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GK–DIMENSION OF ALGEBRAS WITH MANY GENERIC RELATIONS*

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Abstract. We prove some results on algebras, satisfying many generic relations. As an application we show that there are Golod–Shafarevich algebras which cannot be homomorphically mapped onto infinite dimensional algebras with finite Gelfand–Kirillov dimension. This answers a question of Zelmanov (Some open problems in the theory of infinite dimensional algebras, *J. Korean Math. Soc.* **44**(5) 2007, 1185–1195).

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1. Introduction. In this paper the Gelfand–Kirillov dimension of algebras, satisfying many generic relations, is studied. As an application, we prove some results on the growth of Golod–Shafarevich algebras. In 1964 Golod and Shafarevich proved the theorem given below [2].

THEOREM 1. *Let R_d be a non-commutative polynomial ring of d variables over a field K, and let I be the ideal generated by an infinite sequence of homogeneous elements of a degree larger than one, where the number of elements of degree i is equal to r_i. We put $r_i \leq s_i$. If the coefficients of the power series*

$$\left(1 - dt + \sum_{i=2}^{\infty} s_i t^i\right)^{-1}$$

are all non-negative, then the factor algebra R_d/I is infinite-dimensional.

We say that R_d/I is a Golod–Shafarevich algebra if there is a number $0 < t_0$, such that $H(t) = \sum_{i=2}^{\infty} r_i t^i$ converges at t_0 and $1 - dt_0 + H(t_0) < 0$. Golod–Shafarevich algebras were used to solve the General Burnside problem, Kurosh problem for algebraic algebras and the Class Field Tower problem [**1**, **2**]. It is known that Golod–Shafarevich algebras have exponential growth. In [**4**] Zelmanov asked whether every Golod–Shafarevich algebra can be mapped onto an infinite-dimensional algebra with finite Gelfand–Kirillov dimension. We show that the following result holds.

THEOREM 2. *Let K be a field of infinite transcendence degree. Then there is a Golod–Shafarevich algebra R such that every infinite-dimensional homomorphic image of R has exponential growth.*

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This answers a question of Zelmanov [**4**, Problem 5]. It is not known if a similar result holds for algebras over fields of finite transcendence degree. It is also not known if finitely presented Golod–Shafarevich algebras can be homomorphically mapped onto infinite-dimensional algebras with polynomial growth. The next result gives some information about quadratic Golod–Shafarevich algebras.

THEOREM 3. *Let K be a field of infinite transcendence degree, and let $m > 8$. Then there exists a graded algebra $A = A_1 + A_2 + \dots$ generated by A_1, with $\dim_K A_1 = m$ and presented by less than $m^2/4$ quadratic relations, such that for every i, the subalgebra of A generated by A_i cannot be epimorphically mapped onto the polynomial ring $K[t]$.*

This answers another question of Zelmanov [**4**, Conjecture3]). It is not known if in arbitrary quadratic Golod–Shafarevich algebras almost all Veronese subalgebras can be mapped onto algebras with linear growth or onto polynomial-identity algebras [E. Zelmanov, private communication].

For a general information about the Golod–Shafarevich algebras we refer the reader to [**4**] and about the Gelfand–Kirillov dimension to [**3**].

2. The main result. In this paper K is a field, and F is the prime subfield of K. Let R be a K-algebra. Given subsets S, Q of R, let us denote $S + Q = \{s + q : s \in S, q \in Q\}$, $SQ = \{\sum_{i=1}^{n} s_i q_i : s_i \in s, q_i \in Q$, where n is a natural number$\}$. Given a subset S of K, by $F[S]$ we denote the field extension of F generated by elements from S and by FS the linear space over F spanned by elements from S. Given set S, $card(S)$ will denote the cardinality of S. We start with the lemma given next.

LEMMA 1. *Let K be a field and F be a prime subfield of K. Let R be a K-algebra and M be a subset of R. Let $N_1 = M$, and for each $i > 1$, let N_i be a subset of FM^i, such that $KM^i = KN_i$. Denote $\alpha_i = card(N_i)$. Then there are subsets $S_i \subseteq K$ such that $S_1 = \{1\}$, $card(S_{i+1}) \leq card(S_i) + \alpha_{i+1}\alpha_i\alpha_1$ and $M^i \subseteq F[S_i]N_i$ for all i.*

