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ABELIAN STEINER TRIPLE SYSTEMS

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1. Introduction. A *neofield* of order v, $N_v(+, \cdot)$, is an algebraic system of v elements including 0 and 1, $0 \neq 1$, with two binary operations $+$ and $\cdot$ such that $(N_v, +)$ is a loop with identity element 0; $(N_v^*, \cdot)$ is a group with identity element 1 (where $N_v^* = N_v \backslash \{0\}$) and every element of N_v is both right and left distributive (i.e., $(y + z)x = yx + zx$ and $x(y + z) = xy + xz$ for all $y, z \in N_v$). From this we can derive: $0 \cdot x = x \cdot 0 = 0$ for all $x \in N_v$. A neofield N_v has the inverse property (IP) and is called an *IP neofield* if for all $y \in N_v$ there is an element $z \in N_v$ such that $(x + y) + z = x$ and $z + (y + x) = x$ for all $x \in N_v$. It readily follows that z is the unique two-sided negative of y, $-y$. Moreover, we note that $-y = (-1)y$ for all $y \in N_v$, where -1 is the unique two-sided negative of 1. In particular, $(-1)^2 = 1$. A neofield N_v is said to be *commutative* when $(N_v, +)$ is a commutative loop, and it is said to be *abelian* when $(N_v^*, \cdot)$ is an abelian group. An abelian neofield with the inverse property is called an *AIP neofield*. It is easy to show [**2**] that an AIP neofield is always commutative, from which it readily follows that an AIP neofield contains at most one element of multiplicative order 2, namely -1.

In the first part of this paper we give a characterization of an AIP neofield N_v in terms of a certain partition of the elements of the abelian group $A = (N_v^*, \cdot)$ and show that the existence of an AIP neofield having $(N_v^*, \cdot) = A$ is equivalent to the existence of such a partition of A.

In Section 3 we use the above mentioned characterization to show by direct constructions that an abelian group A of order n, n odd, is admissible as the multiplicative group of nonzero elements of an IP neofield if and only if $n \equiv 1$ or 3 (mod 6) and $A \neq C_9$.

In the last section we use the constructions of Section 3 to obtain existence results for abelian Steiner triple systems of all orders $n \equiv 1$ or 3 (mod 6). (A *Steiner triple system* (STS) of order n, $\mathscr{T}_n = [S, \mathscr{S}]$ is an arrangement of the elements of an n-set S into a set $\mathscr{S}$ of triples such that every pair of elements in S occur together in exactly one triple of $\mathscr{S}$. A necessary and sufficient condition for the existence of an STS of order n is that $n \equiv 1$ or 3 (mod 6). An STS is called *abelian* if it has a sharply transitive automorphism group which is abelian.) Finally, we show that the number of nonisomorphic abelian STS's of order n ($n \equiv 1$ or 3 (mod 6)) goes to infinity with n, even as a certain decomposition of the automorphism groups retains a fixed size.

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2. AIP neofields and admissible partitions. Suppose that N_v is an AIP neofield with multiplicative group $(N_v^*, \cdot) = A$. Then N_v is completely characterized by its addition table and since $y + z = w$ if and only if $1 + zy^{-1} = wy^{-1}$ it follows that N_v is completely characterized by the map $p: N_v \to N_v$ given by $p(x) = 1 + x$. We call the map p a *presentation map* for N_v. Clearly $p(0) = 1$ and $p(-1) = 0$. Now suppose that for $x \in N_v \backslash \{0, -1\}$

(1) $p(x) = -y.$

Then clearly $y \in N_v\backslash\{0, -1\}$ and moreover, using the inverse property and the commutativity of addition it follows that

(2) $p(y) = -x.$

It also follows immediately that (1) implies

(3) $p(yx^{-1}) = -x^{-1}$

and therefore

(4) $p(x^{-1}) = -yx^{-1}$

as well as

(5) $p(y^{-1}) = -xy^{-1}$

and therefore

(6) $p(xy^{-1}) = -y^{-1}.$

Thus, the action of p on the set $\theta(x) = \{x, y, yx^{-1}, x^{-1}, y^{-1}, xy^{-1}\}$, $y = -p(x)$, is determined by the action of p on x (or on any other element of $\theta(x)$).

Note that if we let $A_1 = \{x, y\}$, $A_2 = \{yx^{-1}, x^{-1}\}$ and $A_3 = \{y^{-1}, xy^{-1}\}$ then each $\theta(x) = \{x, y, yx^{-1}, x^{-1}, y^{-1}, xy^{-1}\}$ satisfies:

(*) $A_i \cap A_j = \emptyset$ or $A_i = A_j$ for $i, j = 1, 2, 3.$

Moreover, if $w \in \theta(x)$ then $\theta(w) = \theta(x)$ and we have therefore that the sets $\theta(x)$ $(x \neq 0, -1)$ partition $N_v\backslash\{0, -1\}$. This leads to the following definition.

Definition 2.1. Let A be a finite abelian group with identity 1 and having at most one element of order two, an let l denote such an element if it exists, $l = 1$ otherwise. A subset of $A\backslash\{l\}$ of the form $\theta = \{x, y, yx^{-1}, x^{-1}, y^{-1}, xy^{-1}\}$ which satisfies condition (*) is called an *admissible class* in A. A partition of A consisting of $\{l\}$ and admissible classes is called an *admissible partition* of A.

We now prove the main result of this section:

THEOREM 2.2. *Let A be a finite abelian group of order n having at most one element of order two. There exists a neofield N_v or order $v = n + 1$ having $(N_v^*, \cdot) = A$ if and only if there exists an admissible partition of A.*

Proof. If N_v is an AIP neofield having $(N_v^*, \cdot) = A$, letting $l = -1$ and

$y = -(1 + x)$ we have seen that the sets $\theta = \{x, y, yx^{-1}, x^{-1}, y^{-1}, xy^{-1}\}$ form an admissible partition of A.

Conversely, suppose that there exists an admissible partition π of A. Let $N = \{0\} \cup A$. We extend the multiplication of A to N by defining $x \cdot 0 = 0$ for all $x \in N$. We now define addition in N as follows:

(A.1) $x + 0 = 0 + x = x$ for all $x \in N$.

(A.2) $1 + l = 0$ where l is the unique element of order 2 in A if such an element exists; $l = 1$ otherwise.

(A.3) $x + y = (1 + yx^{-1})x$ for all $x, y \in A$.

It only remains to define the additions $1 + x$ for all $x \in A \backslash \{l\}$:

(A.4) For each admissible class $\theta = \{x, y, yx^{-1}, x^{-1}, y^{-1}, xy^{-1}\}$ in the admissible partition π of A we define:

(i) $1 + x = ly$;

(ii) $1 + y = lx$;

(iii) $1 + yx^{-1} = lx^{-1}$;

(iv) $1 + x^{-1} = lyx^{-1}$;

(v) $1 + y^{-1} = lxy^{-1}$ and

(vi) $1 + xy^{-1} = ly^{-1}$.

