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Structure of fine Selmer groups over $\mathbb{Z}_p$-extensions

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Abstract

This paper is concerned with the study of the fine Selmer group of an abelian variety over a $\mathbb{Z}_p$-extension which is not necessarily cyclotomic. It has been conjectured that these fine Selmer groups are always torsion over $\mathbb{Z}_p[[\Gamma]]$, where Γ is the Galois group of the $\mathbb{Z}_p$-extension in question. In this paper, we shall provide several strong evidences towards this conjecture. Namely, we show that the conjectural torsionness is consistent with the pseudo-nullity conjecture of Coates–Sujatha. We also show that if the conjecture is known for the cyclotomic $\mathbb{Z}_p$-extension, then it holds for almost all $\mathbb{Z}_p$-extensions. We then carry out a similar study for the fine Selmer group of an elliptic modular form. When the modular forms are ordinary and come from a Hida family, we relate the torsionness of the fine Selmer groups of the specialization. This latter result allows us to show that the conjectural torsionness in certain cases is consistent with the growth number conjecture of Mazur. Finally, we end with some speculations on the torsionness of fine Selmer groups over an arbitrary p-adic Lie extension.

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1. *Introduction*

The fine Selmer group has been ever-present in Iwasawa theory. Namely, it has been an object of frequent occurrence in the formulation (and proof) of the Iwasawa main conjecture (see [**18**, **22**, **45**, **50**]). Despite this, it was only until the turn of the millennium that a systematic study of the said group was first undertook by Coates and Sujatha [**3**, **4**], and a little later by Wuthrich [**54**, **55**], where they named it as we know today. Initial studies mainly revolved around fine Selmer groups attached to abelian varieties (see loc. cit.; also see [**12**, **23**, **31**]). Subsequently, there have been much interest on the fine Selmer group of a modular form (for instance, see [**14**, **16**, **17**]) or even more general classes of Galois representations (see [**21**, **26**, **27**, **32**]). A common feature in these cited works is that they are mainly concerned with working over the cyclotomic $\mathbb{Z}_p$-extension.

Let p be an odd prime. The aim of the paper is to consider the case of a $\mathbb{Z}_p$-extension which is not the cyclotomic $\mathbb{Z}_p$-extension. In this context, it has been conjectured that the fine Selmer group should be cotorsion over the ring $\mathbb{Z}_p[[\Gamma]]$, where Γ denotes the Galois group of the $\mathbb{Z}_p$-extension (see [**28**, **45**, **54**]). This will be called Conjecture Y in the paper.

One of our approaches towards studying this conjecture is guided by the following perspective which we now describe. Let L_∞ be a $\mathbb{Z}_p^2$-extension of a number field F, which is assumed to contain F^{cyc}, the cyclotomic $\mathbb{Z}_p$-extension of F. We shall write G for the Galois group $\mathrm{Gal}(L_\infty/F)$. Let A be an abelian variety over F. A conjecture of Coates-Sujatha [**4**, conjecture B] (also see body of this paper) then predicts that the Pontryagin dual of the fine Selmer group, denoted by $Y(A/L_\infty)$, over the $\mathbb{Z}_p^2$-extension L_∞ is pseudo-null over the ring $\mathbb{Z}_p[[G]]$. Roughly speaking, this conjecture is saying that $Y(A/L_\infty)$ is "quite small". Therefore, in according to this conjecture of Coates–Sujatha, one would expect that the fine Selmer group at each $\mathbb{Z}_p$-extension of F contained in L_∞ should not be too "large". More concretely, we have the following observation.

PROPOSITION 1·1 (Proposition 3·8). *Let A be an abelian variety defined over a number field F, and L_∞ a $\mathbb{Z}_p^2$-extension of F which contains F^{cyc}. Denote by $\Phi(L_\infty/F)$ the set of all $\mathbb{Z}_p$-extensions of F contained in L_∞. Suppose that $Y(A/L_\infty)$ is pseudo-null over $\mathbb{Z}_p[[Gal(L_\infty/F)]]$. Then $Y(A/\mathcal{L})$ is a torsion $\mathbb{Z}_p[[Gal(\mathcal{L}/F)]]$-module for every $\mathcal{L} \in \Phi(L_\infty/F)$.*

Although there are some numerical examples where pseudo-nullity of $Y(A/L_\infty)$ has been verified (for instance, see [**16**, **24**, **27**]), the general statement remains wide open. The first of our main result is the following, which gives a way of obtaining infinite classes of $\mathbb{Z}_p$-extensions, where Conjecture Y is valid.

THEOREM 1·2 (Theorem 3·9). *Let A be an abelian variety defined over a number field F, and L_∞ a $\mathbb{Z}_p^2$-extension of F which contains F^{cyc}. Suppose that $Y(A/F^{\mathrm{cyc}})$ is torsion over $\mathbb{Z}_p[[Gal(F^{\mathrm{cyc}}/F)]]$. Then for all but finitely many $\mathcal{L} \in \Phi(L_\infty/F)$, $Y(A/\mathcal{L})$ is torsion over $\mathbb{Z}_p[[Gal(\mathcal{L}/F)]]$.*

The torsionness of $Y(A/F^{\mathrm{cyc}})$ is known when A is an elliptic curve over $\mathbb{Q}$ and F is an abelian extension of $\mathbb{Q}$ (see [**18**, **22**]). Hence the above theorem applies in these cases, where Conjecture Y is valid (also see Section 6 for some examples, where F is not necessarily abelian over $\mathbb{Q}$ and the theorem applies). This therefore provides a strong evidence to Conjecture Y, and at the same time, a weak partial support towards the pseudo-nullity prediction of Coates–Sujatha.

As mentioned in the opening paragraph, there has been much interest in the study of the fine Selmer group of a modular form (for instance, see [**14**, **16**, **17**]). This will be the next theme of the paper which we describe briefly here.

Let f be a normalised new cuspidal modular eigenform of even weight $k \geq 2$, level N and nebentypus ϵ. Write A_f for the Galois module attached to f (see body of the paper for its precise definition) which is defined over the ring of integers of $K_{f,\mathfrak{p}}$. Here K_f is the the number field obtained by adjoining all the Fourier coefficients of f to $\mathbb{Q}$, and $K_{f,\mathfrak{p}}$ is the localisation of K_f at some fixed prime $\mathfrak{p}$ of K_f above p. Let F_∞ be a $\mathbb{Z}_p$-extension of F with F_n being the intermediate subfield satisfying $|F_n{:}F| = p^n$. We write $R(A_f/F_n)$ for the fine Selmer group defined over the field F_n for $1 \leq n \leq \infty$, and $Y(A_f/F_n)$ for its Pontryagin dual. We then formulate an analogue of Conjecture Y for $Y(A_f/F_\infty)$, which will be called Conjecture Y_f (see Conjecture 4·1). Since an analogue of Conjecture B has been postulated and studied by Jha [**16**], we therefore have a natural analogue of Proposition 3·8 for fine Selmer groups of modular forms (see Proposition 4·3). This in turn inspires the analogue

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of Theorem 3·9 (see Theorem 4·4). The work of Kato [18] again supplies many classes of examples, where $Y(A_f/F^{\mathrm{cyc}})$ is torsion, thus allowing one to apply Theorem 4·4 to obtain validity of Conjecture Y_f in these cases.

We then follow up the above discussion by proving the following control theorem.

THEOREM 1·3 (Theorem 4·6). *Notations as above. Let S be the set of primes of F containing those dividing pN and the infinite primes. Write S_{fd} for the set of primes in S which is finitely decomposed in F_∞/F. Suppose further that either of the following statements is valid*:

(i) *for every prime $v \in S_{fd}$ and prime v_n of F_n above v, $H^0\left(F_{n,v_n}, A_f\right)$ is finite;*

(ii) *$K_{f,\mathfrak{p}} \cap \mathbb{Q}_p(\mu_{p^\infty}) = \mathbb{Q}_p$ and $H^0(F_v, A_f)$ is finite for every prime $v \in S_{fd}$.*

Then the restriction map

$$r_n : R\left(A_f/F_n\right) \longrightarrow R\left(A_f/F_\infty\right)^{\Gamma_n}$$

has finite kernel and cokernel which are bounded independently of n.

For an abelian variety, a control theorem of such has been established by the author in [28, theorem 3·3]. The above is therefore an analogue of this in the context of modular forms. Note that in this modular form context, there is an extra finiteness hypothesis on $H^0(F_v, A_f)$, and this arises due to a lack of an analogue of Mattuck's theorem [35] for a modular form. We do however remark that although a recent work of Hatley–Kundu–Lei–Ray [14] has provided some sufficient conditions for this finiteness hypothesis to hold, it would seem that the general situation seems out of reach at the moment. We also note that in the event that the level N is not divisible by p, then the finiteness is valid for all primes v above p (cf. [6]).

We say a little more on the finiteness hypothesis on $H^0(F_v, A_f)$. As mentioned in the preceding paragraph, this is imposed on us by the lack of an analogue of Mattuck's theorem. In the proof of the control theorem, since we are estimating the kernel and cokernel at every intermediate F_n, the situation necessitates us to work al prior with a possible stronger hypothesis, namely, $H^0\left(F_{n,v_n}, A_f\right)$ is finite for every v_n above v. As it turns out, in the event that $K_{f,\mathfrak{p}} \cap \mathbb{Q}_p(\mu_{p^\infty}) = \mathbb{Q}_p$, the finiteness hypothesis at the base field suffices. In fact, we shall show that finiteness hypothesis at the base field F will imply the finiteness hypothesis at every intermediate subfield F_n (see Lemma 4·8 and proof of Theorem 4·6). This latter observation seems interesting in its own right.

