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THE GENERIC MULTIVERSE IS NOT GOING AWAY

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Abstract. The generic multiverse was introduced in [74] and [81] to explicate the portion of mathematics which is immune to our independence techniques. It consists, roughly speaking, of all universes of sets obtainable from a given universe by forcing extension. Usuba recently showed that the generic multiverse contains a unique definable universe, assuming strong large cardinal hypotheses. On the basis of this theorem, a non-pluralist about set theory could dismiss the generic multiverse as irrelevant to what set theory is really about, namely that unique definable universe. Whatever one's attitude towards the generic multiverse, we argue that certain impure proofs ensure its ongoing relevance to the foundations of set theory. The proofs use forcing-fragile theories and absoluteness to prove ZFC theorems about simple "concrete" objects.

1. Introduction

Purity of mathematical proof as an ideal dates at least to Aristotle's *Posterior Analytics*. Aristotle believed that proofs appealing to concepts outside of the domain of the claim to be shown—*impure* proofs—could not reveal the true grounds of the claim, and even cites proving geometrical theorems with the aid of arithmetic as impure methodology.[1] Two millennia later, Arana and Detlefsen describe a *pure* proof as one in which the only resources used in the proof are in some sense intrinsic to the theorem proved [5, p. 1]. The central topic of this paper is a class of wildly impure proofs—indeed, it is difficult to conceive of a less pure style of proof. Suppose we want to prove a statement φ in Zermelo-Fraenkel set theory with the axiom of choice (ZFC). We work in a universe of set theory. Then we force some theory, e.g. ZFC + the Continuum Hypothesis (CH), which will be useful to us in proving φ. We "step into" the resulting forcing extension—another universe of set theory—and show that φ holds there. Assuming φ is of low enough logical complexity, it will be absolute between universes of set theory. So, since φ holds in the forcing extension satisfying CH we have passed into, we may conclude that it already held in the universe we started working in—even if the original universe thinks 2^ω is $\aleph_{54}$. The φ's we discuss are from the fields of computability theory, definable equivalence relations, number theory, graph theory, combinatorics, and functional analysis.

The title of this paper is the claim we aim to emphasize, for it seems to get somewhat lost in the pluralism/nonpluralism debate in the philosophy of set theory. We will define the generic multiverse precisely in §2. Suffice it to say, in terms of motivation, that the generic multiverse arises from the phenomenon of forcing in set theory. A universe W of set theory is a *ground* of a universe U if U is a forcing extension of W. Starting with a universe W of set theory, the generic multiverse is the collection of universes of set theory which are either realizable as forcing extensions of W or are grounds of such extensions. Forcing and, implicitly,

[1] See 75a29-75b12.

the generic multiverse have greatly influenced discussions of mathematical truth. To use the prototypical example, there seem to be "equally preferable" universes of set theory U and W such that CH is true in U and false in W. Pluralists about set theory take this to be the case. At the least, this seeming motivates the question: Does CH have a determinate truth value? There are disparate views on this question, but one which arises naturally from the generic multiverse is *(generic) multiversism*, the view according to which the meaningful statements of set theory are those which are true or false in all models of the generic multiverse. (On this view, CH is indeterminate.) By establishing the titular claim, we establish that there is a weak sense in which multiversism is right: The generic multiverse *is* relevant to any conception of set theoretic truth, even nonpluralist or "universist" ones.

What we are after are results which use *forcing fragile* pieces of higher set theory—theories which can be forced to hold and forced to fail. We want proofs which show that forcing extensions can be useful outside of higher set theory. The characteristic properties of the proofs we discuss are (1) the use of a forcing extension satisfying a theory extending ZFC and (2) absoluteness.[2] The proofs are not applications of the Baire category method to establish ZFC theorems.[3]

The project can be contrasted with H. Friedman's. Friedman aims to produce non-metamathematical statements of number theory and analysis—ideally statements which number theorists and analysts would want to know the truth values of—whose proofs require (the consistency of) large cardinals, thereby showing that large cardinals are necessary for non-set theoretic mathematics.[4] The aim of this paper is much less ambitious. We aim only to establish that *forcing-and-absoluteness* proofs show that the generic multiverse provides problem solving tools for proving ZFC theorems, and insofar as it is useful in this way, it is here to stay as an object of foundational relevance.