Proof. We will proceed by induction on i. For $i = 1$ it is true because $N_1 = M$. Suppose the result holds for some i. We will show it is true for $i + 1$. Observe that M^{i+1} consists of finite sums of elements $m_{i+1} = m_i m_1$ for some $m_i \in M^i$, $m_1 \in M$. By the inductive assumption $m_i \subseteq F[S_i]N_i$. Therefore, $m_{i+1} \subseteq F[S_i]N_i N_1$. Recall that $N_i N_1 \subseteq KM^{i+1} = KN_{i+1}$. Consequently, every element $n_i n_1$ with $n_i \in N_i$ and $n_1 \in N_1$ can be written as a linear combination over K of elements from N_{i+1}. Namely $n_i n_1 = \sum_{n_{i+1} \in N_{i+1}} k_{n_{i+1},n_i,n_1} n_{i+1}$ for some $k_{n_{i+1},n_i,n_1} \in K$. Denote $K_{i+1} = \{k_{n_{i+1},n_i,n_1} : n_{i+1} \in N_{i+1}, n_i \in N_i, n_1 \in N_1\}$. Observe that $N_i N_1 \subseteq F[K_{i+1}]N_{i+1}$. Denote $S_{i+1} = S_i \cup K_{i+1}$. Then, $M^{i+1} \subseteq F[S_i]N_i N_1 \subseteq F[S_{i+1}]N_{i+1}$. Note that $card(S_{i+1}) \leq card(S_i) + card(K_{i+1})$. Hence, $card(S_{i+1}) \leq card(S_i) + \alpha_{i+1}\alpha_i\alpha_1$.

Let K be a field, and let F be the prime subfield of K. We say that elements $a_1, a_2, \dots, a_n$ are algebraically independent over F if the algebra generated over F by elements $a_1, a_2, \dots, a_n$ is free. □

The main result of this paper is the theorem given next.

THEOREM 4. *Let K be a field, and let F be the prime subfield of K. Let R be a K-algebra, and let M be a finite subset of R. Denote $\alpha_1 = card(M)$ and for $i > 1$, $\alpha_i = \dim_K KM^i$ for all i. Let m, n, t be natural numbers, and let $x_1, \dots, x_t \in FM^m$ and $m > 1$. Assume that there are elements $k_{i,j} \in K$ which are algebraically independent over*

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F, such that for all $i \le n$ we have

$$\sum_{j=1}^{t} k_{i,j} x_j = 0.$$

If $n > 1 + \sum_{i=2}^{m} \alpha_i \alpha_{i-1} \alpha_1$, then

$$x_1 = x_2 = \ldots = x_t = 0.$$

Proof. Suppose the contrary, and let γ be the smallest number, such that $x_\gamma \neq 0$. We can assume that $\gamma = 1$ and $x_1 \neq 1$. Consider subsets $N_1 \subseteq FM, \ldots N_m \subseteq FM^m$ such that $x_1 \in N_m$. Moreover, assume that $N_1 = M$, and for $1 < i \le m$ elements from the set N_i are linearly independent over K. By Lemma 1, there is set $S_m \subseteq K$ with cardinality not exceeding $c = 1 + \sum_{i=2}^{m} \alpha_i \alpha_{i-1} \alpha_1$, such that $FM^m \subseteq F[S_m]N_m$. This implies that there are elements $\xi_{i,q} \in F[S_m]$ for $2 \le i \le t$ and $q \in N_m$, such that $x_i = \sum_{q \in N_m} \xi_{i,q} q$. By substituting these expressions for elements x_i for the equations $\sum_{j=1}^{t} k_{i,j} x_j = 0$, we get $k_{i,1} x_1 + \sum_{j=2}^{t} k_{i,j} (\sum_{q \in N_m} \xi_{j,q} q) = 0$. Elements $q \in N_m$ are linearly independent over K; therefore the sum of the coefficients by x_1 should be 0, since $x_1 \in N_m$. It follows that $k_{i,1} + \sum_{j=2}^{t} k_{i,j} \xi_{j,x_1} = 0$, for $i = 1, 2, \ldots, n$. Denote $V = \{k_{i,j} : i = 1, 2, \ldots, n, j = 2, 3, \ldots, t\}$ and $E = F[V]$. By the above equations, we get $E[k_{1,1}, k_{2,1}, \ldots, k_{n,1}] \subseteq E[S_m]$. Note that the field $E[k_{1,1}, k_{2,1}, \ldots, k_{n,1}]$ has transcendence degree n over the field E, by the assumptions. On the other hand, the transcendence degree of the field $E[S_m]$ over E doesn't exceed the cardinality of S_m, which is smaller than n, by the assumptions – which is a contradiction. □

3. Golod–Shafarevich algebras. Let K be a field, and let $R_d = K[x_1, \ldots, x_d]$ be the non-commutative polynomial ring of d variables over a field K. Assigning the degree one for elements $x_1, \ldots, x_d$, let us define a gradation on R_d. We say that $f \in R_d$ is a homogeneous element in R_d if f is a sum of monomials of the same degree. Let I be the ideal in R_d, generated by homogeneous elements $f_1, f_2, \ldots$ of degrees larger than one. Suppose that the number of elements of degree i among $f_1, f_2, \ldots$ is r_i. Denote $H(t) = \sum_{i=2}^{\infty} r_i t^i$. Then R_d/I is a Golod–Shafarevich algebra if there is $0 < t_0$, such that $H(t)$ converges at t_0 and $1 - dt_0 + H(t_0) < 0$. By the Golod–Shafarevich theorem, every Golod–Shafarevich algebra has an exponential growth [**1**, **2**, **4**].