It is clear that the operations $\cdot$ and $+$ are well defined. We claim that $(N, +, \cdot)$ is an AIP neofield with $-1 = l$. First, we note that addition is commutative. From (A.4) we have that for all $z \in A \backslash \{l\}$, $1 + z = lw$ if and only if $1 + z^{-1} = lwz^{-1}$ and by (A.3) we obtain $z + 1 = lw$. Also since $l^{-1} = l$, $l + 1 = l(1 + l) = 0$. Thus, $z + w = (1 + wz^{-1})z = (wz^{-1} + 1)z = w + z$ for all $z, w \in A$. Since $z + 0 = 0 + z$ for all $z \in N$ we have that $(N, +)$ is commutative.

We next show $(N, +)$ is a loop with identity 0. Let $z, w \in N$ be given. We must show that there exists a unique $x \in N$ such that

$$(L)\ z + x = w$$

holds. If $z = 0$ choose $x = w$ and if $w = 0$ choose $x = lz$ (then $z + lz = (1 + l)z = 0 \cdot z = 0$). Suppose now $z, w \in A$ and $z \neq w$ (if $z = w$ choose $x = 0$). Then $lwz^{-1} \neq l$ and by (A.4) there exists a unique $x^* \in A \backslash \{l\}$ such that $1 + x^* = l \cdot lwz^{-1} = wz^{-1}$. Letting $x = zx^*$ we have $z + x = w$. From (A.2) it immediately follows now that $l = -1$.

The distributive laws follow immediately from (A.3) and commutativity of multiplication. It only remains to show that the inverse property

$$(IP)\ (z + w) + (-w) = z$$

holds for all $z, w \in N$. First note that for all $x \in A \backslash \{l\}$ we have by (A.4) that $1 + x = -y$ if and only if $1 + y = -x$ and thus $(x + 1) + (-1) = (-y) + (-1) = x$. Then, $(z + w) + (-w) = [(zw^{-1} + 1) + (-1)]w = (zw^{-1})w = z$ for all $z, w \in A$, $zw^{-1} \neq l = -1$. If $z = 0$, $w = 0$ or $z = -w$, (IP) holds trivially. This completes the proof of Theorem 1.2.

We now examine the structure of the admissible partition of an abelian group A of order n in more detail. If n is odd then $l = 1$ and for every admissible class $\theta = \{x, y, yx^{-1}, x^{-1}, y^{-1}, xy^{-1}\}$, $\theta \subset A\backslash\{1\}$ and therefore $x \neq y$. If $y \neq x^2$ then $yx^{-1} \neq x$ and by $(*)$, $x \neq y^{-1}$ and it can be readily verified that $|\theta| = 6$. If $y = x^2$ then $yx^{-1} = x$ and by $(*)$ $x^{-1} = y$, whence $x^3 = 1$ and $\theta = \{x, x^2\}$. Note that this can only occur when $3|n$, i.e., $n \equiv 3 \pmod 6$. From this it immediately follows that when $n \equiv 5 \pmod 6$ every admissible class is of size six and thus no admissible partition of A can exist.

If n is even and A has only one element l of order two then $\theta \subset A\backslash\{l\}$ for each admissible class $\theta = \{x, y, yx^{-1}, x^{-1}, y^{-1}, xy^{-1}\}$. The unique class θ_l containing the identity is of the form $\theta_l = \{1\}$ or $\theta_l = \{1, x, x^{-1}\}$, $x \neq 1, l$. Every remaining admissible class is again of size six if $y \neq x^2$ or of the form $\theta = \{x, x^2\}$ if $y = x^2$, in which case $x^3 = 1$ and $n \equiv 0 \pmod 6$. (Henceforth we will refer to an admissible class of size six as an *admissible sextuple* and an admissible class of the form $\theta = \{x, x^2\}$ with $x^3 = 1$ as an *admissible pair.*)

By means of a simple counting argument we can now summarize our conclusions in the following lemma:

LEMMA 2.3. *Let N_v be an AIP neofield of order v with $(N_v^*, \cdot) = A$, and let π be the admissible partition of A induced by N_v.*

(1) *When $v \equiv 0 \pmod 6$ N_v does not exist.*

(2) *When $v \equiv 2 \pmod 6$, π consists of $\{1\}$ and $\frac{1}{6}(v - 2)$ admissible sextuples.*

(3) *When $v \equiv 4 \pmod 6$, then π consists of $\{1\}$, h admissible pairs where $h \equiv 1 \pmod 3$ and $\frac{1}{6}(v - 2 - 2h)$ admissible sextuples.*

(4) *When $v \equiv 3 \pmod 6$ then π consists of $\{-1\}$, $\theta_l = \{1\}$ and $\frac{1}{6}(v - 3)$ admissible sextuples.*

(5) *When $v \equiv 5 \pmod 6$ then π consists of $\{-1\}$, $\theta_l = \{1, x, x^{-1}\}$ $(x \neq 1)$ and $\frac{1}{6}(v - 5)$ admissible sextuples.*

(6) *When $v \equiv 1 \pmod 6$ then π consists of $\{-1\}$, θ_l, h admissible pairs where $h \equiv 1 \pmod 3$ if $|\theta_l| = 3$; $h \equiv 2 \pmod 3$ if $|\theta_l| = 1$ and $\frac{1}{6}(v - 2 - 2h - |\theta_l|)$ admissible sextuples.*

3. AIP neofields of even order. From Lemma 2.3(1) we know that when $v \equiv 0 \pmod 6$ no *AIP* neofield of order v can exist. In addition, it can be easily verified that there is no neofield N of order 10 having $(N^*, \cdot) = C_9$ (see [**1**, p. 39]). In this section we show that there are no other exceptions to the existence of even ordered *AIP* neofields, i.e., there exists an even ordered *AIP* neofield N_v having $(N_v^*, \cdot) \cong A$ if and only if $v \equiv 2$ or $4 \pmod 6$ and $A \neq C_9$.