§2 introduces the generic multiverse and why one might think it is dispensable. §3 describes the collection of proofs. We discuss what these proofs do and do not support, philosophically, in §4. In §5 we consider an objection arising from the impurity of forcing and absoluteness proofs. We suggest that impurity of proof has a role to play in increasing our confidence in the coherence of the generic multiverse and set theory generally.

2. Dispensing with the generic multiverse

Suppose we want to prove that a statement φ is relatively consistent with ZFC. Given a ground universe W satisfying ZFC, we find an appropriate partial order $\mathbb{P}$ in W, and construct from a filter G meeting every dense subset of $\mathbb{P}$ to obtain the forcing extension $W[G]$. If φ is true in $W[G]$, then we have shown that if ZFC is consistent, then so is $\mathrm{ZFC} + \varphi$.

Forcing has been used to show that many mathematical propositions are not decided on the basis of the ZFC axioms via such relative consistency proofs. For example, Borel's conjecture that any set of strong measure zero is countable and Kaplansky's conjecture that every homomorphism on $C[0,1]$ is continuous are each

[2]This class of proofs was mentioned by Martin in [56] as evidence that "the effect of independence proofs on mathematics is not entirely negative" [p. 83]. That is, the techniques can prove ZFC theorems, not only relative consistency results.

[3]See [60].

[4]See the forthcoming [30].

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independent of ZFC.[5] The intractability of CH is infamous. Not only can it be forced and unforced, but the same is true in the presence of large cardinal axioms, and, by a theorem of Solovay [72], the cardinality of the continuum "can be anything it ought to be." On the basis of these results about CH in particular, forcing has led some philosophers and set theorists to reject the idea that there is an intended universe V of set theory. To them, it no longer makes sense to ask whether Kaplansky's conjecture is true simpliciter—one must ask whether it holds in this or that universe of set theory.

This relativism is in some sense a recurrence of an earlier worry about intended interpretations of set theoretic discourse. That worry was precipitated by Skolem's paradox in the 1920s. Skolem saw the non-absoluteness of countability as a reason to give up set theoretic foundations altogether, since its axioms cannot pin down a unique structure, and von Neumann took it to indicate that higher set theory is meaningless [80, p. 413]. While with hindsight we can say that these reactions were predicated on an unrealistic expectation (that our foundational theory should have a categorical domain), the power of forcing as a model construction technique and the mathematically central statements it showed to be independent inspired a deeper "second generation" of this worry. Logicians had become comfortable with the non-absoluteness of countability. But forcing showed more than the non-absoluteness of a notion. It showed that there are infinitely many foundational theories which disagree about mathematical propositions, and their models are interesting.[6] Some set theorists and philosophers, including Cohen [16], Shelah [68], Field [28], and Hamkins [38], began to feel that the idea that set theory is about a unique universe V is at odds with the practice.

Views on the importance of forcing extensions to the question of whether there is an intended interpretation, or to the intended interpretation itself, can be articulated with reference to the *generic multiverse*, which is given by the following theory.[7][8] The *multiverse language* is the language of set theory with sorts for universes of set theory and for sets.

Definition 1 (Steel)**.** The *multiverse theory* consists of the following axioms.

(1) φ^W, for each axiom φ of ZFC and each universe W.

(2) (a) Every universe is a transitive proper class, and an object is a set iff it belongs to some universe.
  (b) If U is a universe, and $U = W[G]$ for some G which is $\mathbb{P}$-generic over W, then W is a universe.
  (c) If W is a universe and $\mathbb{P} \in W$ is a forcing notion, then there is a universe of the form $W[G]$, where G is $\mathbb{P}$-generic over W.
  (d) If U, W are universes, then there are G, H which are generic over them such that $W[G] = U[H]$.

The multiverse language is a sublanguage of the language of set theory, and the multiverse theory is conservative over ZFC.

[5]By work of Luzin and Laver in the former case, and Solovay and Woodin in the latter.

[6]Moreover, when forcing over countable transitive models, forcing extensions are *standard*.

[7]We work with Steel's conception, but note that Steel's and Woodin's give the same set of generic multiverse truths, those statements which are true in every universe in the generic multiverse. Woodin's version does not have the amalgamability property, and it is not formally axiomatizable.

[8][53] is an extended discussion of Steel's conception.