When the modular form arises from a specialization of an ordinary Hida deformation, we attach a fine Selmer group to the Hida deformation (denoted by $R(\mathcal{A}/F_\infty)$; whose Pontryagin dual is denoted by $Y(\mathcal{A}/F_\infty)$) and formulate an analogous conjecture which we call Conjecture $\mathcal{Y}$ (see Conjecture 5·2). Our main result in this context is the following which can be thought as a "horizontal" variant of Theorems 3·9 and 4·3 (we refer readers to the body of the paper for the definitions of the objects and hypotheses appearing in the theorem).

THEOREM 1·4 (Theorem 5·3). *Assume that* **(H1)** *and* **(H2)** *are valid. Suppose that there exists $\eta \in \mathfrak{X}_{arith}\left(h^{ord}_{\mathcal{F}}\right)$ which satisfies the following properties;*

(a) for every prime $v \in S'_{cd}$, we have $H^0\left(F_{n,v_n}, A_f\right)$ being finite:

(b) $R(A_{f_\eta}/F_\infty)$ is cotorsion over $\mathcal{O}_\eta[[\Gamma]]$.

Then Conjecture $\mathcal{Y}$ is valid, or equivalently, $R(\mathcal{A}/F_\infty)$ is cotorsion over $\mathcal{R}[[\Gamma]]$. Furthermore, for all but finitely many $\lambda \in \mathfrak{X}_{arith}\left(h^{ord}_{\mathcal{F}}\right)$, $R\left(A_{f_\lambda}/F_\infty\right)$ is cotorsion over $\mathcal{O}_\lambda[[\Gamma]]$.

Note that here again, the lack of an analogue of Mattuck's theorem necessitates us to work under assumption (*a*) (for a different set of primes). We should mention that the above theorem is inspired by the work of Jha [16]. Now, looking at Theorems 4·3 and 5·3, one can't help posing the following question.

Question $\mathcal{A}$. Is $R(\mathcal{A}/F_\infty)^\vee$ pseudo-null over $\mathcal{R}[[\Gamma]]$?

Unfortunately, we do not have an answer to this. In fact, to the best knowledge of the author, even in the context of a cyclotomic $\mathbb{Z}_p$-extension F^{cyc}, the structure of $R(\mathcal{A}/F^{\mathrm{cyc}})$ does not seem well-understood (but see a very recent work of Lei-Palvannan [25] for some discussion in this direction).

It should be evident to the readers that much of the discussion in this paper may be extended to fine Selmer groups attached to even broader classes of Galois representations of interest as studied in [21, 27, 32]. We have decided to restrict our attention to the context of the paper to simplify the presentation. Furthermore, we believe that even in the modular form or Hida deformation context, the occurrence of certain interesting phenomenon deserves further future studies. (For instance, the lack of an analogue of Mattuck's theorem definitely requires further investigation and this sort of issues will naturally come up if one wants to study fine Selmer groups of more general Galois representations.)

Although the focus of our paper is to formulate variants of Conjecture Y, we should remark that it would be interesting to study the variation of the Iwasawa invariants of the $\mathbb{Z}_p$-specialisations (either horizontally or vertically). We will not pursue this here but refer readers to [7, 11, 20, 39] for some discussion in the vertical direction. While we have nothing to say about this variational aspect, we shall end by formulating and investigating a generalised Conjecture Y (see Conjecture 7·1) over an arbitrary p-adic Lie extension of dimension > 1, and give some conceptual evidences towards the paucity of this generalised conjecture (see Remark 7·2 and Proposition 7·5).

We now give an outline of our paper. In Section 2, we collect several results on modules over regular local rings which will be required in our arithmetic discussion. In Section 3, we introduce the fine Selmer groups of abelian varieties and establish Theorem 3·9. Section 4 is where we study the fine Selmer group of a modular form. The control theorem for the fine Selmer group of a modular form will be proved in this section. In Section 5, we investigate the relationship between the Conjecture $\mathcal{Y}$ on the fine Selmer group for a Hida family and the corresponding Conjecture Y_f for the specialisations. We also discuss a situation showing that the conjectural torsionness is consistent with a growth number conjecture of Mazur (see Theorem 5·6). In Section 6, we give several examples to illustrate the results of the paper. Finally, in Section 7, we formulate a generalised Conjecture Y for fine Selmer groups over an arbitrary p-adic Lie extension which does not necessarily contain the cyclotomic $\mathbb{Z}_p$-extension. Here we will show that this conjecture is consistent with the pseudo-nullity conjecture of Coates–Sujatha (see Proposition 7·5).

2. *Review of some commutative algebra*

We collect certain commutative algebraic results that will be required for the discussion of the paper. Throughout this section, Λ will always denote a regular local ring. (For our arithmetic purposes, we are usually concerned with regular local rings of the form $\mathcal{O}[[T_1, T_2, ..., T_r]]$, where $\mathcal{O}$ is some integral domain which is finite flat over $\mathbb{Z}_p$.) It's a standard fact that Λ is therefore a unique factorisation domain (cf. [**34**, theorem 20·3]). In particular, it follows from [**34**, theorem 20·1] that every prime ideal of Λ of height one is principal.

Recall that a finitely generated Λ-module M is said to be torsion if for every element $m \in M$, there exists $x \in \Lambda$ such that $xm = 0$. Equivalently, this is saying that $\mathrm{Hom}_\Lambda(M, \Lambda) = 0$. The module M is said to be pseudo-null if the localisation $M_\mathfrak{p}$ of M at every prime ideal $\mathfrak{p}$ of height ≤ 1 is trivial. The latter is equivalent to $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i = 0, 1$ (for instance, see [**34**, chapter. 5] or [**40**, chapter. V, section 1]).

We now present a useful lemma (compare with [**44**, section 1·3, lemme 4])

LEMMA 2·1. *Let x be an element in Λ which is a generator of a prime ideal of Λ of height one. Write $\Omega := \Lambda/x$ for the quotient ring which is also a regular local ring. Then the following statements are valid:*

(i) *if y is another prime element of Λ which is coprime to x, then $\Lambda/(x, y^n)$ is a torsion Ω-module for every $n \geq 1$;*

(ii) *if M is a pseudo-null Λ-module, then both $M[x]$ and M/x are torsion over Ω;*

(iii) *if M is a Λ-module with M/x being torsion over Ω, then M is torsion over Λ and $M[x]$ is torsion over Ω.*

Proof. We begin proving assertion (i). In fact, we shall establish a slightly stronger assertion: namely, if z is an element of Λ which is coprime to x, then $\Lambda/(x, z)$ is a torsion Ω-module. Since z is coprime to x, it does not lie in the ideal (x). Hence $z + (x)$ is a nonzero element of Ω, and it plainly annihilates $\Lambda/(x, z)$. Therefore, this proves our first assertion.

For the proof of (ii), we shall write N for either $M[x]$ or M/x. Since the module N is annihilated by x, it may be viewed as a Ω-module. Consider the following spectral sequence

$$\mathrm{Ext}^i_\Omega\big(N, \mathrm{Ext}^j_\Lambda(\Omega, \Lambda)\big) \Longrightarrow \mathrm{Ext}^{i+j}_\Lambda(N, \Lambda) \tag{1}$$

(cf. [**53**, exercise 5·6·3]). From the Λ-free resolution

$$0 \longrightarrow \Lambda \longrightarrow \Lambda \longrightarrow \Omega \longrightarrow 0$$

of Ω, we see that

$$\mathrm{Ext}^j_\Lambda(\Omega, \Lambda) = \begin{cases} \Omega, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the spectral sequence (1) degenerates yielding the isomorphism

$$\mathrm{Ext}^i_\Omega(N, \Omega) \cong \mathrm{Ext}^{i+1}_\Lambda(N, \Lambda)$$

for $i \geq 0$. In particular, we have

$$\mathrm{Hom}_{\Omega}(N, \Omega) = \mathrm{Ext}^1_{\Lambda}(N, \Lambda) = 0,$$

where the final zero follows from the assumption that M is pseudo-null over Λ. This in turn implies that N is torsion over Ω and we have the second assertion.

The final assertion follows from [44, section 1·3, lemme 2] or [27, corollary 4·13].

Now, let M be a finitely generated torsion Λ-module. By [40, proposition 5·1·7], there is a pseudo-isomorphism

$$\varphi : M \longrightarrow \bigoplus_{i \in I} \Lambda / x_i^{n_i}, \tag{2}$$

where I is a finite indexing set, each x_i is a generator of a prime ideal of height one and n_i is a non-negative integer. Note that the prime ideals Λx_i and integers n_i are determined by the module M.

LEMMA 2·2. *Notation as above. Suppose that x is a prime element in Λ which is coprime to all the x_i's. Then M/x is a torsion module over the ring $\Omega = \Lambda/x$.*

Proof. Let $P_1 = \ker \varphi$, $P_2 = \mathrm{coker}\, \varphi$ and $Q = \mathrm{im}\, \varphi$, where φ is given as in (2). From which, we have the following two short exact sequences

$$0 \longrightarrow P_1 \longrightarrow M \longrightarrow Q \longrightarrow 0,$$

$$0 \longrightarrow Q \longrightarrow \bigoplus_{i \in I} \Lambda / x_i^{n_i} \longrightarrow P_2 \longrightarrow 0.$$

From which, we have

$$P_1/x \longrightarrow M/x \longrightarrow Q/x \longrightarrow 0,$$

$$P_2[x] \longrightarrow Q/x \longrightarrow \bigoplus_{i \in I} \Lambda / (x_i^{n_i}, x) \longrightarrow P_2/x \longrightarrow 0.$$

By Lemma 2·1, the modules P_1/x, $P_2[x]$ and $\bigoplus_{i \in I} \Lambda/(x_i^{n_i}, x)$ are torsion over Ω. Putting these observations into the above two exact sequences, we see that so is M/x.