In the forthcoming constructions we make use of the following lemma:

LEMMA 3.1. *Let A_1, A_2 be abelian groups of odd order having admissible partitions π_1, π_2 respectively. Then there exists an admissible partition of $A_1 \times A_2$.*

Proof. For each admissible class (pair or sextuple) $\theta_1 = \{x_1, y_1, y_1x_1^{-1}, x_1^{-1},$

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$y_1^{-1}, x_1y_1^{-1}\}$ of π_1 and $\theta_2 = \{x_2, y_2, y_2x_2^{-1}, x_2^{-1}, y_2^{-1}, x_2y_2^{-1}\}$ of π_2 we form the following classes in $A_1 \times A_2 \backslash \{(1, 1)\}$:

$$\tau_1 = \{(x_1, 1), (y_1, 1), (y_1x_1^{-1}, 1), (x_1^{-1}, 1), (y_1^{-1}, 1), (x_1y_1^{-1}, 1)\}$$
$$\tau_2 = \{(1, x_2), (1, y_2), (1, y_2x_2^{-1}), (1, x_2^{-1}), (1, y_2^{-1}), (1, x_2y_2^{-1})\}$$
$$\tau_3 = \{(x_1, x_2), (y_1, y_2), (y_1x_1^{-1}, y_2x_2^{-1}), (x_1^{-1}, x_2^{-1}), (y_1^{-1}, y_2^{-1}), (x_1y_1^{-1}, x_2y_2^{-1})\}$$
$$\tau_4 = \{(x_1, y_2), (y_1, y_2x_2^{-1}), (y_1x_1^{-1}, x_2^{-1}), (x_1^{-1}, y_2^{-1}), (y_1^{-1}, x_2y_2^{-1}), (x_1y_1^{-1}, x_2)\}$$
$$\tau_5 = \{(x_1, y_2x_2^{-1})(y_1, x_2^{-1}), (y_1x_1^{-1}, y_2^{-1}), (x_1^{-1}, x_2y_2^{-1}), (y_1^{-1}, x_2), (x_1y_1^{-1}, y_2)\}$$
$$\tau_6 = \{(x_1, x_2^{-1}), (y_1, y_2^{-1}), (y_1x_1^{-1}, x_2y_2^{-1}), (x_1^{-1}, x_2), (y_1^{-1}, y_2), (x_1y_1^{-1}, y_2x_2^{-1})\}$$
$$\tau_7 = \{(x_1, y_2^{-1}), (y_1, x_2y_2^{-1}), (y_1x_1^{-1}, x_2), (x_1^{-1}, y_2), (y_1^{-1}, y_2x_2^{-1}), (x_1y_1^{-1}, x_2^{-1})\}$$
$$\tau_8 = \{(x_1, x_2y_2^{-1}), (y_1, x_2), (y_1x_1^{-1}, y_2), (x_1^{-1}, y_2x_2^{-1}), (y_1^{-1}, x_2^{-1}), (x_1y_1^{-1}, y_2^{-1})\}$$

The classes thus obtained are clearly equal or disjoint and therefore yield an admissible partition of $A_1 \times A_2$. Note that the six classes $\tau_3, \tau_4, \ldots, \tau_8$ partition to set $\theta_1 \times \theta_2$ into admissible classes. We call this the *direct product* of θ_1 and θ_2.

If A is an abelian group of odd order then by the fundamental theorem of finite abelian groups we can write $A = C_{n_1} \times C_{n_2} \times \ldots \times C_{n_t}$ where $n_i | n_{i-1}$ ($i = 2, \ldots, t$) and n_i odd. Letting a_i be a generator of C_{n_i} ($i = 1, 2, \ldots, t$) we have $A = \{a_1^{k_1}a_2^{k_2} \ldots a_t^{k_t} | k_i \in Z_{n_i}\}$ and $A \cong Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_t}$ under the canonical map $\varphi(a_1^{k_1}a_2^{k_2} \ldots a_t^{k_t}) = (k_1, k_2, \ldots, k_t)$. In particular $\varphi(1) = (0, 0, \ldots, 0)$ and corresponding to an admissible class

$$\theta = \{x, y, yx^{-1}, x^{-1}, y^{-1}, xy^{-1}\}$$

in an admissible partition of A we have an admissible class

$$\theta' = \{k, j, j - k, -k, -j, k - j\}$$

(where $k = (k_1, k_2, \ldots, k_t)$, $j = (j_1, j_2, \ldots, j_t)$, $x = a_1^{k_1}a_2^{k_2} \ldots a_t^{k_t}$ and $y = a_1^{j_1}a_2^{j_2} \ldots a_t^{j_t}$) in an admissible partition of $Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_t}$.

We now construct admissible partitions for $Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_t}$ by induction on t. The case $t = 1$ corresponds to the cyclic constructions given in [**1**, pp. 39-51], where it is proved that Z_n has an admissible partition for all $n \equiv 1$ or 3 (mod 6), $n \neq 9$. To simplify the analysis we give cases $t = 2$ and $t > 2$ as separate theorems.

THEOREM 3.2. *Let* $A = C_{n_1} \times C_{n_2}$ *where* $n_2 | n_1$, $n_2 > 1$, $n = n_1 \cdot n_2 \equiv 1$ *or* 3 (mod 6). *Then* A *has an admissible partition.*

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Proof. Since $n_2 | n_1$ and $n_1 \cdot n_2 \equiv 1$ or 3 (mod 6) we must have one of the following:

(1) $n_1 \equiv 1 \pmod 6$ and $n_2 \equiv 1 \pmod 6$
(2) $n_1 \equiv 3 \pmod 6$ and $n_2 \equiv 1 \pmod 6$
(3) $n_1 \equiv 3 \pmod 6$ and $n_2 \equiv 3 \pmod 6$
(4) $n_1 \equiv 3 \pmod 6$ and $n_2 \equiv 5 \pmod 6$
(5) $n_1 \equiv 5 \pmod 6$ and $n_2 \equiv 5 \pmod 6$.

Cases (1), (2) and (3) (with $n_1, n_2 \neq 9$) follow from the cyclic case and Lemma 3.1. For the remaining cases we will use the following notation:

(a) For an arbitrary $r \in Z_{n_1} \backslash \{0, n_1/3, 2\, n_1/3\}$ and $s \in Z_{n_2} \backslash \{0\}$ we let $A_{r,s}$ denote the following admissible sextuple in $Z_{n_1} \times Z_{n_2}$:

$$A_{r,s} = \alpha = \{(r, s), (-r, s), (-2r, 0), (-r, -s), (r, -s), (2r, 0)\}$$

	0	s	$-s$
r		α	α
$-r$		α	α
$2r$	α		
$-2r$	α		

(b) For an arbitrary $r \in Z_{n_1} \backslash \{0\}$ and $s \in Z_{n_2} \backslash \{0, n_2/3, 2\, n_2/3\}$ we let $B_{r,s}$ denote the following admissible sextuple in $Z_{n_1} \times Z_{n_2}$:

$$B_{r,s} = \beta = \{(r, s), (r, -s), (0, -2s), (-r, -s), (-r, s), (0, 2s)\}.$$

	s	$-s$	$2s$	$-2s$
0			β	β
r	β	β		
$-r$	β	β		

(c) For an arbitrary $r \in Z_{n_1} \backslash \{0, n_1/3, 2\, n_1/3\}$ and $s \in Z_{n_2} \backslash \{0, n_2/3, 2\, n_2/3\}$ we let $C_{r,s}$ denote the union of the following two admissible sextuples in $Z_{n_1} \times Z_{n_2}$:

$$\gamma = \{(r, s), (2r, -s), (r, -2s), (-r, -s), (-2r, s), (-r, 2s)\}$$
$$\gamma' = \{(r, -s), (2r, s), (r, 2s), (-r, s), (-2r, -s), (-r, -2s)\}$$

	s	$-s$	$2s$	$-2s$
r	γ	γ'	γ'	γ
$-r$	γ'	γ	γ	γ'
$2r$	γ'	γ		
$-2r$	γ	γ'		