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Steel [74] describes three views one can have on the generic multiverse and its relation to the purported intended interpretation of set theory. *Weak Relativism* is the view that all statements in the language of set theory are expressible in the language of the multiverse. The generic multiverse is all there is to set theory, in essentially the way that generic multiversism holds that the truths of set theory are those which are invariant in the generic multiverse. *Strong Absolutism* holds that there is an intended universe V of set theory, but the generic multiverse has no bearing on it. The generic multiverse language is too impoverished to capture the meaning of talk about V: It makes sense to talk about V, but this sense cannot be captured in the multiverse language. *Weak Absolutism* is the view that one can talk meaningfully about V in the language of the generic multiverse, and hence V is "present" in the multiverse. Precisely speaking, weak absolutism is the mathematical claim that there is a unique, definable world in the multiverse.[9] This is equivalent to the generic multiverse containing a model from which all others arise as forcing extensions, a "core." Steel concludes that, unless and until we are in a position to specify V, it makes sense—for the weak relativist and weak absolutist at least—to conceive of set theory as taking place in the generic multiverse, rather than in an intended model.

It is now a theorem of Usuba [79] that if there exists an extendible cardinal, then the weak absolutist thesis is true: The generic multiverse has a core. Usuba's theorem can be interpreted as grounds for doing away with the generic multiverse.[10] After all, the generic multiverse is a formalization of a conception of set theory (i) which was well supported by the independence phenomena and (ii) in which no universe of set theory, on the face of it, stands out over any other. For the Relativist had a point: not enough had been done to specify the V which non-pluralists insisted exists. Whether there is such a V should be treated as an open question, and Steel put forth the generic multiverse as a framework for developing set theory in which an investigation into this question could take place. Usuba's theorem shows that within this neutral framework, one *can* uniquely identify a world. And it is a consequence of the Definability of Grounds[11] that if the generic multiverse has a core, then the core is V. By a theorem of Woodin [81], the theories of forcing extensions in the generic multiverse reduce to the theory of the core. So V can figure out these theories; the generic multiverse is unnecessary. With Usuba's theorem in hand, the framework that allowed V to be identified can be kicked away.

[9] In contrast to the other views, the content of the weak absolutist view is a mathematical claim. Steel calls it the *Weak Absolutist Thesis*.

[10] Weak Relativism only claims that the meaningful statements of set theory are those expressible in the language of the multiverse. Since Usuba's theorem implies that "$\dot{V}$" is meaningful in that language, the Weak Relativist assents to talking about it. Results about definability in the language of the generic multiverse remain irrelevant to the Strong Absolutist, since she has antecedently decided that the multiverse language has no bearing on V. As a consequence, the Strong Absolutist cannot immediately infer that the core is V.

[11] Laver [52] and Woodin [81] independently showed that if $W[G]$ is a forcing extension of a ground universe W, then W is definable in $W[G]$ from parameters, namely the powerset of the cardinality of the forcing. The property of being a forcing extension is thus first order (so forcing extensions in the generic multiverse "know" they are not the core). In fact there is a uniform definition of the grounds of a forcing extension, see [31].

If the Weak Absolutist adds the statement "V is the HOD of a model of determinacy"[12] to the multiverse theory, and if that statement is consistent with all Σ_2-definable large cardinal notions, then it is hard to see that we lose any mathematics. We get a complete conception of V which founds everything mathematics uses. Why not adopt this theory and turn attention to the core, dispensing with the generic multiverse and the rival theories whose interconnections it allows us to study?[13]

We will argue that this interpretation of Usuba's theorem is too quick by illustrating the utility of figuratively "hopping around the multiverse" in acquiring mathematical knowledge relevant to all parties, relativist or absolutist. Specifically, this paper aims to (1) push back on the sentiment recorded in [58] that the generic multiverse is dispensable or assailable and (2) document a body of forcing-and-absoluteness proofs which use higher set theory to prove ZFC theorems about concrete objects.

3. Forcing to prove theorems about the core

An immediate corollary of Shoenfield's Absoluteness Theorem is that the truth value of a logically simple enough statement cannot be changed by passing to a forcing extension.

Theorem 2 (Shoenfield). *(ZFC) If φ is a Σ^1_2 formula and $\mathbb{P}$ is a forcing notion, then*

$$W[G] \models \varphi \Leftrightarrow V \models \varphi,$$

where G is $\mathbb{P}$-generic over W.

Such a formula φ is *absolute* between transitive models of set theory. In the generic multiverse setting, Shoenfield's theorem says that all worlds agree on Σ^1_2 sentences. This opens the door to investigating problems in carefully tailored forcing extensions of W.