3. *Fine Selmer groups of abelian varieties*

3·1. *Fine Selmer groups*

We begin reviewing the fine Selmer groups of abelian varieties following [3, 4, 12, 28, 31, 54–56]. Fix an odd prime p. Let A be an abelian variety defined over a number field F. Let S be a finite set of primes of F containing the primes above p, the bad reduction primes of A and the infinite primes. Denote by F_S the maximal algebraic extension of F which is unramified outside S. For every extension $\mathcal{L}$ of F contained in F_S, we write $G_S(\mathcal{L}) = \mathrm{Gal}(F_S/\mathcal{L})$, and denote by $S(\mathcal{L})$ the set of primes of $\mathcal{L}$ above S.

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Let L be a finite extension of F contained in F_S. The fine Selmer group of A over L is defined by

$$R(A/L) = \ker\left(H^1(G_S(L), A[p^\infty]) \longrightarrow \bigoplus_{v\in S(L)} H^1(L_v, A[p^\infty])\right).$$

We remark that the above definition is independent of the choice of S (see [31, lemma 4·1]). Just as the classical p-primary Selmer group, the fine Selmer group sits in the following analogous short exact sequence

$$0 \longrightarrow \mathcal{M}(A/L) \longrightarrow R(A/L) \longrightarrow Ж(A/L) \longrightarrow 0, \tag{3}$$

where $\mathcal{M}(A/F)$ is the fine (p-)Mordell–Weil group and $Ж(A/F)$ is the fine (p-) Tate–Shafarevich group in the sense of Wuthrich [56]. The fine Mordell–Weil group $\mathcal{M}(A/F)$ is defined to be the subgroup of $A(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ consisting of those elements which are mapped to zero in $A(F_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ for all primes v above p. It can be shown that $\mathcal{M}(A/F)$ injects into $R(A/F)$. The fine Tate–Shafarevich group $Ж(A/F)$ is then defined to be the cokernel of this injection (see [56, section 2] for details).

Let F_∞ be a (not necessarily cyclotomic) $\mathbb{Z}_p$-extension of F, whose Galois group $\mathrm{Gal}(F_\infty/F)$ will be denoted by Γ. If Γ_n denotes the unique subgroup of Γ of index p^n, we write F_n for the fixed field of Γ_n. The fine Selmer group of A over F_∞ is defined to be $R(A/F_\infty) = \varinjlim_n R(A/F_n)$ which comes naturally equipped with a $\mathbb{Z}_p[[\Gamma]]$-module structure. The $\mathbb{Z}_p[[\Gamma]]$-modules $\mathcal{M}(A/F_\infty)$ and $Ж(A/F_\infty)$ are similarly defined by taking limit of the corresponding objects over the intermediate subfields. We shall write $Y(A/F_\infty)$ for the Pontryagin dual of $R(A/F_\infty)$. Upon taking direct limit of the sequence (3) and following up by taking Pontryagin dual, we obtain

$$0 \longrightarrow Ж(A/F_\infty)^\vee \longrightarrow Y(A/F_\infty) \longrightarrow \mathcal{M}(A/F_\infty)^\vee \longrightarrow 0. \tag{4}$$

It is not difficult to verify that the modules occurring in the exact sequence are finitely generated over $\mathbb{Z}_p[[\Gamma]]$ (for instance, see [28, Lemma 3·2]). In fact, one expects more (see [28, 45, 54]).

CONJECTURE 3·1 (Conjecture Y) *Let A be an abelian variety defined over a number field F and F_∞ a $\mathbb{Z}_p$-extension of F. Then $Y(A/F_\infty)$ is torsion over $\mathbb{Z}_p[[\Gamma]]$.*

Remark 3·2. (1) When F_∞ is the cyclotomic $\mathbb{Z}_p$-extension, the above conjecture is a consequence of a conjecture of Mazur [36] and Schneider [51] on the structure of the classical Selmer groups. The latter is known when A is an elliptic curve over $\mathbb{Q}$ with good reduction at p and F is an abelian extension of $\mathbb{Q}$ (see [18, 22]).

(2) Suppose that E is an elliptic curve defined over $\mathbb{Q}$ with complex multiplication given by a imaginary quadratic field K at which p split completely in $K/\mathbb{Q}$. Let K_∞ be the $\mathbb{Z}_p$-extension of K which is unramified outside one of the primes of K above p. Then the validity of Conjecture Y is a consequence of a result of Coates (see [9, chapter IV, corollary 1·8]; also see the recent papers [1, 19]).

(3) Suppose that E is an elliptic curve defined over $\mathbb{Q}$, and K^{ac} the anti-cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field K. Then the $\mathbb{Z}_p[[\mathrm{Gal}(K^{\mathrm{ac}}/K)]]$-torsionness of $Y(A/K^{\mathrm{ac}})$ is known in many cases (for instance, see [33, 47]).

We record two results taken from [28] which serve as support for Conjecture Y, and will play some role in the subsequent discussion of the paper.

THEOREM 3·3. *Let A be an abelian variety defined over a number field F. Let F_∞ be a $\mathbb{Z}_p$-extension of F. If $Y(A/F)$ is finite, then $Y(A/F_\infty)$ is torsion over $\mathbb{Z}_p[[\Gamma]]$.*

Proof. See [28, corollary 3·5].

PROPOSITION 3·4. *Let A be an abelian variety defined over a number field F, and F_∞ a $\mathbb{Z}_p$-extension of F. Suppose that $Ш(A/F_n)$ is finite for every n. Then $Ш(A/F_\infty)$ is a cotorsion $\mathbb{Z}_p[[\Gamma]]$-module.*

Proof. See [28, proposition 4·1].

We now record a corollary of the preceding proposition.

COROLLARY 3·5 *Let A be an abelian variety defined over a number field F, and F_∞ a $\mathbb{Z}_p$-extension of F. Suppose that the following statements are valid.*

(a) $Ш(A/F_n)$ *is finite for every n.*

(b) $A(F_\infty)$ *is a finitely generated abelian group.*

Then $Y(A/F_\infty)$ is a torsion $\mathbb{Z}_p[[\Gamma]]$-module.

Proof. From Proposition 3·4, we see that $Ш(A/F_\infty)^\vee$ is torsion over $\mathbb{Z}_p[[\Gamma]]$ under hypothesis *(a)*. Hypothesis *(b)* tells us that $\mathcal{M}(A/F_\infty)^\vee$ is torsion over $\mathbb{Z}_p[[\Gamma]]$. The conclusion follows from these and the short exact sequence (4).

3·2. *Connection with the pseudo-nullity conjecture of Coates–Sujatha*

We now study the relation between our Conjecture Y and the Conjecture B of Coates–Sujatha [4, conjecture B]. As a start, we recall their conjecture, which for now is stated for $\mathbb{Z}_p^2$-extensions; see Conjecture 7·4 below for the general version.

CONJECTURE 3·6 (Conjecture B). *Let L_∞ be a $\mathbb{Z}_p^2$-extension of F which contains F^{cyc}. Then $Y(A/L_\infty)$ is pseudo-null over $\mathbb{Z}_p[[G]]$, where $G = Gal(L_\infty/F)$.*

Remark 3·7. In [4], they formulated their conjecture under the extra assumption which is their so-called Conjecture A (see [4, conjecture A]). In this paper, we do not require this extra hypothesis, and so the above formulation suffices.

Retaining the above notation, we denote by $\Phi(L_\infty/F)$ the set of all $\mathbb{Z}_p$-extensions of F contained in L_∞. For each $\mathcal{L} \in \Phi(L_\infty/F)$, write $\Gamma_{\mathcal{L}} = \mathrm{Gal}(\mathcal{L}/F)$ and $H_{\mathcal{L}} = \mathrm{Gal}(L_\infty/\mathcal{L})$. Fix a topological generator $h_{\mathcal{L}}$ of $H_{\mathcal{L}}$. Then $h_{\mathcal{L}} - 1$ generates a prime ideal of $\mathbb{Z}_p[[G]]$ of height one with

$$\mathbb{Z}_p[[G]]/(h_{\mathcal{L}} - 1) \cong \mathbb{Z}_p[[\Gamma_{\mathcal{L}}]].$$

We can now establish the following observation as mentioned in the introduction.

PROPOSITION 3·8. *Let A be an abelian variety defined over a number field F, and L_∞ a $\mathbb{Z}_p^2$-extension of F which contains F^{cyc}. Suppose that Conjecture B is valid for A over L_∞, or in other words, $Y(A/L_\infty)$ is pseudo-null over $\mathbb{Z}_p[[G]]$. Then $Y(A/\mathcal{L})$ is a torsion $\mathbb{Z}_p[[Gal(\mathcal{L}/F)]]$-module for every $\mathcal{L} \in \Phi(L_\infty/F)$.*

Proof. From the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R(A/\mathcal{L}) \arrow[r] \arrow[d, "s"] & H^1\big(G_S(\mathcal{L}), A[p^\infty]\big) \arrow[d] \\
0 \arrow[r] & R(A/L_\infty)^{H_\mathcal{L}} \arrow[r] & H^1\big(G_S(L_\infty), A[p^\infty]\big)^{H_\mathcal{L}}
\end{tikzcd}$$

we see that $\ker s$ is contained in $H^1(H_\mathcal{L}, A(L_\infty))$ which is cofinitely generated over $\mathbb{Z}_p$. Upon taking Pontryagin dual, we obtain a map

$$Y(A/F_\infty)_{H_\mathcal{L}} \longrightarrow Y(A/\mathcal{L}),$$

whose cokernel is finitely generated over $\mathbb{Z}_p$. Therefore, for a given $\mathcal{L} \in \Phi(L_\infty/F)$, whenever $Y(A/L_\infty)_{H_\mathcal{L}}$ is torsion over $\mathbb{Z}_p[[\Gamma_\mathcal{L}]]$, so is $Y(A/\mathcal{L})$.