(d) For an arbitrary admissible sextuple $\theta = \{m, n, n - m, -m, -n, m - n\}$ in Z_{n_1} an arbitrary $s \in Z_{n_2}\backslash\{0, n_2/3, 2\, n_2/3\}$ we let $D_{\theta,s}$ denote the union of the following two admissible sextuples in $Z_{n_1} \times Z_{n_2}$:

$$\delta = \{(m, s), (n, 2s), (n - m, s), (-m, -s), (-n, -2s), (m - n, s)\}$$
$$\delta' = \{(m, -s), (n, -2s), (n - m, -s), (-m, s), (-n, 2s), (m - n, s)\}$$

	s	$-s$	$2s$	$-2s$
m	δ	δ'		
$-m$	δ'	δ		
n			δ	δ'
$-n$			δ'	δ
$n - m$	δ	δ'		
$m - n$	δ'	δ		

(e) For an arbitrary $r \in Z_{n_1}\backslash\{0, n_1/3, 2\, n_1/3\}$ and an arbitrary admissible sextuple $\theta' = \{p, q, q - p, -p, -q, p - q\}$ in Z_{n_2} we let $E_{r,\theta'}$ denote the union of the following two admissible sextuples in $Z_{n_1} \times Z_{n_2}$:

$$\epsilon = \{(r, p), (2r, q), (r, q - p), (-r, -p), (-2r, -q), (-r, p - q)\}$$
$$\epsilon' = \{(-r, p), (-2r, q), (-r, q - p), (r, -p), (2r, -q), (r, p - q)\}$$

	p	$-p$	q	$-q$	$q-p$	$p-q$
r	ϵ	ϵ'			ϵ	ϵ'
$-r$	ϵ'	ϵ			ϵ'	ϵ
$2r$			ϵ	ϵ'		
$-2r$			ϵ'	ϵ		

We now construct admissible partitions for the remaining subcases of *Case* (3): $n_1 \equiv 3 \pmod 6$, $n_2 \equiv 3 \pmod 6$. If $n_1 = 9$ we must have $n_2 = 3$ or $n_2 = 9$. If $n_2 = 3$ then the admissible sextuples $A_{1,1}$, $A_{2,1}$ and $A_{4,1}$ together with the admissible pairs $\{(0, 1), (0, 2)\}$; $\{(3, 0), (6, 0)\}$; $\{(3, 1), (6, 2)\}$ and $\{(3, 2), (6, 1)\}$ give an admissible partition of $Z_9 \times Z_3$. If $n_2 = 9$ then the admissible sextuples $A_{1,3}$, $A_{2,3}$, $A_{4,3}$; $B_{3,1}$, $B_{3,2}$, $B_{3,4}$; $C_{1,1}$, $C_{2,2}$, $C_{4,4}$ together with the admissible pairs $\{(0, 3), (0, 6)\}$, $\{(3, 0), (6, 0)\}$, $\{(3, 3), (6, 6)\}$ and $\{(3, 6), (6, 3)\}$ give an admissible partition of $Z_9 \times Z_9$.

If $n_2 = 9$, $n_1 > 9$, then there exists an admissible partition π of Z_{n_1} consisting of $\{0\}$, $\{n_1/3, 2\, n_1/3\}$ and admissible sextuples of the form

$$\theta = \{m, n, n - m, -m, -n, m - n\},$$

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[**1**]. For each such θ we construct the following admissible sextuples in $Z_{n_1} \times Z_9$:

$$\tau_0 = \{(m, 0), (n, 0), (n - m, 0), (-m, 0), (-n, 0), (m - n, 0)\}$$
$$\tau_1 = \{(m, 3), (n, 6), (n - m, 3), (-m, 6), (-n, 3), (m - n, 6)\}$$
$$\tau_2 = \{(m, 6), (n, 3), (n - m, 6), (-m, 3), (-n, 6), (m - n, 3)\}$$

as well as $D_{\theta,1}$, $D_{\theta,2}$, $D_{\theta,4}$. Note that this accounts for all the elements of $\theta \times Z_9$ (see Figure 1). In addition, we construct the admissible sextuples

$$B_{n_1/3,1}, B_{n_1/3,2}, B_{n_1/3,4}$$

which together with the admissible pairs

$$P^0 = \{(0, 3), (0, 6)\}, \quad P_0 = \{(n_1/3, 0), (2\,n_1/3, 0)\}$$
$$P_1 = \{(n_1/3, 3), (2\,n_1/3, 6)\}, \quad P_2 = \{(n_1/3, 6), (2\,n_1/3, 3)\}$$

account for all the elements of $\{0, n_1/3, 2\,n_1/3\} \times Z_9 \backslash \{(0, 0)\}$.

	0	3	6	1 8	2 7	4 5
0	—	P^0	P^0	$B_{n_1/3,4}$	$B_{n_1/3,1}$	$B_{n_1/3,2}$
$n_1/3$	P_0	P_1	P_2	$B_{n_1/3,1}$	$B_{n_1/3,2}$	$B_{n_1/3,4}$
$2n_1/3$	P_0	P_2	P_1			
θ: m	τ_0	τ_1	τ_2	$D_{\theta,1}$	$D_{\theta,2}$	$D_{\theta,4}$
$-m$	τ_0	τ_2	τ_1			
n	τ_0	τ_1	τ_2	$D_{\theta,4}$	$D_{\theta,1}$	$D_{\theta,2}$
$-n$	τ_0	τ_2	τ_1			
$n-m$	τ_0	τ_1	τ_2	$D_{\theta,1}$	$D_{\theta,2}$	$D_{\theta,4}$
$m-n$	τ_0	τ_2	τ_1			

FIGURE 1

Case (4): $n_1 \equiv 3 \pmod 6$, $n_2 \equiv 5 \pmod 6$. Here $n_2 \geqq 5$ whence $n_1 \geqq 15$ and there is an admissible partition of Z_{n_1} consisting of $\{0\}$, $\{n_1/3, 2\,n_1/3\}$ and admissible sextuples $\theta = \{m, n, n - m, -m, -n, m - n\}$, [**1**]. We also partition $Z_{n_2}\backslash\{0\}$ into sets of the form $\sigma_s = \{\pm s, \pm 2s, \pm 2^2 s, \ldots, \pm 2^t s\}$ where $2^{t+1}s = s$ or $-s$. Note that for all $s \neq 0$, $|\sigma_s| \geqq 4$ since $s \neq -s$ (n_2 odd), $2s \neq s$ and $2s \neq -s$ ($3 \nmid n_2$).