Building on Shoenfield's work, Platek proved metamathematical results which give conditions under which one can eliminate appeals to GCH and the axiom of choice in proofs of ZF theorems.

Theorem 3 (Platek [63]). *Let φ be a sentence in the language of set theory.*

(1) *If φ is Π^1_4 and $\mathrm{ZFC}+\mathrm{GCH} \vdash \varphi$, then $\mathrm{ZF} \vdash \varphi$.*
(2) *If φ is Π^2_1 and $\mathrm{ZFC}+\mathrm{GCH} \vdash \varphi$, then $\mathrm{ZF} + \aleph_1\text{-}\mathsf{DC} \vdash \varphi$.*
(3) *If φ is Π^{n+2}_1 and $\mathrm{ZFC}+$ "GCH holds for $\kappa \geq \beth_n$" $\vdash \varphi$, then $\mathrm{ZFC} \vdash \varphi$.*

Remark. As observed by an anonymous reviewer, item (2) can be improved, using a forcing-and-absoluteness proof, to show that φ is provable in $\mathrm{ZF}+\mathrm{DC}$. Let φ be Π^2_1 and $A \subseteq \mathbb{R}$. Generically wellorder the reals of $L(A,\mathbb{R})$ via G. Then $L(A,\mathbb{R})[G]$ is a model of ZFC, and hence $L(A,\mathbb{R})[G] \models \varphi$. Then φ is downward absolute to $L(A,\mathbb{R})$. DC is required so that there are no new reals in $L(A,\mathbb{R})[G]$. ⊣

[12] Woodin's axiom $V =$ Ultimate-L, which says that there is a proper class of Woodin cardinals and every true Σ_2 sentence reflects into the HOD of $L(A,\mathbb{R})$ where A is universally Baire, is one way of formally explicating this. See [82, Definition 7.14]. This is expressible in the multiverse language.

[13] Meadows' reading [58] of Woodin's arguments in [81] construes Woodin's position as rejecting the generic multiverse in this way, albeit on the basis of different arguments.

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Each result is best possible for the respective levels of logical complexity. (Item (1) cannot be improved because there is a form of choice which is Σ^1_4 and unprovable in ZF [44], and item (2) cannot be improved because CH is Σ^2_1 and unprovable in ZF.) Platek's theorem captures the reason why we cannot claim in general that the proofs to be described show that the generic multiverse is formally necessary: By their very nature, any such appeal to a world in the generic multiverse may be eliminable.

Following [74], by *concrete* mathematics, we mean mathematics that is about the natural numbers, the real numbers, or absolutely definable sets of real numbers. We refer to set theories extending ZFC collectively as *higher set theory.* The structure of the proofs to follow is: We begin in a ground universe W of ZFC. We want to prove a statement in concrete mathematics, say of complexity Σ^1_2. We pass to a forcing extension $W[G]$ of higher set theory, e.g. ZFC+$\neg$CH. We show that $W[G] \models \varphi$, and we conclude via absoluteness that $W \models \varphi$. We have used forcing and absoluteness to prove a ZFC proposition about concrete mathematics using higher set theory.

Here is a simple example. Recall that a real x is *Turing reducible* to a real y ($x \leq_T$ y) if there is a Turing machine which, given information about the bits of y, can compute x. A *Turing degree* is an equivalence class of real numbers x, y which can compute each other ($x \leq_T y$ and $y \leq_T x$).

Theorem 4 (Kleene-Post)**.** *There are $\leq_T$-incomparable Turing degrees.*

We give a folklore proof. If there were no incomparable Turing degrees, then $\leq_T$ would be a linear order. Any Turing degree can compute only countably many Turing degrees, so the cofinality of the degrees could be at most ω_1. Then there must be ω_1-many Turing degrees, and since there are continuum-many Turing degrees, we get that CH holds. Now we pass into a forcing extension $V[G]$ in which $2^\omega > \omega_1$. Let φ be the Σ^1_1 statement that there are two incomparable Turing degrees. Clearly $V[G] \models \varphi$, since $V[G] \models \neg$CH. Moreover, φ is sufficiently simple so that Theorem 2 applies. Hence there are incomparable Turing degrees. $\dashv$

There is no content in the theorem which suggests anything beyond recursion theoretic tools is needed for its proof, and indeed the original proof is purely recursion theoretic, using an oracle for $\mathbf{0}'$ and a finite extension construction. The original proof explicitly constructs incomparable degrees so that one sees what disrupts the computation of one degree from the other. So we have a case in which Aristotle was right: the pure proof is preferable. The proof above is something that a set theorist with little knowledge of recursion theory might come up with when told that there are incomparable degrees.[14]

In the proof, we passed to an extension satisfying ZFC + $\neg$CH. Suppose for the sake of discussion that CH is actually true (whatever that means). Then we proved a true statement using a false theory. If statements of higher set theory have determinate truth values, some subset of the proofs to come do the same. This is a Hilbertian circumstance: mere consistency (of the higher set theory) implies existence (of the concrete state of affairs to be shown).