Now, in view of the hypothesis that $Y(A/L_\infty)$ is pseudo-null over $\mathbb{Z}_p[[G]]$, we may apply Lemma 2·1(ii) to conclude that $Y(A/F_\infty)_{H_\mathcal{L}}$ is torsion over $\mathbb{Z}_p[[\Gamma_\mathcal{L}]]$. Combining this with the assertion in the previous paragraph, we have the conclusion.

We now state and prove the following.

THEOREM 3·9. *Let A be an abelian variety defined over a number field F, and L_∞ a $\mathbb{Z}_p^2$-extension of F which contains F^{cyc}. Suppose that $Y(A/F^{cyc})$ is torsion over $\mathbb{Z}_p[[Gal(F^{cyc}/F)]]$. Then for all but finitely many $\mathcal{L} \in \Phi(L_\infty/F)$, Conjecture Y is valid for $Y(A/\mathcal{L})$ or, in other words, $Y(A/\mathcal{L})$ is torsion over $\mathbb{Z}_p[[\Gamma_\mathcal{L}]]$.*

Proof. By [27, proposition 7·2], it follows from the $\mathbb{Z}_p[[\mathrm{Gal}(F^{\mathrm{cyc}}/F)]]$-torsionness of $Y(A/F^{\mathrm{cyc}})$ that $Y(A/L_\infty)$ is torsion over $\mathbb{Z}_p[[G]]$. The structure theorem of $\mathbb{Z}_p[[G]]$-modules (cf. [40, proposition 5·1·7]) then implies that there is a pseudo-isomorphism

$$Y(A/L_\infty) \sim \bigoplus_{i\in I} \mathbb{Z}_p[[G]]/Q_i^{n_i}$$

of $\mathbb{Z}_p[[G]]$-modules, where I is some finite indexing set and each Q_i is a (principal) prime ideal of $\mathbb{Z}_p[[G]]$ of height one. Since there are only finitely many Q_i's, the element $h_\mathcal{L} - 1$ is coprime to these Q_i's for all but finitely many $\mathcal{L} \in \Phi(L_\infty/F)$. For each of such element $h_\mathcal{L} - 1$, it then follows from Lemma 2·2 that $Y(A/L_\infty)_{H_\mathcal{L}} = Y(A/L_\infty)/(h_\mathcal{L} - 1)$ is torsion over $\mathbb{Z}_p[[\Gamma_\mathcal{L}]]$. By a similar argument to that in Proposition 3·8, we see that $Y(A/\mathcal{L})$ is torsion over $\mathbb{Z}_p[[\Gamma_\mathcal{L}]]$ for every such $\mathcal{L}$. This yields the required conclusion of the theorem.

We record one case, where we can obtain many cases of validity of Conjecture Y.

COROLLARY 3·10. *Let E be an elliptic curve defined over $\mathbb{Q}$ and F a finite abelian extension of $\mathbb{Q}$. Let L_∞ be a $\mathbb{Z}_p^2$-extension of F which contains F^{cyc}. Then for all but finitely many $\mathcal{L} \in \Phi(L_\infty/F)$, $Y(E/\mathcal{L})$ is cotorsion over $\mathbb{Z}_p[[\Gamma_\mathcal{L}]]$.*

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Proof. A well-known theorem of Kato [18] (also see [22]) asserts that $R(E/F^{\text{cyc}})$ is cotorsion over $\mathbb{Z}_p[[\text{Gal}(F^{\text{cyc}}/F)]]$. The corollary is now an immediate consequence of this and Theorem 3·9.

We also refer readers to Section 6 for examples, where F is not necessarily abelian over $\mathbb{Q}$ but Theorem 3·9 applies.

4. *Fine Selmer groups of elliptic modular forms*

As before, p will denote a fixed odd prime. Let f be a normalised new cuspidal modular eigenform of even weight $k \geq 2$, level N and nebentypus ϵ. Let $\mathcal{K}_f$ be the number field obtained by adjoining all the Fourier coefficients of f to $\mathbb{Q}$. Throughout, we shall fix a prime $\mathfrak{p}$ of $\mathcal{K}_f$ above p, and let V_f denote the corresponding two-dimensional $\mathcal{K}_{f,\mathfrak{p}}$-linear Galois representation attached to f in the sense of Deligne [8]. Writing $\mathcal{O} = \mathcal{O}_{\mathcal{K}_{f,\mathfrak{p}}}$ for the ring of integers of $\mathcal{K}_{f,\mathfrak{p}}$, we fix a $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stable $\mathcal{O}$-lattice T_f in V_f. We then set $A_f = V_f/T_f$. Note that A_f is isomorphic to $\mathcal{K}_{f,\mathfrak{p}}/\mathcal{O} \oplus \mathcal{K}_{f,\mathfrak{p}}/\mathcal{O}$ as $\mathcal{O}$-modules.

Let F be a finite extension of $\mathbb{Q}$. Denote by S a finite set of primes of F containing those dividing pN and all the infinite primes. Let F_∞ be a $\mathbb{Z}_p$-extension of F. Following [16, 17], we define the fine Selmer group of A_f over F_∞ to be $\varinjlim_n R(A_f/F_n)$, where F_n is the intermediate subfield of F_∞/F with $|F_n{:}F| = p^n$ and $R(A_f/F_n)$ is defined by

$$R(A_f/F_n) = \ker\left(H^1(G_S(F_n), A_f) \longrightarrow \bigoplus_{v \in S(F_n)} H^1(F_{n,v}, A_f)\right).$$

The Pontryagin dual of $R(A_f/F_\infty)$ is then denoted by $Y(A_f/F_\infty)$. As before, one can similarly show that $Y(A_f/F_\infty)$ is finitely generated over $\mathcal{O}[[\Gamma]]$. The following conjecture is the natural analogue of Conjecture Y for modular forms.

CONJECTURE 4·1 (Conjecture Y_f). *Let A_f be defined as above, and F_∞ a $\mathbb{Z}_p$-extension of a number field F. Denote by Γ the Galois group $Gal(F_\infty/F)$. Then $Y(A_f/F_\infty)$ is torsion over $\mathcal{O}[[\Gamma]]$.*

We now present natural analogue of Proposition 3·8 and Theorem 3·9 for the fine Selmer group of a modular form. As a start, we recall the following analogue of Conjecture B which was first studied by Jha [16].

CONJECTURE 4·2. *Let L_∞ be a $\mathbb{Z}_p^2$-extension of F which contains F^{cyc}. Then $Y(A_f/L_\infty)$ is pseudo-null over $\mathcal{O}[[G]]$, where $G = Gal(L_\infty/F)$.*

As in Section 3, denote by $\Phi(L_\infty/F)$ the set of all $\mathbb{Z}_p$-extensions of F contained in L_∞. For each $\mathcal{L} \in \Phi(L_\infty/F)$, write $\Gamma_{\mathcal{L}} = \text{Gal}(\mathcal{L}/F)$ and $H_{\mathcal{L}} = \text{Gal}(L_\infty/\mathcal{L})$. From which, a similar argument to that in Proposition 3·8 yields the following.

PROPOSITION 4·3 *Suppose that $Y(A_f/L_\infty)$ is pseudo-null over $\mathcal{O}[[G]]$. Then $Y(A_f/\mathcal{L})$ is a torsion $\mathcal{O}[[Gal(\mathcal{L}/F)]]$-module for every $\mathcal{L} \in \Phi(L_\infty/F)$.*

Similarly, we can establish the following by a similar argument to that in Theorem 3·9.

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THEOREM 4·4 *Notations as above. Suppose that $R(A_f/F^{cyc})$ is cotorsion over $\mathcal{O}[[Gal(F^{cyc}/F)]]$. Then for all but finitely many $\mathcal{L} \in \Phi(L_\infty/F)$, Conjecture Y_f is valid for $Y(A_f/\mathcal{L})$.*

Combining the above with Kato' result [18], we have the following analogue of Corollary 3·10.

COROLLARY 4·5. *Suppose that F is an abelian extension of $\mathbb{Q}$ and L_∞ a $\mathbb{Z}_p^2$-extension of F which contains F^{cyc}. Then for all but finitely many $\mathcal{L} \in \Phi(L_\infty/F)$, $Y(E/\mathcal{L})$ is cotorsion over $\mathcal{O}[[\Gamma_{\mathcal{L}}]]$.*

We end the section by establishing a control theorem for the fine Selmer groups of elliptic modular forms, which is the analogue to that in [28, theorem 3·3] proved for abelian varieties. From now on, we let S_{fd} denote the set of primes in S which do not split completely in F_∞/F. We shall also write $W(L) = W^{\mathrm{Gal}(F_S/L)}$ for any $F \subseteq L \subseteq F_S$. Similarly, for each $v \in S$, we write $W(L) = W^{\mathrm{Gal}(\bar{F}_v/L)}$ for any $F_v \subseteq L \subseteq \bar{F}_v$.

THEOREM 4·6. *Let f be a normalized new cuspidal modular eigenform of even weight $k \geq 2$, level N and nebentypus ϵ. Write A_f for the Galois module attached to f defined as above. Let F_∞ be a $\mathbb{Z}_p$-extension of F with F_n being the intermediate subfield satisfying $|F_n : F| = p^n$. Suppose that either of the following statements is valid.*

(i) *For every prime $v \in S_{fd}$ and v_n of F_n dividing v, then $H^0\left(F_{n,v_n}, A_f\right)$ is finite.*

(ii) *$K_{f,\mathfrak{p}} \cap \mathbb{Q}_p(\mu_{p^\infty}) = \mathbb{Q}_p$ and $H^0(F_v, A_f)$ is finite for every prime $v \in S_{fd}$.*

Then the restriction map

$$r_n : R(A_f/F_n) \longrightarrow R(A_f/F_\infty)^{\Gamma_n}$$

has finite kernel and cokernel which are bounded independently of n.