For each θ in Z_{n_1} and σ_s in Z_{n_2} defined as above we construct the following admissible sextuples in $Z_{n_1} \times Z_{n_2}$:

$$\tau_0 = \{(m, 0), (n, 0), (n - m, 0), (-m, 0), (-n, 0), (m - n, 0)\}$$

$$D_{\theta,s}, D_{\theta,2s}, \ldots, D_{\theta,2^ts}.$$

In addition, we construct the admissible sextuples

$$B_{n_1/3,s}, B_{n_1/3,2s}, \ldots, B_{n_1/3,2^ts}.$$

Those, together with the admissible pair $P = \{(n_1/3, 0), (2\, n_1/3, 0)\}$ yield an admissible partition of $Z_{n_1} \times Z_{n_2}$ (see Figure 2).

	0	$s \quad -s$	$2s \quad -2s$	$\cdots$	$2^ts \quad -2^ts$
0	$-$	$B_{n_1/3,2^ts}$	$B_{n_1/3,s}$	$\cdots$	$B_{n_1/3,2^{t-1}s}$
$n_1/3$	P	$B_{n_1/3,s}$	$B_{n_1/3\ 2s}$	$\cdots$	$B_{n_1/3\ 2^ts}$
$2\, n_1/3$	P				
m	τ_0	$D_{\theta,s}$	$D_{\theta,2s}$	$\cdots$	$D_{\theta,2^ts}$
$-m$	τ_0				
n	τ_0	$D_{\theta,2^ts}$	$D_{\theta,s}$	$\cdots$	$D_{\theta,2^{t-1}s}$
$-n$	τ_0				
$n - m$	τ_0	$D_{\theta,s}$	$D_{\theta,2s}$	$\cdots$	$D_{\theta,2^ts}$
$m - n$	τ_0				

FIGURE 2

Case (5): $n_1 \equiv 5 \pmod 6$, $n_2 \equiv 5 \pmod 6$. The construction in this case is based on the following lemma, the proof of which is given in the Appendix.

LEMMA 3.3. *For all* $n \equiv 5 \pmod 6$ *except* $n = 11$ *there exists a partition of* $Z_n \backslash \{0, w, -w, 2w, -2w\}$ (*where* $w = (n + 1)/6$) *into* $(n - 5)/6$ *admissible sextuples.*

If $n_1 \neq 11$, $n_2 \neq 11$ we partition $Z_{n_1} \backslash \{0, w_1, -w_1, 2w_1, -2w_1\}$ ($w_1 = (n_1 + 1)/6$) into admissible sextuples θ_i and $Z_{n_2} \backslash \{0,, w_2, -w_2, 2w_2, -2w_2\}$ ($w_2 = (n_2 + 1)/6$) into admissible sextuples θ_j'. For each θ_i, θ_j' thus obtained we construct the six admissible sextuples given by the direct product of θ_i and θ_j'

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(see Lemma 3.1), as well as the admissible sextuples given by D_{θ_i, w_2} and $E_{w_1, \theta_j'}$ (see Figure 3).

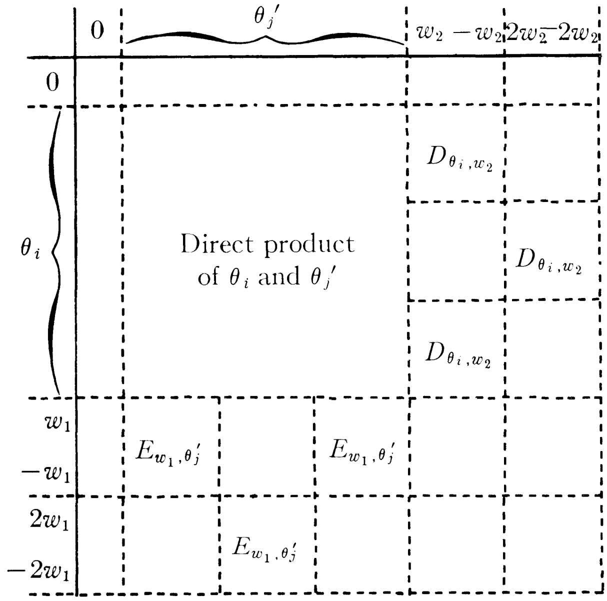

FIGURE 3

Note now that for each pair $\{r, -r\}$ in Z_{n_1} where $r \neq 0, \pm w_1, \pm 2w_1$ there is exactly one pair $\{x_r, -x_r\}$ in $Z_{n_2} \backslash \{0\}$ such that the elements of $\{r, -r\} \times \{x_r, -x_r\}$ have not been accounted for. We construct the sextuples A_{r, x_r} for each pair $\{r, -r\}$, $r \neq 0, \pm w_1, \pm 2w_1$, and in addition we construct A_{w_1, w_2}, $A_{2w_1, 2w_2}$.

We now have that for each pair $\{s, -s\}$, $s \in Z_{n_2} \backslash \{0\}$ there is exactly one pair of elements $\{y_s, -y_s\}$ $y_s \in Z_{n_1} \backslash \{0\}$ such that the elements of the set $\{y_s, -y_s\} \times \{s, -s\}$ have not been accounted for. We construct the sextuples $B_{y_s, s}$ for all pairs $\{s, -s\}$, $s \in Z_{n_2} \backslash \{0\}$. This accounts for all the remaining elements of $Z_{n_1} \times Z_{n_2} \backslash \{(0, 0)\}$.

In the case $n_1 = 11$ we must have $n_2 = 11$ as well. Here an admissible partition of $Z_{11} \times Z_{11}$ is given by the admissible sextuples:

$A_{1,4}, A_{2,8}, A_{4,5}, A_{8,1}, A_{5,2}$;

$B_{4,1}, B_{8,2}, B_{5,4}, B_{1,8}, B_{2,5}$;

$C_{1,1}, C_{2,2}, C_{4,4}, C_{8,8}$ and $C_{5,5}$.

If $n_1 > 11$, $n_2 = 11$ we use Lemma 3.3 to obtain a partition of $Z_{n_1} \backslash \{0, w_1, -w_1, 2w_1, -2w_1\}$, $w_1 = (n_1 + 1)/6$, into admissible sextuples θ_i. For each θ_i thus obtained we construct $D_{\theta_i,1}, D_{\theta_i,2}, D_{\theta_i,4}, D_{\theta_i,8}$. Corresponding to the admissible sextuple $\varphi = \{2, 3, 1, 9, 8, 10\}$ in Z_{11} we construct $E_{w_1, \varphi}$.

In addition we construct the admissible sextuples $B_{2w_1,1}$, $B_{2w_1,2}$, $B_{2w_1,4}$, $B_{w_1,8}$, $B_{w_1,5}$.