3.1. **The proofs.** This section documents some *forcing-and-absoluteness proofs* of ZFC theorems about concrete objects. Others are collected in Appendix A. Since

[14] In the context of hyperdegrees, if one did not know Spector's measure theoretic proof that there are incomparable hyperdegrees [73], it is plausible that one would come up with a proof like the above using Σ^1_2 absoluteness.

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my foremost purpose is to show that the method has something to tell us about the relevance of the generic multiverse to the foundations of set theory, and to foreground ensuing discussions in §4 and §5, I have opted to give diverse examples accompanied by remarks about their individuating features. The nonexpert reader can skim this section. The key takeaway is that there are forcing-and-absoluteness proofs such that

- bespoke forcing extensions are used (Theorems 18, 19, 22, 23)
- bespoke generic absoluteness theorems are invoked (Theorems 11, 21, 35)
- generic absoluteness is invoked multiple times (Theorems 18, 35)
- distinct forms of generic absoluteness are invoked (Theorem 15)
- successive forcing extensions are taken (Theorem 22)
- a forcing extension satisfying ZFC is appealed to in proving a ZF+DC+AD theorem (Theorem 8)
- mutual genericity is used to pull a fact into the ground universe from two forcing extensions (Theorem 20)
- the language of forcing arguably allows the result proved to be stated in its full generality, and in a readily applicable way (Theorem 20)
- CH is not a source of counterexamples but of useful structure (Theorems 8, 11, 29, 37)
- ¬CH is essential to the argument, perhaps in the form of MA + ¬CH (Theorems 22, 35, 16, 31)
- the argument adapts *mutatis mutandis* to prove stronger theorems in an extension of ZFC (Theorems 5, 14, 18, 32)
- an object is constructed generically and shown, by absoluteness, to exist in an inner model of the ground universe (Theorem 28)
- a finite object is generically constructed (Theorem 24)
- large cardinals are used (Theorem 33)
- the result established shows that forcing-and-absoluteness may suffice to solve a further problem (Theorem 28)
- their pure or classical counterparts predated them (Theorems 4, 5, Theorems 20, 21, 22, 23)
- their pure counterparts postdated them (Theorems 6, 35, 18, 37)
- no pure proof is currently known/published (Theorems 28, 8, 11, 31, 32, 33, 24)
- they are preferred to their pure counterparts by experts for epistemic or methodological reasons (Theorems 5, 6, 18, 19, 20, 21, 22, 23)

The quotable forms of generic absoluteness appealed to include Shoenfield absoluteness, Martin-Solovay's Σ^1_3-absoluteness, and Stern's absoluteness. The theorems come from computability theory, definable equivalence relations, the partition calculus, number theory,[15] analysis, graph theory, and functional analysis.

3.1.1. *Harrington's proof of the Halpern-Läuchli Theorem.* The first example is not quite a forcing-and-absoluteness proof, but it has implications for the study of purity. Halpern and Läuchli wanted to produce a model of ZF+¬AC + "every Boolean algebra contains a prime ideal." At the time, it was known that the statement "every Boolean algebra contains a prime ideal" cannot be proved in ZF alone, but is

[15]The example below, the Ax-Kochen theorem, was proved by working in Gödel's L and applying absoluteness. A forcing extension satisfying CH works just as well.

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provable with AC. The goal was to show that the theory is of intermediate strength between ZF and ZFC. The Halpern-Läuchli Theorem, a Ramsey-like Theorem for finite products of finitely branching infinite trees without terminal nodes, is a step toward that goal. It was proved with metamathematical methods in the 1966 paper [37], where the authors characterized their proof as "dissatisfying" due to its indirectness. In 1978, Harrington gave a proof which, despite the current variety of alternative proofs, "is regarded as providing the most insight" by Ramsey theorists [20, p. 2], for whom the theorem is a widely applicable tool. It was Harrington's argument that was generalized by Shelah to establish a version of Halpern-Läuchli for higher cardinals [66] and which motivates a research program on variations of Halpern-Läuchli by Dobrinen and Hathaway and others (see [21, 22]).