Remark 4·7. In the case when $p \nmid N$, one automatically has the finiteness of $H^0\left(F_{n,v_n}, A_f\right)$ for every v_n above p (see [6]).

Before proving Theorem 4·6, we establish the following preliminary lemma.

LEMMA 4·8. *Let M be a $\mathcal{O}[[\Gamma]]$-module which is finitely generated over $\mathcal{O}$. Suppose that either of the following statements is valid:*

(a) M_{Γ_n} is finite for every n;

(b) $K_{f,\mathfrak{p}} \cap \mathbb{Q}_p(\mu_{p^\infty}) = \mathbb{Q}_p$ and M_Γ is finite with $\mathrm{rank}_{\mathcal{O}}(M) < p-1$.

Then the homology group $H_1(\Gamma_n, M)$ is finite with order bounded independently of n.

Proof. Suppose that hypothesis (a) is valid. Since M is plainly torsion as a $\mathcal{O}[[\Gamma_n]]$-module, one therefore has

$$0 = \mathrm{rank}_{\mathcal{O}[[\Gamma_n]]}(M) = \mathrm{rank}_{\mathcal{O}}\left(M_{\Gamma_n}\right) - \mathrm{rank}_{\mathcal{O}}\left(M^{\Gamma_n}\right), \tag{5}$$

where the second equality follows from [40, proposition 5·3·20]. Combining these observations, we see that each M^{Γ_n} is finite. This in turn implies that M^{Γ_n} is contained in

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$M[p^\infty]$. But since M is finitely generated over $\mathcal{O}$, the latter is finite, and hence we conclude that M^{Γ_n} is finite with order bounded independently of n. Finally, since $\Gamma_n \cong \mathbb{Z}_p$, we have the identification $M^{\Gamma_n} \cong H_1(\Gamma_n, M)$, thus proving the lemma under the validity of the hypothesis (a).

Now suppose that hypothesis (b) is valid. Identify $\mathcal{O}[[\Gamma]]$ with $\mathcal{O}[[T]]$ under a choice of generator of Γ. Since M_Γ is finite, we see that T has to be coprime to the characteristic polynomial of M. Since $K_{f,\mathfrak{p}} \cap \mathbb{Q}_p(\mu_{p^\infty}) = \mathbb{Q}_p$, every other cyclotomic polynomial is irreducible over $\mathcal{O}[[T]]$. Since such a polynomial has degree $\geq p-1$, it has to be coprime to the characteristic polynomial of M. Consequently, M_{Γ_n} is finite for every n. We are therefore in the situation of hypothesis (a), and so the conclusion follows from the above discussion.

We can now give the proof of Theorem 4·6.

Proof of Theorem 4·6. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R(A_f/F_n) \arrow[r] \arrow[d, "r_n"] & H^1\ G_S(F_n), A_f) \arrow[r] \arrow[d, "h_n"] & \bigoplus_{v_n \in S(F_n)} H^1(F_{n,v_n}, A_f) \arrow[d, "g_n = \oplus g_{n,v_n}"] \\
0 \arrow[r] & R(A_f/F_\infty)^{\Gamma_n} \arrow[r] & H^1\ G_S(F_\infty), A_f)^{\Gamma_n} \arrow[r] & \left(\bigoplus_{w \in S(F_\infty)} H^1(F_{\infty,w}, A_f)\right)^{\Gamma_n}
\end{tikzcd}$$

with exact rows. Since Γ_n has p-cohomological dimension 1, the restriction-inflation sequence tells us that h_n is surjective and that $\ker h_n = H^1\big(\Gamma_n, A_f(F_\infty)\big)$. It therefore remains to show the finiteness and boundness of $\ker h_n$ and $\ker g_n$.

We begin by showing the finiteness and boundness of $\ker g_n$. For each $v_n \in S(F_n)$, fix a prime of F_∞ above v_n which is denoted by w_n, and write v for the prime of F below v_n. Write Γ_{w_n} for the decomposition group of w_n in Γ. By the Shapiro's lemma and the restriction-inflation sequence, we have

$$\ker\left(\bigoplus_{v_n \in S(F_n)} g_{n,v_n}\right) = \bigoplus_{v_n \in S(F_n)} H^1\left(\Gamma_{w_n}, A_f(F_{\infty,v_n})\right).$$

If v is a prime of F below w_n such that v splits completely in F_∞/F, then $\Gamma_{w_n} = 0$ and so one has $H^1\left(\Gamma_{w_n}, A_f\left(F_{\infty,w_n}\right)\right) = 0$. Thus, it remains to consider the primes $v \in S$ which do not split completely in F_∞/F. Since S is a finite set, the number of such possibly nonzero summands $\bigoplus H^1\left(\Gamma_{w_n}, A_f\left(F_{\infty,w_n}\right)\right)$ is therefore finite and bounded independently of n. Hence it remains to show that each $H^1\left(\Gamma_{w_n}, A_f\left(F_{\infty,w_n}\right)\right)$ is finite and bounded independently for those primes lying above v which do not decompose completely in F_∞/F. For this, one just needs to verify that either (a) or (b) of Lemma 4·8 is valid. We note that hypothesis (a) is a direct consequence of hypothesis (i) of the theorem. It remains to show that hypothesis (ii) of our theorem yields (b) of the said lemma. For this, it suffices to show that $\mathrm{corank}_{\mathcal{O}}\left(A_f\left(F_{\infty,w_n}\right)\right) < p-1$. We first consider the case when v does not divide p. In this setting, F_{∞,w_n} is the cyclotomic $\mathbb{Z}_p$-extension of F_v which is unramified. Since v divides N, A_f cannot be an unramified $\mathrm{Gal}(\bar{F}_v/F_v)$-module and so A_f cannot be realised over

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F_{∞,w_n}. Therefore, we must have $\mathrm{corank}_{\mathcal{O}}\left(A_f\left(F_{\infty,w_n}\right)\right) \leq 1$. As the prime p is assumed to be odd, this in turn implies that $\mathrm{corank}_{\mathcal{O}}\left(A_f\left(F_{\infty,w_n}\right)\right) < p-1$. Now suppose that v divides p. It is well-known that $F_v(A_f)$ is a p-adic Lie extension of F_v of dimension at least 2 (see [**49**]). It follows from this that A_f cannot be realised over F_{∞,w_n}, and so we have $\mathrm{corank}_{\mathcal{O}}\left(A_f\left(F_{\infty,w_n}\right)\right) \leq 1$.

We now show that $\ker h_n = H^1(\Gamma_n, A_f(F_\infty))$ is finite and bounded independent of n. Now since F_∞/F is a $\mathbb{Z}_p$-extension, it must have at least one prime v above p which is ramified in F_∞/F. In view of the discussion in the preceding paragraph, we have $H^0(F_{n,w}, A_f)$ being finite, where w is a prime of F_∞ above v. This in turn implies that $H^0(G_S(F_n), A_f)$ is finite for every n. The desired conclusion is now a consequence of Lemma 4·8.

Theorem 4·6 has the following natural corollary.

COROLLARY 4·9. *Retain the settings of Theorem 4·6. Assume further that $R(A_f/F)$ is finite. Then $Y(A_f/F_\infty)$ is torsion over $\mathcal{O}[[\Gamma]]$.*

5. *Hida deformations*

Let us briefly introduce certain notion and facts arising from the work of Hida [**15**]. Denote by Γ' the group of diamond operators for the tower of modular curves $\{Y_1(p^r)\}$. There is a natural identification of Γ' with $1+p\mathbb{Z}_p$ which we denote by $\kappa : \Gamma' \xrightarrow{\sim} 1+p\mathbb{Z}_p$. For an integer N coprime to p, we write $h_{\mathcal{F}}^{\mathrm{ord}}$ for the quotient of the universal ordinary Hecke algebra with conductor N corresponding to an ordinary $\mathbb{Z}_p[[\Gamma']]$-adic eigenform $\mathcal{F}$. The ring $h_{\mathcal{F}}^{\mathrm{ord}}$ is a local integral domain which is finite flat over $\mathbb{Z}_p[[\Gamma']]$. In [**15**], Hida constructed an irreducible representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}_{h_{\mathcal{F}}^{\mathrm{ord}}}(\mathcal{T}_{\mathcal{F}})$$

which is unramified outside Np, and where $\mathcal{T}_{\mathcal{F}}$ is a finitely generated torsion-free module of generic rank two over $h_{\mathcal{F}}^{\mathrm{ord}}$. We now impose two standing assumptions on the pair $\left(\mathcal{T}_{\mathcal{F}}, h_{\mathcal{F}}^{\mathrm{ord}}\right)$.

- (**H1**) The ring $h_{\mathcal{F}}^{\mathrm{ord}}$ is isomorphic to $\mathcal{O}[[\Gamma']]$ for the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$.
- (**H2**) Denote by $\mathfrak{m}$ the maximal ideal of $h_{\mathcal{F}}^{\mathrm{ord}}$. The residual representation $\bar{\rho} \longrightarrow \mathrm{Aut}_{h_{\mathcal{F}}^{\mathrm{ord}}/\mathfrak{m}}(\mathcal{T}_{\mathcal{F}}/\mathfrak{m}\mathcal{T}_{\mathcal{F}})$ is absolutely irreducible as a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module.