	0	1 10	2 9	3 7	8 3	5 6
0	—	$B_{w_1,5}$	$B_{2w_1,1}$	$B_{2w_1,2}$	$B_{2w_1,4}$	$B_{w_1,8}$
θ_i		$D_{\theta_i,1}$	$D_{\theta_i,2}$	$D_{\theta_i,4}$	$D_{\theta_i,8}$	
			$D_{\theta_i,1}$	$D_{\theta_i,2}$	$D_{\theta_i,4}$	$D_{\theta_i,8}$
		$D_{\theta_i,1}$	$D_{\theta_i,2}$	$D_{\theta_i,4}$	$D_{\theta_i,8}$	
w_1, $-w_1$		$E_{w_1,\varphi}$	$E_{w_1,\varphi}$		$B_{w_1,8}$	$B_{w_1,5}$
$2w_1$, $-2w_1$		$B_{2w_1,1}$	$B_{2w_1,2}$	$B_{2w_1,4}$	$E_{w_1,\varphi}$	

FIGURE 4

We now note that for each pair $\{r, -r\}$ in $Z_{n_1}\backslash\{0\}$ there is exactly one pair $\{x_r, -x_r\}$ in $Z_{11}\backslash\{0\}$ such that the elements of the set $\{r, -r\} \times \{x_r, -x_r\}$ have not been accounted for. We then construct the admissible sextuples A_{r,x_r} for each pair $\{r, -r\}$ in $Z_{n_1}\backslash\{0\}$, thus completing an admissible partition of $Z_{n_1} \times Z_{11}$. This concludes the proof of Theorem 3.2.

THEOREM 3.4. *Let* $A = C_{n_1} \times C_{n_2} \times \ldots \times C_{n_t}$ *where* $n_i | n_{i-1}$, $i = 2, \ldots, t$, $n_t > 1$, $t \geqq 3$ *and* $n = n_1 \cdot n_2 \ldots n_t \equiv 1$ *or* $3 \pmod 6$. *Then* A *has an admissible partition.*

Proof. We consider three cases according to the residue class of $n_1 \pmod 6$.

Case (1): $n_1 \equiv 1 \pmod 6$. Here we must have $n_2 \ldots n_t \equiv 1 \pmod 6$ since $n_2 \ldots n_t \equiv 3 \pmod 6$ implies $n_j \equiv 3 \pmod 6$ some $j = 2, \ldots, t$ and since $n_j | n_1$, $n_1 \equiv 3 \pmod 6$, a contradiction. There exists an admissible partition of C_{n_1} [**1**] and an admissible partition of $C_{n_2} \times \ldots \times C_{n_t}$ (by induction). From Lemma 3.1 we obtain an admissible partition of A.

Case (2): $n_1 \equiv 5 \pmod 6$. Here we must have $n_2 \ldots n_t \equiv 5 \pmod 6$ and thus there exists n_k, $k = 2, \ldots, t$, such that $n_k \equiv 5 \pmod 6$. By Theorem 3.2 there exists an admissible partition of $C_{n_1} \times C_{n_k}$ and by induction there exists an admissible partition of $C_{n_2} \times \ldots \times C_{n_{k-1}} \times C_{n_{k+1}} \times \ldots \times C_{n_t}$, since $n_2 \ldots n_{k-1} \cdot n_{k+1} \ldots n_t \equiv 1 \pmod 6$. Again using Lemma 3.1 we obtain an admissible partition of $(C_{n_1} \times C_{n_k}) \times (C_{n_2} \times \ldots C_{n_{k-1}} \times C_{n_{k+1}} \times \ldots \times C_{n_t}) \cong A$.

Case (3): $n_1 \equiv 3 \pmod 6$. Here we can have $n_2 \ldots n_t \equiv 1, 3$ or $5 \pmod 6$. If $n_2 \ldots n_t \equiv 1 \pmod 6$ then $n_1 \neq 9$ and we repeat the argument of Case (1); if $n_2 \ldots n_t \equiv 5 \pmod 6$ then again $n_1 \neq 9$ and we proceed as in Case (2).

Suppose now that $n_2 \ldots n_t \equiv 3 \pmod 6$. If $n_1 \neq 9$ then there exists an admissible partition of C_{n_1} by [**1**] and an admissible partition of $C_{n_2} \times \ldots \times C_{n_t}$ (by induction) and thus there exists an admissible partition of A by Lemma 3.1.

If $n_1 = 9$ and $n_t = 3$ then there exists an admissible partition of $C_{n_1} \times \ldots \times C_{n_{t-1}}$ and an admissible partition of C_3 and again by Lemma 3.1 there exists an admissible partition of A.

If $n_1 = 9$ and $n_t = 9$ then $n_i = 9, i = 1, 2, \ldots, t$ and we consider separately the cases $t \geqq 4$ and $t = 3$. If $t \geqq 4$ then there exists an admissible partition of $C_9 \times C_9$ and an admissible partition of $C_9 \times \ldots \times C_9$ $(t - 2$ times) which by Lemma 3.1 yield an admissible partition of A. For $t = 3$ an admissible partition of $(Z_9 \times Z_9) \times Z_9 \cong A$ is given in Figure 5.

		0	3	6	1 8	2 7	4 5
	(0, 0)	—	P^0	P^0	$B_{(3,3),4}$	$B_{(3,3),1}$	$B_{(3,3),2}$
	(3, 3)	P_0	P_1	P_2	$B_{(3,3),1}$	$B_{(3,3),2}$	$B_{(3,3),4}$
	(6, 6)	P_0	P_2	P_1			
φ	(0, 3)	φ_0	φ_1	φ_2	$D_{\varphi,1}$	$D_{\varphi,2}$	$D_{\varphi,4}$
φ	(0, 6)	φ_0	φ_2	φ_1			
φ	(3, 0)	φ_0	φ_1	φ_2	$D_{\varphi,4}$	$D_{\varphi,1}$	$D_{\varphi,2}$
φ	(6, 0)	φ_0	φ_2	φ_1			
φ	(3, 6)	φ_0	φ_1	φ_2	$D_{\varphi,1}$	$D_{\varphi,2}$	$D_{\varphi,4}$
φ	(6, 3)	φ_0	φ_2	φ_1			
θ		θ_0	θ_1	θ_2	$D_{\theta,1}$	$D_{\theta,2}$	$D_{\theta,4}$
θ		θ_0	θ_2	θ_1			
θ		θ_0	θ_1	θ_2	$D_{\theta,4}$	$D_{\theta,1}$	$D_{\vartheta,2}$
θ		θ_0	θ_2	θ_1			
θ		θ_0	θ_1	θ_2	$D_{\theta,1}$	$D_{\theta,2}$	$D_{\theta,4}$
θ		θ_0	θ_2	θ_1			

FIGURE 5

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Note 1: The elements of $Z_9 \times Z_9$ denoting the rows of Figure 5 appear partitioned into admissible sextuples θ as given in Theorem 3.2, and the additional admissible sextuple $\varphi = \{(0, 3), (3, 0), (3, 6), (0, 6), (6, 0), (6, 3)\}$ obtained by combining the admissible pairs $\{(0, 3), (0, 6)\}$, $\{(3, 0), (6, 0)\}$ and $\{(3, 6), (6, 3)\}$.

Note 2: The notation in Figure 5 is analogous to that of Theorem 3.2, where the elements in the first projection are elements in $Z_9 \times Z_9$.

