 2×2 principal submatrix

$$A' = \begin{bmatrix} h_1 + k_1\mathbf{i} & q \\ -\bar{q} & h_2 + k_2\mathbf{i} \end{bmatrix}$$

of $H + K$. Since $W(A') \subset W(H + K) = W(A)$, it follows that $[l', r'] = W(A') \cap \mathbf{R} \subset W(A) \cap \mathbf{R} = [l, r] \subset [h_1, h_n]$. Hence $h_1 \le l \le l'$. By Lemma 2.1, the hypothesis

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$k_1(h_2 - h_1) = 0$ implies that $l' = h_1$ and so $h_1 = l$. To finish the proof, we consider the 2×2 principal submatrix

$$A'' = \begin{bmatrix} h_n + k_{n-1}\mathbf{i} & p \\ -\overline{p} & h_n + k_n\mathbf{i} \end{bmatrix}$$

of $H + K$. Applying an analogous argument as above, we have $r'' \le r \le h_n$ where $[l'', r''] = W(A'') \cap \mathbf{R}$. By the Lemma 2.1, $k_n(h_n - h_{n-1}) = 0$ implies that $r'' = h_n$ and so $h_n = r$. Consequently, we have $[h_1, h_n] = [l, r]$ which implies the convexity of $W(A)$ by Proposition 1.4. ∎

Note that if A is normal then the eigenvalues of A are quasi-diagonal elements of A. Hence Theorem 1.1 is recovered as a corollary.

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