Proof of Theorem 2. Let $R_d = K[x_1, \ldots, x_d]$ be the non-commutative polynomial ring of d variables over a field K. Denote $M = \{x_1, \ldots, x_d\}$. Let $k_{i,n_j} \in K$ be algebraically independent over F elements of K, for $j = 2, 3, \ldots$, $n_j \in M^j$, $i = 1, 2, \ldots, 2^j$. Let I be the ideal in R_d, generated by 2^j generic relations of degree j, for all $j > 1$, namely by relations

$$\sum_{n_j \in M^j} k_{i,n_j} n_j,$$

for $j > 1$, $1 \le i \le 2^j$. Assume that $d > 16$. Notice that if $t_0 = 1/8$, then $H(t_0) = \sum_{i=2}^{n} 2^i t_0^i < 1/8$, and so $1 - dt_0 + H(t_0) < 1 - (d/8) + (1/8) < 0$. It follows that R_d/I is a Golod–Shafarevich algebra. Suppose now that Q is an ideal in $A = R_d/I$, such that A/Q is infinite-dimensional. Given $n_j \in M^j$ let $\bar{n}_j$ denote the image of n_j in A/Q and $\bar{M}$ denote the image of M in A/Q. Then for every number j, there is element $n_j \in M^j$ such

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that $\bar{n}_j \neq 0$, because A/Q is infinite-dimensional and generated in degree one. Observe that algebra A/Q satisfies the following relations:

$$\sum_{n_j \in M^j} k_{i,n_j} \bar{n}_j,$$

for $j > 1$, $1 \leq i \leq 2^j$. By Theorem 4, applied to the algebra $R = A/Q$ and the set $\bar{M} \subset R$, we get

$$2^i < 1 + \sum_{j=2}^{i} \alpha_j \alpha_{j-1} \alpha_1,$$

where $\alpha_1 = card(M) = d$, and for $j > 1$, $\alpha_j = dim_K K\bar{M}^j$ (because there is $\bar{n}_i \neq 0$ for every i). It follows that $2 \leq [limsup_{i\to\infty} log(dim_K K\bar{M}^i)]^2$. It also follows that $limsup_{i\to\infty} log(dim_K K\bar{M}^i) \geq \sqrt{2}$, and hence $R = A/Q$ has exponential growth.

4. Quadratic algebras. In this section we will prove Theorem 3.

Proof of Theorem 3. Let R_m be the free K-algebra, generated by elements $x_1, \ldots, x_m$. Denote $y_i = \sum_{j=1}^{m} d_{i,j} x_j$, where $d_{i,j} \in K$ are algebraically independent over F. Let I be the ideal in R_m generated by relations $y_i^2 = 0$ for $i = 1, \ldots, 2m$. Denote $A = R_m/I$. Let a_i be the image of x_i in R_m/I and c_i the image of y_i in R_m/I. Then $a_1, \ldots, a_m$ are generators of A, and $A = A_1 + A_2 + \ldots$, where $A_1 = Ka_1 + \ldots + Ka_m$ and $A_t = A_1^t$. We will show that for every t, the subalgebra $S(A_t)$ generated by A_t cannot be mapped onto a domain, and so $S(A_t)$ cannot be mapped onto $K[t]$. Suppose the contrary, and let t be a natural number and $f : S(A_t) \to D$ be a ring homomorphism onto a domain D. Then, $0 = f(rc_ic_ir') = f(rc_i)f(c_ir') = 0$ for every $i \leq 2m$ and every $r, r' \in A_{t-1}$. (If $t = 1$ take $r, r' \in K$.)

Since D is a domain, it follows that for each i, either $f(c_iA_{t-1}) = 0$ or $f(A_{t-1}c_i) = 0$. (We put $A_0 = K$.) Hence, there is a set $E \subseteq \{1, \ldots, 2m\}$ of cardinality at least m, such that either $f(A_{t-1}c_i) = 0$ for all $i \in E$ or $f(c_iA_{t-1}) = 0$ for all $i \in E$. Observe that for every $k \leq m$, $a_k \in \sum_{i\in E} Kc_i$, because elements $d_{i,j}$ are algebraically independent over F. (So the determinant of the related matrix is not zero.)

Hence, if $f(A_{t-1}c_i) = 0$ for all $i \in E$, then $f(A_{t-1}a_k) = 0$ for every $k \leq m$. Consequently, $f(A_t) = 0$. Similarly, if $f(c_iA_{t-1}) = 0$ for all $i \in E$, then $f(a_kA_{t-1}) = 0$ for every $k \leq m$ – which is a contradiction, since $f(A_t)$ generates D.

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