4. AIP neofields and steiner triple systems. Let N_v be an AIP neofield of order $v \equiv 2$ or $4 \pmod 6$ with $(N_v^*, \cdot) = A$. From [**3**, Theorem 2.1] we have that N_v is equivalent to a Steiner triple system τ_n of order $n = v - 1$ having a regular (i.e., sharply transitive) automorphism group isomorphic to A. It immediately follows from the results of the previous section that *every abelian group A of order $n \equiv 1$ or $3 \pmod 6$, $A \neq C_9$, is a regular automorphism group for some Steiner triple system τ_n.* In this section we discuss nonisomorphic Steiner triple systems having the same abelian regular automorphism group.

Let N_v and N_v' be two AIP neofields based on the same set of elements N and having the same multiplicative group $A = C_{n_1} \times C_{n_2} \times \ldots \times C_{n_t}$ where $n_1 \cdot n_2 \ldots n_t = v - 1 \equiv 1$ or $3 \pmod 6$, $n_i | n_{i-1}$ $(i = 2, \ldots, t)$, $n_t > 1$. If N_v and N_v' are isomorphic under an isomorphism φ, φ must induce an automorphism of A and for each generator a_i of C_{n_i} $(i = 1, 2, \ldots, t)$ the order of $\varphi(a_i)$ in A must equal the order of a_i in A—which is n_i. It follows that *the number of distinct presentations of an AIP neofield N_v (based on the same set N) is at most the number w_t of t-tuples $(x_1, x_2, \ldots, x_t)$ where x_i is of order n_i in A.*

From Theorems 3.2, 3.4 and Lemma 1.3 we know that when $v \equiv 2 \pmod 6$ an admissible partition of A always exists and it contains $(v - 2)/6$ admissible sextuples. For $v \equiv 4 \pmod 6$ it can be easily verified that the constructions of Theorems 3.2, 3.4 can be slightly changed to give admissible partitions consisting of $(v - 4)/6$ admissible sextuples. Thus, for any $v \equiv 2$ or $4 \pmod 6$ a neofield N_v can be constructed having $[v/6]$ admissible sextuples in the admissible partition of its multiplicative group ($[x]$ denotes greatest integer smaller-equal than x).

In [**3**, Theorem 3.8] it is shown that given an admissible partition of an abelian group A consisting of σ admissible sextuples we can construct 2^σ AIP neofields having multiplicative group A. From the above remarks we get:

LEMMA 4.1. *Let $A = C_{n_1} \times C_{n_2} \times \ldots \times C_{n_t}$, $n_i | n_{i-1}$ $(i = 2, \ldots, t)$, $n_1 \cdot n_2 \ldots n_t = v - 1 \equiv 1$ or $3 \pmod 6$, $A \neq C_9$. Then there are at least*

$$(1) \quad \frac{2^{[v/6]}}{w_t}$$

nonisomorphic AIP neofields having multiplicative group A.

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We now observe that nonisomorphic AIP neofields having the same multiplicative group A may also have isomorphic additive loops. We wish to determine therefore a lower bound for the number of nonisomorphic AIP neofields having a given multiplicative group A and nonisomorphic additive loops, for these correspond to nonisomorphic Steiner triple systems having the same regular automorphism group A [**3**].

Let $\tau_n = [S, \mathcal{S}]$ be a Steiner triple system of order n having an abelian regular automorphism group $A = C_{n_1} \times C_{n_2} \times \ldots \times C_{n_t}, n_i | n_{i-1}$ $(i = 2, \ldots, t)$. The action of A on the elements of S is determined by the action of the automorphisms $a_1, a_2, \ldots, a_t$ (a_i is a generator of C_{n_i}) on S, and this itself is completely determined by the action of $a_1, a_2, \ldots, a_t$ on a maximal generating set Ω of τ_n. Let $\Omega = \{s_1, s_2, \ldots, s_\alpha\} \subset S$. From [**3**, Lemma 5.1] we know that $\alpha \leqq \log_2 (n + 1)$. Now each a_i maps the generating set $\{s_1, s_2, \ldots, s_\alpha\}$ into another generating set $\{s_1', s_2', \ldots, s_\alpha'\}$. This yields at most $n(n - 1) \ldots (n - \alpha + 1)$ choices for the action of a_i and there are therefore at most $(n(n - 1) \ldots (n - \alpha + 1))^t$ choices for a tuple $(a_1, a_2, \ldots, a_t)$ where a_i is a generator of C_{n_i}. Thus, there are at most $\varphi_t = (n(n - 1) \ldots (n - \alpha + 1))^t / w_t$ ways in which A can act as a regular automorphism group on τ_n and therefore from the arguments given in [**3**, p. 13] there are at most φ_t nonisomorphic AIP neofields having multiplicative group A and isomorphic additive loops. This, together with Lemma 4.1 implies that the number of nonisomorphic AIP *neofields having multiplicative group A and non-isomorphic additive loops is at least*

$$(2) \quad \frac{2^{[v/6]}}{w_t \varphi_t} = \frac{2^{[v/6]}}{(n(n-1)\ldots(n-\alpha+1))^t} = \frac{2^{[v/6]}}{((v-1)(v-2)\ldots(v-\alpha))^t} > \frac{2^{[v/6]}}{v^{\alpha t}} \geqq \frac{2^{[v/6]}}{v^{t.\log_2 v}} = \frac{2^{[v/6]}}{2^{(\log_2 v)^2 . t}} > \frac{2^{[v/6]}}{2^{(\log_2 v)^3}}$$

Since $2^{[v/6]}/2^{(\log_2 v)^3} \to \infty$ as $v \to \infty$ we have:

THEOREM 4.2. *Let t be a fixed positive integer. If we consider abelian groups of the form* $A = C_{n_1} \times C_{n_2} \times \ldots \times C_{n_t}$ *where* n_i *are integers bigger than one,* $n_i | n_{i-1}$ $(i = 2, \ldots, t)$ *and* $n = n_1 \cdot n_2 \ldots n_t \equiv 1$ *or* 3 (mod 6), *the number of nonisomorphic Steiner triple systems having the abelian group A for a regular automorphism group goes to infinity with n.*

Appendix.

LEMMA 3.3. *For all* $u \equiv 5$ (mod 6) *except* $u = 11$ *there exists a partition of* $Z_u \backslash \{0, w, -w, 2w, -2w\}$ (*where* $w = (u + 1)/6$) *into* $(u - 5)/6$ *admissible sextuples of the form* $(m, n, n - m, -m, -n, m - n)$.

Proof. We consider four cases according to the residue class of u (mod 24).

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(We use the following notation:

$$S(m, n) = S_{n-m} = (m, n, n - m, -m, -n, m - n).)$$

Case (1) $u = 24k + 5$ $(w = 4k + 1)$. For $k = 0$ the lemma holds vacuously.

For $k = 1$, we have $u = 29$, $w = 5$ and the desired admissible sextuples are: $S_1 = S(6, 7)$; $S_2 = S(12, 14)$; $S_3 = S(8, 11)$ and $S_4 = S(9, 13)$.