Under **(H2)**, the module $\mathcal{T}_{\mathcal{F}}$ is free over $h_{\mathcal{F}}^{\mathrm{ord}}$ (see [**38**, section 2, corollary 6]). Let $\mathfrak{X}_{\mathrm{arith}}\left(h_{\mathcal{F}}^{\mathrm{ord}}\right)$ be the set consisting of $\mathbb{Z}_p$-algebra homomorphism $\lambda : h_{\mathcal{F}}^{\mathrm{ord}} \longrightarrow \bar{\mathbb{Q}}_p$ which satisfies the property that there exists an open subgroup U of Γ' and a non-negative integer w such that $\lambda(u) = \kappa^w(u)$ for every $u \in U$. For each of such λ, we shall write w_λ for the integer w appearing the above definition. We also write P_λ for the kernel of λ which is a principal prime ideal of $h_{\mathcal{F}}^{\mathrm{ord}}$ of height one. In fact, identifying $h_{\mathcal{F}}^{\mathrm{ord}} \cong \mathcal{O}[[T]]$ (recall that we are assuming **(H1)**), the prime ideal P_λ can be viewed as lying above the prime ideal of $\mathbb{Z}_p[[T]]$ generated by $(1+p)^{w_\lambda} - (1+T)$. We shall write p_λ for a generator of the prime ideal P_λ.

By Hida theory, for each $\lambda \in \mathfrak{X}_{\mathrm{arith}}\left(h_{\mathcal{F}}^{\mathrm{ord}}\right)$, there exists a normalised cuspidal eigenform f_λ of weight $w_\lambda + 2$ such that $\mathcal{T}_{\mathcal{F}}/P_\lambda \cong T_{f_\lambda}$, where T_{f_λ} is the lattice of the Galois representation attached to f_λ as in the sense of Deligne. Here T_{f_λ} is a free $\mathcal{O}_\lambda$-module of rank two, where $\mathcal{O}_\lambda = h_{\mathcal{F}}^{\mathrm{ord}}/P_\lambda$.

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From now on, to simplify notation, we sometimes write $\mathcal{T}$ for $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{R}$ for $h_{\mathcal{F}}^{\mathrm{ord}}$. Recall that $\mathcal{R} \cong \mathcal{O}[[\Gamma']]$ by our standing assumption **(H1)**. Set $\mathcal{A} = \mathcal{T} \otimes_{\mathcal{R}} (\mathcal{R}^{\vee})$. The next lemma is left to the reader as an exercise.

LEMMA 5·1. *One has $\mathcal{A}[p_\lambda] \cong A_{f_\lambda}$, where A_{f_λ} is the Galois module attached to f_λ as in Section 4.*

Let F be a finite extension of $\mathbb{Q}$. Denote by S a finite set of primes of F containing those dividing pN and all the infinite primes. For a $\mathbb{Z}_p$-extension F_∞ of F, the fine Selmer group $R(\mathcal{A}/F_\infty)$ of $\mathcal{A}$ over F_∞ is defined to be $\varinjlim_n R(\mathcal{A}/F_n)$, where F_n is the intermediate subfield of F_∞/F with $|F_n{:}F| = p^n$ and $R(\mathcal{A}/F_n)$ is given by

$$R(\mathcal{A}/F_n) = \ker\left(H^1(G_S(F_n), \mathcal{A}) \longrightarrow \bigoplus_{v_n \in S(F_n)} H^1(F_{n,v_n}, \mathcal{A}) \right).$$

We can now state the following analogue of Conjecture Y for $\mathcal{A}$.

CONJECTURE 5·2 (Conjecture $\mathcal{Y}$) *Retain settings as above. Denote by $Y(\mathcal{A}/F_\infty)$ the Pontryagin dual of $R(\mathcal{A}/F_\infty)$. Then $Y(\mathcal{A}/F_\infty)$ is torsion over $\mathcal{R}[[\Gamma]]$, where $\Gamma = Gal(F_\infty/F)$.*

We now prove a "hortizontal" analogue of Theorems 3·9 and 4·4. In the subsequent discussion, we write S'_{cd} for the set of primes of S which does not divide p and split completely in F_∞/F.

THEOREM 5·3. *Assume that* (H1) *and* (H2) *are valid. Suppose that there exists $\eta \in \mathfrak{X}_{arith}\left(h_{\mathcal{F}}^{ord}\right)$ which satisfies all of the following properties:*

(a) for every prime $v \in S'_{cd}$, the group $H^0(F_v, A_{f_\eta})$ is finite;

(b) $Y(A_{f_\eta}/F_\infty)$ is torsion over $\mathcal{O}_\eta[[\Gamma]]$.

Then Conjecture $\mathcal{Y}$ is valid, or equivalently, $Y(\mathcal{A}/F_\infty)$ is torsion over $\mathcal{R}[[\Gamma]]$. Furthermore, for all but finitely many $\lambda \in \mathfrak{X}_{arith}\left(h_{\mathcal{F}}^{ord}\right)$, $Y\left(A_{f_\lambda}/F_\infty\right)$ is torsion over $\mathcal{O}_\lambda[[\Gamma]]$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R(A_{f_\eta}/F_\infty) \arrow[r] \arrow[d, "r_\eta"] & H^1\left(G_S(F_\infty), A_{f_\eta}\right) \arrow[r] \arrow[d, "h_\eta"] & \bigoplus_{w \in S(F_\infty)} H^1(F_{\infty,w}, A_{f_\eta}) \arrow[d, "l=\oplus l_w"] \\
0 \arrow[r] & R(\mathcal{A}/F_\infty)[p_\eta] \arrow[r] & H^1\left(G_S(F_\infty), \mathcal{A}\right)[p_\eta] \arrow[r] & \left(\bigoplus_{w \in S(F_\infty)} H^1(F_{\infty,w}, \mathcal{A}) \right)[p_\eta]
\end{tikzcd}$$

with exact rows and vertical maps induced by the following short exact sequence

$$0 \longrightarrow A_{f_\eta} \longrightarrow \mathcal{A} \xrightarrow{p_\eta} \mathcal{A} \longrightarrow 0.$$

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We shall first show that the kernel and cokernel of r_η are cotorsion over $\mathcal{O}_\eta[[\Gamma]]$. To start off, we see that h_η is surjective with $\ker h_\eta = \mathcal{A}(F_\infty)/P_\eta$. Since $\mathcal{A}(F_\infty)/P_\eta$ is cofinitely generated over $\mathcal{O}_\eta$, it is cotorsion over $\mathcal{O}_\eta[[\Gamma]]$. It therefore remains to show that $\ker l$ is cotorsion over $\mathcal{O}_\eta[[\Gamma]]$. For this, we decompose $l = \oplus_{w\in S(F_\infty)} l_w = \oplus_{v\in S}(\oplus_{w|v} l_w)$ and show that $\ker\left(\oplus_{w|v} l_w\right)$ is cotorsion over $\mathcal{O}_\eta[[\Gamma]]$ for each $v \in S$. For each w, we have $\ker l_w = \mathcal{A}(F_{\infty,w})/P_\eta$. Now if v is finitely decomposed in F_∞, then the sum $\oplus_{w|v}$ is finite, and so $\ker\left(\oplus_{w|v} l_w\right)$ is cofinitely generated over $\mathcal{O}_\eta$ for these v's. Now suppose that v splits completely in F_∞. As noted in Remark 4·7, $\mathcal{A}(F_{\infty,w})[p_\eta] = A_{f_\eta}(F_{\infty,w})$ is finite for all v dividing p. For those primes not dividing p, the finiteness follows from assumption (a). Consequently, $\mathcal{A}(F_{\infty,w})/P_\eta$ is finite by Lemma 2·1(iii) and is hence annihilated by some powers of p. This power of p annihilates $\ker\left(\oplus_{w|v} l_w\right)$. Therefore, we also have $\ker\left(\oplus_{w|v} l_w\right)$ being cotorsion over $\mathcal{O}_\eta[[\Gamma]]$ for these primes. Hence we have shown that the kernel and cokernel of r_η are cotorsion over $\mathcal{O}_\eta[[\Gamma]]$.

Combining this observation with hypothesis (b) of the theorem, we see that $R(\mathcal{A}/F_\infty)[p_\eta]$ is cotorsion over $\mathcal{O}_\eta[[\Gamma]]$. It then follows from Lemma 2·1(iii) that $R(\mathcal{A}/F_\infty)$ is cotorsion over $\mathcal{R}[[\Gamma]]$. This proves the first assertion of the theorem.

In view of **(H1)**, the ring $\mathcal{R}[[\Gamma]]$ is isomorphic to $\mathcal{O}[[W,T]]$, where $1+W$ (resp., $1+T$) corresponds to a topological generator of Γ (resp., a topological generator of Γ'). By the structure theorem (cf. [**40**, proposition 5·1·7]), we then have a pseudo-isomorphism

$$Y(\mathcal{A}/F_\infty)^\vee \sim \bigoplus_{i\in I} \mathcal{R}[[\Gamma]]/Q_i^{n_i}$$

of $\mathcal{R}[[\Gamma]]$-modules, where I is some finite indexing set and each Q_i is a principal prime ideal of $\mathcal{R}[[\Gamma]]$ of height one. Since there are only finitely many Q_i's, for all but finitely many $\lambda \in \mathfrak{X}_{\mathrm{arith}}\left(h_{\mathcal{F}}^{\mathrm{ord}}\right)$, we have $Y(\mathcal{A}/F_\infty)/P_\lambda$ being torsion over $\mathcal{O}_\lambda[[\Gamma]]$. From the above discussion, we see that

$$Y(\mathcal{A}/F_\infty)/P_\lambda \longrightarrow Y(A_{f_\lambda}/F_\infty)$$

has cokernel which is torsion over $\mathcal{O}_\lambda[[\Gamma]]$. It then follows that $Y\left(A_{f_\lambda}/F_\infty\right)$ is torsion over $\mathcal{O}_\lambda[[\Gamma]]$ for these λ's.

The finiteness condition $H^0(F_v, A_{f_\eta})$ is known to hold in several cases (see [**14**, section 5·2]). In particular, if η comes from an elliptic curve, then this is always true, and so we have the following.