Let $T(n)$ be the nth level of T, i.e. the set of sequences in T of length n. We refer the reader to [20] for the definition of *strong subtree*.

Theorem 5 (Halpern-Läuchli [37]; Harrington (see [20],[24])). *Let $T_i \subseteq \omega^{<\omega}$, for each $i < d < \omega$, be a finitely branching tree, and let*

$$c = \bigcup_{n<\omega} \prod_{i<d} T_i(n) \to k,$$

for some $k < \omega$, be a coloring. Then there are an infinite set of levels $L \subseteq \omega$ and strong subtrees $S_i \subseteq T_i$, each with nodes exactly at the levels of L, such that c is monochromatic on

$$\bigcup_{n\in L} \prod_{i<d} S_i(n).$$

Harrington's proof uses the Cohen forcing adding $d \times \kappa$-many Cohen reals to add κ-many infinite paths through the trees T_i, where $\kappa = \beth_{2d}$. It uses the forcing and the forcing language to inductively search for the next monochromatic levels of the strong subtrees.

The reason that Harrington's proof does not fit the forcing-and-absoluteness proof template exactly is that it neither uses absoluteness (although part of the proof is an argument about what is preserved in c.c.c. forcing) nor requires actually working in a forcing extension. It only requires combinatorially analyzing the forcing. Yet it is highly impure.

3.1.2. *Silver's Dichotomy.* Silver's proof of his well-known theorem on coanalytic equivalence relations used forcing and absoluteness.

Theorem 6 (Silver [70]). *Let E be a coanalytic equivalence relation on ω^ω with uncountably many equivalence classes. Then there is a perfect set of mutually E-inequivalent reals.*

Silver's Dichotomy extends to the coanalytic sets Suslin's theorem that every analytic set has the perfect set property. It shows that coanalytic equivalence relations satisfy Vaught's Conjecture and is regarded as a first step historically in understanding the hierarchy of definable equivalence relations. Silver's proof uses GCH despite being about definable relations on reals. This appeal to higher set theory was quite unusual, as proofs of other descriptive set theoretic dichotomies were local. Silver shows that Theorem 6 is a Π^1_3 statement and appeals to the folklore result that any statement of this complexity proved in ZFC + GCH is provable in ZF [70, §3].

Silver observed that

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> Since our theorem can be formulated in the language of so-called "analysis" or "second-order number theory", it might be expected that it could be proved within the usual axiomatic system for second-order number theory. Such is the case for almost all other statements of second-order number theory which are known to be provable in ordinary set theory and do not have a metamathcntatical content. In fact, the only statements known to be counterexamples to this rule relate to Borel determinacy... On the other hand, for all we know, it may yet be possible to find a proof of the theorem of this paper within second-order number theory. [70, p. 2]

Before [70] went to press, Harrington found such a proof using effective descriptive set theory.[16] This is the proof that is taught, and it is presented in the standard reference on set theory.[17] It uses what is now called the Gandy-Harrington forcing—the topology generated by the effectively analytic sets—which has become a standard tool in proving dichotomy theorems in the theory of definable equivalence relations (the Glimm-Effros dichotomy, Solecki's dichotomy, and Hjorth's Turbulence Theorem), and it admits lightface relativizations. It also generalizes to prove Silver's theorem for Π^1_3 equivalence relations. Silver's theorem is another in which the pure proof simply has more virtues than its impure counterpart.

Burgess extended Silver's theorem to analytic equivalence relations:

Theorem 7 (Burgess [13])**.** *Any Σ^1_1 equivalence relation E on ω^ω either has at most ω_1 equivalence classes or a perfect set of E-inequivalent reals.*

His proof directly appeals to Silver's theorem, and Burgess' theorem predated Harrington's "reproof" of it,[18] so this can be viewed as another significant use of forcing and absoluteness. Harrington and Shelah [40] extended Harrington's argument to prove a theorem that subsumes and generalizes Silver's and Burgess' theorems.