For $k \geqq 2$ we obtain the desired sextuples by partitioning $\{1, 2, \ldots, 4k\} \cup \{4k + 2, \ldots, 8k + 1\} \cup \{8k + 3, 8k + 4, \ldots, 12k + 2\}$ into triples of the form $(m, n, n - m)$ as follows:

m	n	$n - m$
$8k + 4$	$12k + 2$	$4k - 2$
$8k + 5$	$12k + 1$	$4k - 4$
.	.	.
.	.	.
.	.	.
$10k + 1$	$10k + 5$	4
$10k + 2$	$10k + 4$	2
$4k + 2$	$8k + 1$	$4k - 1$
$4k + 3$	$8k$	$4k - 3$
.	.	.
.	.	.
.	.	.
$5k - 1$	$7k + 4$	$2k + 5$
$5k$	$7k + 3$	$2k + 3$
$5k + 3$	$7k + 2$	$2k - 1$
$5k + 4$	$7k + 1$	$2k - 3$
.	.	.
.	.	.
.	.	.
$6k$	$6k + 5$	5
$6k + 1$	$6k + 4$	3
$6k + 2$	$8k + 3$	$2k + 1$
$6k + 3$	$10k + 3$	$4k$
$5k + 1$	$5k + 2$	1

Case (2) $u = 24k + 23$ $(w = 4k + 4)$. For $k = 0$, we have $u = 23$, $w = 4$

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and the desired admissible sextuples are: $S_1 = S(5, 6)$; $S_2 = S(9, 11)$ and $S_3 = S(7, 10)$.

For $k = 1$, we have $u = 47$, $w = 8$ and the desired admissible sextuples are: $S_1 = S(10, 11)$; $S_2 = S(19, 21)$; $S_3 = S(12, 15)$; $S_4 = S(18, 22)$; $S_5 = S(9, 14)$; $S_6 = S(17, 23)$ and $S_7 = S(13, 20)$.

For $k \geqq 2$ we obtain the desired sextuples by partitioning $\{1, 2, \ldots, 4k + 3\} \cup \{4k + 5, \ldots, 8k + 6, 8k + 7\} \cup \{8k + 9, \ldots, 12k + 11\}$ into triples $(m, n, n - m)$ as follows:

m	n	$n - m$
$8k + 9$	$12k + 11$	$4k + 2$
$8k + 10$	$12k + 10$	$4k$
.	.	.
.	.	.
.	.	.
$10k + 8$	$10k + 12$	4
$10k + 9$	$10k + 11$	2
$4k + 5$	$8k + 6$	$4k + 1$
$4k + 6$	$8k + 5$	$4k + 1$
.	.	.
.	.	.
.	.	.
$5k + 3$	$7k + 8$	$2k + 5$
$5k + 4$	$7k + 7$	$2k + 3$
$5k + 7$	$7k + 6$	$2k - 1$
$5k + 8$	$7k + 5$	$2k - 3$
.	.	.
.	.	.
.	.	.
$6k + 4$	$6k + 9$	5
$6k + 5$	$6k + 8$	3
$6k + 6$	$8k + 7$	$2k + 1$
$6k + 7$	$10k + 10$	$4k + 3$
$5k + 5$	$5k + 6$	1

Case (3) $u = 24k + 11$ $(w = 4k + 2)$. For $k = 0$, $Z_{11}\backslash\{0, 2, 9, 4, 7\} = \{1, 3, 5, 6, 8, 10\}$ can never be arranged into an admissible sextuple.

For $k = 1$ we have $u = 35$, $w = 6$ and the desired admissible sextuples are: $S_1 = S(7, 8)$; $S_2 = S(9, 11)$; $S_3 = S(13, 16)$; $S_4 = S(14, 18)$ and $S_5 = S(10, 15)$.

For $k = 2$ we have $u = 59$, $w = 10$ and the desired sextuples are: $S_1 = S_1(27, 28)$; $S_2 = S(14, 16)$; $S_3 = S(22, 25)$; $S_4 = S(13, 17)$; $S_5 = S(21, 26)$; $S_6 = S(12, 18)$; $S_7 = S(23, 30)$; $S_8 = S(11, 19)$ and $S_9 = S(15, 24)$.

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For $k \geqq 3$ we obtain the desired sextuples by partitioning the set

$$\{1, 2, \ldots, 4k+1\} \cup \{4k+3, \ldots, 8k+3\} \cup \{8k+5, \ldots, 12k+4, -(12k+5)\}$$

into the following triples $(m, n, n-m)$: (Note that we have chosen $-(12k+5) = 12k+6$ instead of the more natural $12k+5$ as the last element listed.)

m	n	$n-m$
$4k+3$	$8k+3$	$4k$
$4k+4$	$8k+2$	$4k-2$
.	.	.
.	.	.
.	.	.
$6k+1$	$6k+5$	4
$6k+2$	$6k+4$	2
$8k+5$	$12k+4$	$4k-1$
$8k+6$	$12k+3$	$4k-3$
.	.	.
.	.	.
.	.	.
$9k+1$	$11k+8$	$2k+7$
$9k+2$	$11k+7$	$2k+5$
$9k+3$	$11k+4$	$2k+1$
$9k+4$	$11k+3$	$2k-1$
.	.	.
.	.	.
.	.	.
$10k+1$	$10k+6$	5
$10k+2$	$10k+5$	3
$10k+3$	$12k+6$	$2k+3$
$6k+3$	$10k+4$	$4k+1$
$11k+5$	$11k+6$	1

Case (4) $u = 24k+17$ $(w = 4k+3)$. For $k = 0$ we have $u = 17$, $w = 3$ and the desired sextuples are $S_1 = S(4, 5)$ and $S_2 = S(7, 9)$.

For $k \geqq 1$ we partition into triples $(m, n, n-m)$ the set $\{1, 2, \ldots, 4k+2\}$

$$\cup \{4k+4, \ldots, 8k+4, 8k+5\}$$
$$\cup \{8k+7, \ldots, 11k+7, -(11k+8), 11k+9, \ldots, 12k+8\}$$

as follows. (Note that in the place of $11k+8$ we have $-(11k+8) = 13k+9$.

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m	n	$n - m$
$4k + 4$	$8k + 4$	$4k$
$4k + 5$	$8k + 3$	$4k - 2$
.	.	.
.	.	.
.	.	.
$6k + 2$	$6k + 6$	4
$6k + 3$	$6k + 5$	2
$8k + 7$	$12k + 8$	$4k + 1$
$8k + 8$	$12k + 7$	$4k - 1$
.	.	.
.	.	.
.	.	.
$9k + 5$	$11k + 10$	$2k + 5$
$9k + 6$	$11k + 9$	$2k + 3$
$9k + 8$	$11k + 7$	$2k - 1$
$9k + 9$	$11k + 6$	$2k - 3$
.	.	.
.	.	.
.	.	.
$10k + 6$	$10k + 9$	3
$10k + 7$	$10k + 8$	1
$6k + 4$	$8k + 5$	$2k + 1$
$9k + 7$	$13k + 9$	$4k + 2$

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