COROLLARY 5·4. *Assume that* (**H1**) *and* (**H2**) *are valid. Suppose that there exists* $\eta \in \mathfrak{X}_{arith}\left(h_{\mathcal{F}}^{ord}\right)$ *such that* $\mathcal{A}[P_\eta] \cong E[p^\infty]$ *for some elliptic curve* E *with* $R(E/F_\infty)$ *cotorsion over* $\mathbb{Z}_p[[\Gamma]]$.

Then $R(\mathcal{A}/F_\infty)$ *is cotorsion over* $\mathcal{R}[[\Gamma]]$. *Furthermore, for all but finitely many* $\lambda \in \mathfrak{X}_{arith}\left(h_{\mathcal{F}}^{ord}\right)$, $R\left(A_{f_\lambda}/F_\infty\right)$ *is cotorsion over* $\mathcal{O}_\lambda[[\Gamma]]$.

Proof. It remains to show that $H^0(F_v, E[p^\infty])$ is finite, but this is an immediate consequence of Mattuck's theorem that $E(F_v)$ is finitely generated over $\mathcal{O}_{F_v}$ (see [**35**]).

Comparing Theorems 4·3 and 5·3, one is naturally led to the following question.

Question $\mathcal{A}$. Is $Y(\mathcal{A}/F_\infty)$ pseudo-null over $\mathcal{R}[[\Gamma]]$?

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To the best knowledge of the author, the structure of $R(\mathcal{A}/F_\infty)$ does not seem to be well-understood. Even in the context of a cyclotomic $\mathbb{Z}_p$-extension, it is still an open question whether $R(\mathcal{A}/F_\infty)^\vee$ is finitely generated over $\mathcal{R}$ (see [**17**, conjecture 1]). There are some recent studies on this structure by Lei–Palvannan [**25**] in this direction. However, to the best knowledge of the author, the subject of the fine Selmer group of a Hida deformation over a general $\mathbb{Z}_p$-extension does not seem to be covered in the literature. We can only hope to revisit this subject in a future study.

We end the section by specialising to the case of an imaginary quadratic field, where we explain how a conjecture of Mazur implies the various Conjecture Y's. As a start, we recall the said conjecture of Mazur [**37**].

CONJECTURE 5·5 (Growth Number Conjecture). *Let E be an elliptic curve defined over $\mathbb{Q}$ and let K denote an imaginary quadratic field. The Mordell–Weil rank of E stays bounded along any $\mathbb{Z}_p$-extension of K, unless the extension is anticyclotomic and the root number is negative.*

THEOREM 5·6. *Assume that* (**H1**) *and* (**H2**) *are valid. Suppose that there exists $\eta \in \mathfrak{X}_{arith}\left(h^{ord}_{\mathcal{F}}\right)$ such that $\mathcal{A}[P_\eta] \cong E[p^\infty]$ for some elliptic curve E. Let K be an imaginary quadratic field and K_∞ a $\mathbb{Z}_p$-extension of K. Suppose that all of the following statements are valid.*

(*a*) *If K_∞ is the anticyclotomic $\mathbb{Z}_p$-extension, assume further that the root number is positive.*

(*b*) *The assertion of the growth number conjecture of Mazur is valid for the pair (E, K_∞). In other words, the Mordell-Weil rank of E stays bounded along the $\mathbb{Z}_p$-extension K_∞.*

(*c*) *The fine Tate–Shafarevich group $Ж(E/K_n)$ is finite for every n, where K_n is the intermediate subextension of K_∞/K with $|K_n{:}K| = p^n$.*

Then $Y(\mathcal{A}/K_\infty)$ is torsion over $\mathcal{R}[[\Gamma]]$. Moreover, for all but finitely many $\lambda \in \mathfrak{X}_{arith}\left(h^{ord}_{\mathcal{F}}\right)$, $Y(A_{f_\lambda}/K_\infty)$ is torsion over $\mathcal{O}_\lambda[[\Gamma]]$.

Proof. The growth number conjecture of Mazur implies that $E(K_\infty)$ is a finitely generated abelian group. Therefore, taking assumption (c) into account, we may apply Corollary 3·5 to conclude that $Y(E/F_\infty)$ is torsion over $\mathbb{Z}_p[[\Gamma]]$. The theorem is now a consequence of a combination of this latter assertion and Corollary 5·4.

Remark 5·7. One can of course prove variants of Proposition 4·3 and Theorem 4·4 for fine Selmer groups of $\mathcal{A}$. We shall leave the details for the readers to fill in.

6. *Examples*

We give some examples to illustrate our results.

(i) Let E be the elliptic curve $y^2 + y = x^3 - x$. Let $p = 5$. It is well-known that $\mathrm{Sel}(E/\mathbb{Q}(\mu_{5^\infty})) = 0$ (cf. [**5**, theorem 5·4]), where $\mathrm{Sel}(E/\mathbb{Q}(\mu_{5^\infty}))$ is the classical Selmer group. Let F be a finite Galois p-extension of $\mathbb{Q}(\mu_5)$. Then by [**13**, corollary 3·4], $\mathrm{Sel}(E/F^{\mathrm{cyc}})$ is cotorsion over $\mathbb{Z}_p[[\mathrm{Gal}(F^{\mathrm{cyc}}/F)]]$. Since the fine Selmer group $R(E/F^{\mathrm{cyc}})$ is contained in $\mathrm{Sel}(E/F^{\mathrm{cyc}})$, this in turn implies that $R(E/F^{\mathrm{cyc}})$ is cotorsion over $\mathbb{Z}_5[[\mathrm{Gal}(F^{\mathrm{cyc}}/F)]]$. Therefore, Theorem 3·9 applies. In other words, for any

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given $\mathbb{Z}_5^2$-extension L_∞ of F containing F^{cyc}, for all but finitely many $\mathbb{Z}_5$-extension F_∞ of F contained in L_∞, we have that $R(E/F_\infty)$ is cotorsion over $\mathbb{Z}_5[[\mathrm{Gal}(F_\infty/F)]]$. In view of Conjecture Y, we expect that $R(E/F_\infty)$ is cotorsion over $\mathbb{Z}_5[[\mathrm{Gal}(F_\infty/F)]]$ for all $\mathbb{Z}_5$-extensions of F. But this seems to be out of reach in general. (Note that Proposition 3·8 cannot apply here, since we do not have a good way of determining pseudo-nullity at our current state of knowledge.)

We however describe how one may obtain the absolute validity of Conjecture Y for the elliptic curve E in question over certain classes of 5-extensions of $\mathbb{Q}(\mu_5)$. Let F be a finite Galois 5-extension of $\mathbb{Q}(\mu_5)$ such that every ramified prime v of $F/\mathbb{Q}(\mu_5)$ outside 5 is neither a split multiplicative reduction prime of E nor a good reduction prime of E with $E(\mathbb{Q}(\mu_5)_v)[5] \neq 0$. By Kida's formula for elliptic curves (cf. [**13**, theorem 3·1]), $\mathrm{Sel}(E/F^{\mathrm{cyc}})$ is finite which in turn implies that $R(E/F^{\mathrm{cyc}})$ is finite. It then follows from [**28**, theorem 3·3] that so is $R(E/F)$. Applying Corollary 3·3, we see that Conjecture Y holds for every $\mathbb{Z}_5$-extension of F. Examples of such F's are $\mathbb{Q}(\mu_{5^n}, 5^{-5^m})$.

(ii) The above discussion can also be applied to the elliptic curve $E{:}y^2 + xy = x^3 - x - 1$ and $p = 7$. Indeed, in this case, one has $\mathrm{Sel}(E/\mathbb{Q}(\mu_{7^\infty})) = 0$ (cf. [**5**, theorem 5·31]). For more examples where the discussion in (i) also applies, we refer readers to the tables in [**10**] (basically look for those with $\mathcal{L}_E(\sigma)$ being a unit).

(iii) Even if $\mathrm{Sel}(E/\mathbb{Q}(\mu_{p^\infty})) \neq 0$, there are many numerical examples (for instance, see [**10**, **46**]), where the group $\mathrm{Sel}(E/\mathbb{Q}(\mu_{p^\infty}))$ can be shown to be cofinitely generated over $\mathbb{Z}_p$. By virtue of [**13**, corollary 3·4], $\mathrm{Sel}(E/F^{\mathrm{cyc}})$ is cofinitely generated over $\mathbb{Z}_p$ for every finite Galois p-extension F of $\mathbb{Q}(\mu_p)$ which in turn implies that $R(E/F)$ is cofinitely generated over $\mathbb{Z}_p[[\mathrm{Gal}(F^{\mathrm{cyc}}/F)]]$. Hence we can at least apply Theorem 3·9 for these examples. We mention that the data in [**46**] also consists of elliptic curves with supersingular reduction at p, where the plus-minus Selmer groups in the sense of Kobayashi [**22**] have been verified to be cofinitely generated over $\mathbb{Z}_p$. As the fine Selmer group is contained in either of the plus-minus Selmer groups, the fine Selmer group is also cofinitely generated over $\mathbb{Z}_p$. This cofinite generation of fine Selmer group is preserved under p-base change of fields (for instance, see the proof of [**31**, theorem 5·5]). Hence Theorem 3·9 applies for these examples.

(iv) Let E be any elliptic curve in 49a and $F = \mathbb{Q}(E[7])$. Take $p = 7$. We shall show that Conjecture Y is valid for every $\mathbb{Z}_7$-extension of F. Indeed, this is plainly true if the $\mathbb{Z}_7$-extension is the cyclotomic $\mathbb{Z}_7$-extension by [**48**, corollary 8] and [**4**, corollary 3·6]. For a non-cyclotomic $\mathbb{Z}_7$-extension F_∞ of F, the compositum of F_∞ and F^{cyc} is a $\mathbb{Z}_7^2$-extension of F. By [**26**, section 4, example (a)], $Y(E/L_\infty)$ is pseudo-null. Therefore, the torsionness of $Y(E/F_\infty)$ follows from this and Proposition 3·8. For more of such examples, we refer readers to [**26**, section 4, example (a)] and [**48**, table 2].

(v) Let E be any elliptic curve in 32a and $F = \mathbb{Q}(\sqrt{-43})$. Take $p = 3$. It has been verified by Lei-Palvannan that $R(E/L_\infty)$ is pseudo-null (see [**24**, section 8·4]), where L_∞ is the $\mathbb{Z}_3^2$-extension of F. Therefore, we may apply Theorem 3·8 to obtain the validity of Conjecture Y for every $\mathbb{Z}_3$-extension of F. [**24**, table 2] provides more examples, where Proposition 3·8 can be applied too.

(vi) Let E be the elliptic curve 79a1 of Cremona's tables given by

$$y^2 + xy + y = x^3 + x^2 - 2x.$$

Let $p=3$ and $F=\mathbb{Q}(\mu_3)$. It has been computed by Wuthrich (see [10, p. 253]) that $E(F)$ has $\mathbb{Z}$-rank 1 and the Tate–Sharafevich group $Ш(E/F)[3^\infty]$ is finite. Under these observations, it follows from a similar argument to that in [2, theorem 12] that one obtains $H^2(G_S(F), E[3^\infty])=0$. By [12, lemma 3·2], this in turn implies that $R(E/F)$ is finite. Let F_∞ be any $\mathbb{Z}_3$-extension of F. Then Theorem 3·3 applies to yield the torsionness of $R(E/F_\infty)$. As noted in [16, p. 362], the residual representation $E[3]$ is irreducible. Let $\mathcal{F}$ be the Hida family associated to E. Applying Corollary 5·4, we see that for all but finitely many $\lambda \in \mathfrak{X}_{\text{arith}}\left(h^{\text{ord}}_{\mathcal{F}}\right)$, $R\left(A_{f_\lambda}/F_\infty\right)$ is cotorsion over $\mathcal{O}_\lambda[[\mathrm{Gal}(F_\infty/F)]]$.

7. *Non-commutative speculation and further remarks*

In this final section, we formulate an extension of Conjecture Y for a (possibly non-commutative) p-adic Lie extension. To simplify the discussion and for better clarity of the ideas behind, we shall restrict our attention to the context of the paper. However, it should be evident that much of the discussion here carries over to broader classes of Galois representations.

We shall let $\bar{A}$ denote either one of the following objects: $A[p^\infty], A_f, \mathcal{A}$, and $\bar{R}$ the corresponding coefficients rings of these objects; namely, resp., $\mathbb{Z}_p, \mathcal{O}, \mathcal{R}$. Note that in the context of $\mathcal{A}$, we still work under the hypotheses **(H1)** and **(H2)**. Let F_∞ be a p-adic Lie extension of F such that $G:=\mathrm{Gal}(F_\infty/F)$ is pro-p with no p-torsion and that F_∞/F is unramified outside a finite set of primes of F. The fine Selmer group of $\bar{A}$ over F_∞, which is defined similarly as before, now has the structure of a $\bar{R}[[G]]$-module. We note that the ring $\bar{R}[[G]]$ is Auslander regular (cf. [52, theorem 3·26]; also see [27, theorem A·1]) and has no zero divisors (cf. [41]). Therefore, there is a well-defined notion of torsion $\bar{R}[[G]]$-modules and pseudo-null $\bar{R}[[G]]$-modules. For our purpose, a $\bar{R}[[G]]$-module M is said to be torsion (resp., pseudo-null) if $\mathrm{Ext}^i_{\bar{R}[[G]]}(M, \bar{R}[[G]])=0$ for $i=0$ (resp., $i=0,1$). The following is then the natural generalization of Conjecture Y for a p-adic Lie extension.

CONJECTURE 7·1 (Generalised Conjecture Y). *For every p-adic Lie extension F_∞ of F, $Y(\bar{A}/F_\infty)$ is torsion over $\bar{R}[[G]]$.*

For an abelian variety, the above conjecture was formulated in [23] for a $\mathbb{Z}_p^d$-extension of F.

Remark 7·2. We now discuss a context closely related to the generalised Conjecture Y. Had we replaced $\bar{A}$ in the definition of the fine Selmer group by $\mathbb{Q}_p/\mathbb{Z}_p(-i)$ $(i>0)$, where $(-i)$ denotes the $(-i)$th Tate twist, we will obtain the so-called étale wild kernel (for instance, see [42, 43]). In this context, the author has established the torsionness of this group over *every* p-adic Lie extension (see [29, proposition 4·1·1] and [30, sections 3·2 and 3·3] for details). Therefore, this provides some optimism towards the generalised Conjecture Y.

A similar argument to that in Theorem 3·9 yields the following.

THEOREM 7·3. *Let $d\geq 2$. Suppose that L_∞ is a $\mathbb{Z}_p^d$-extension of F which contains F^{cyc}. Denote by $\Phi_d(L_\infty/F)$ the set of all $\mathbb{Z}_p^{d-1}$-extensions of F contained in L_∞. Assume that $R(\bar{A}/F^{cyc})$ is cotorsion over $\bar{R}[[Gal(F^{cyc}/F)]]$. Then for all but finitely many $\mathcal{L}\in\Phi_d(L_\infty/F)$, $R(\bar{A}/\mathcal{L})$ is cotorsion over $\bar{R}[[Gal(\mathcal{L}/F)]]$.*

We finally end by providing a conceptual explanation on why the postulation of Conjecture Y over a general p-adic Lie extension is plausible. To give this explanation, we now recall the generalised Conjecture B of Coates–Sujatha [4] and Jha [16].

CONJECTURE 7·4. *If F_∞ is a p-adic Lie extension of F of dimension >1 containing F^{cyc}, then $Y(\bar{A}/F_\infty)$ is pseudo-null over $\bar{R}[[G]]$.*

PROPOSITION 7·5. *Suppose that Conjecture 7·4 is valid for every p-adic Lie extension of F of dimension >1 containing F^{cyc}. Then the generalised Conjecture Y is valid for every p-adic Lie extension of F of dimension >1.*

Proof. We first make the following remark. Let G be a compact p-adic Lie group and M a $\bar{R}[[G]]$-module. For every open subgroup G_0 of G, we then have

$$\mathrm{Ext}^i_{\bar{R}[[G]]}\left(M, \bar{R}[[G]]\right) \cong \mathrm{Ext}^i_{\bar{R}[[G_0]]}\left(M, \bar{R}[[G_0]]\right).$$

(cf. [40, proposition 5·4·17]). Therefore, the question of M being torsion (resp., pseudo-null) over $\bar{R}[[G]]$ is equivalent to M being torsion (resp., pseudo-null) over $\bar{R}[[G_0]]$.

Now let F_∞ be a p-adic Lie extension of F of dimension >1. If F_∞ contains F^{cyc}, then by the assumption of the proposition, $Y(\bar{A}/F_\infty)$ is pseudo-null over $\bar{R}[[G]]$, and hence is also torsion over $\bar{R}[[G]]$. Suppose that F^{cyc} is not contained in F_∞. Then $F^{\mathrm{cyc}} \cap F_\infty$ is a finite extension of F. In view of the the remark in the first paragraph, we see that neither the hypothesis nor the conclusion is affected if we replace F by $F^{\mathrm{cyc}} \cap F_\infty$. Hence, upon relabelling, we might as well assume that $F = F^{\mathrm{cyc}} \cap F_\infty$. Therefore, writing $L_\infty = F^{\mathrm{cyc}} \cdot F_\infty$, we have $\mathcal{G} := \mathrm{Gal}(L_\infty/F) \cong Z \times G$, where $Z \cong \mathrm{Gal}(F^{\mathrm{cyc}}/F)$. Let z be a topological generator of Z. Then for every $\bar{R}[[\mathcal{G}]]$-module M, we have $M_Z = M/(z-1)$ which can be viewed as a $\bar{R}[[G]]$-module. As Z is central in $\mathcal{G}$, M_Z may also be viewed as a $\bar{R}[[\mathcal{G}]]$-module. In particular,

$$0 \longrightarrow \bar{R}[[\mathcal{G}]] \longrightarrow \bar{R}[[\mathcal{G}]] \longrightarrow \bar{R}[[G]] \longrightarrow 0$$

is a $\bar{R}[[\mathcal{G}]]$-free resolution of the $\bar{R}[[\mathcal{G}]]$-module $\bar{R}[[G]]$. Via a spectral sequence argument similar to that in Lemma 2·1, we obtain

$$\mathrm{Ext}^i_{\bar{R}[[G]]}(Y(A/L_\infty)_Z, \bar{R}[[G]]) \cong \mathrm{Ext}^{i+1}_{\bar{R}[[\mathcal{G}]]}(Y(A/L_\infty)_Z, \bar{R}[[\mathcal{G}]]).$$

In particular, one has

$$\mathrm{Hom}_{\bar{R}[[G]]}(Y(\bar{A}/L_\infty)_Z, \bar{R}[[G]]) \cong \mathrm{Ext}^1_{\bar{R}[[\mathcal{G}]]}(Y(\bar{A}/L_\infty)_Z, \bar{R}[[\mathcal{G}]]).$$

Since $Y(\bar{A}/L_\infty)$ is pseudo-null over $\bar{R}[[\mathcal{G}]]$ by hypothesis, the latter is zero. From which, we see that $Y(\bar{A}/L_\infty)_Z$ is torsion over $\bar{R}[[G]]$. On the other hand, a descent argument as in Theorem 3·9 yields a map

$$Y(\bar{A}/L_\infty)_Z \longrightarrow Y(\bar{A}/F_\infty),$$

whose cokernel is a quotient of $H^1(Z, \bar{A}(L_\infty))^\vee$, where the latter is plainly finitely generated over $\bar{R}$. It then follows that $Y(\bar{A}/F_\infty)$ is torsion over $\bar{R}[[G]]$ as required.

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