3.1.3. *Uses of Mokobodzki's theorem.* Slaman and Steel [71] asked: In ZF+DC+AD, if $d \mapsto <_d$ is a map associating to each Turing degree d a linear ordering of d, do the rationals order embed in $<_d$ for almost all d? Kechris [46] showed that the answer is yes. To do so, he introduced the notion of *amenability*.[19] Kechris' theorem uses (i) the *non*-amenability of Turing equivalence and its restrictions to Turing invariant sets and (ii) the following lemma:

Lemma 8 (Kechris)**.** *(*ZF+DC*) Let E be a countable Borel equivalence relation on 2^ω such that E extends $\equiv_T$. Let $C \mapsto <_C$ be a Borel map associating to each E-class C a linear ordering of C. If μ is the standard measure on 2^ω, then for μ-a.e. real x, there is an order-preserving embedding of the rationals into $\langle [x]_E, <_{[x]_E} \rangle$.*

[16] Whether a lightface proof of a boldface theorem is pure is a curious question. The definability analysis of the continuum was motivated by concerns of effectivity before the analyses of intuitively computable functions. The infusion of recursion theoretic techniques into descriptive set theory is naturally construed as a refinement of the notion of effectivity at the heart of the fuller theory. But we will return to this below in §5.

[17] See [45, Ch. 31].

[18] Burgess' dissertation [12] was submitted in 1974, and Harrington's proof was first written down in a set of handwritten notes entitled "A powerless proof of Silver's theorem" in 1976.

[19] A countable Borel equivalence relation E on a Polish space X is *amenable* if to every E-equivalence class $[x]_E$ a finitely additive probability measure defined on all subsets of $[x]_E$ can be associated in a universally measurable way.

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The proof of the lemma uses a CH theorem of Mokobodzki.

Theorem 9 (Mokobodzki)**.** *Assume* CH*. Then there is*

(i) *a universally measurable shift invariant mean on the integers, and*
(ii) *a universally measurable shift invariant mean on the natural numbers.*

Kechris notes that because the property to be shown in Lemma 8 is Π^1_2, one can proceed from ZFC + CH (essentially by Theorem 3). So we pass into a CH world. Then if the lemma is false, there is a Borel E-invariant $X \subseteq 2^\omega$ such that the rationals embed into $\langle [x]_E, <_{[x]_E} \rangle$ order preservingly for all $x \in X$. It follows using the medial mean from Theorem 9 that $E \restriction X$ and $\equiv_T \restriction X$ are amenable, which contradicts that Turing equivalence restricted to any invariant set is not amenable. The lemma follows via absoluteness.

This is a particularly interesting example because absoluteness allows one to obtain the lemma in the ZFC + CH world and transfer it, in the course of the proof of Kechris' main theorem and with some finessing to keep everything relevant Borel, to a ZF + DC + AD world. There is no *prima facie* reason for Choice or CH to be involved in answering a question posed in the effective setting of AD.[20]

Cieśla and Sabok likewise assume CH in order to apply Mokobodzki's theorem in their work on measurable circle squaring.[21] They prove a custom absoluteness lemma which ensures that proving in the forcing extension that there is an equidecomposition of two measurable sets into measurable pieces amounts to proving the same in the ground model.

Lemma 10 (Cieśla-Sabok [15])**.** *Let $U \subseteq W$ be models of* ZFC*. Suppose in U there is a Borel space X with a Borel probability measure μ, two Borel subsets $A, B \subseteq X$ and $\Gamma \curvearrowright (X, \mu)$ is a Borel probability measure preserving action of a countable group Γ. Then the statement that A and B are Γ-equidecomposable μ-a.e. using μ-measurable pieces is absolute between U and W.*

Working in an extension satisfying CH, they show, using the medial means given by Mokobodzki's theorem, the following characterization of circle squaring.

Theorem 11 (Cieśla-Sabok [15, Theorem 2])**.** *Let Γ be a finitely generated abelian group and let Γ be a free probability measure preserving action on a space (X, μ). Suppose $A, B \subseteq X$ are two measurable Γ-equidistributed sets of the same positive measure. Then the following are equivalent:*

(1) *the pair A, B satisfies the Hall condition with respect to Γ μ-a.e.,*
(2) *A and B are Γ-equidecomposable μ-a.e. using μ-measurable sets,*
(3) *A and B are Γ-equidecomposable μ-a.e.*

CH is often characterized as a source of "pathological" constructions of length ω_1. Mokobodzki's theorem is an instance of CH implying *useful* structure.[22]

[20]In ZF + AD, neither ω_1 nor $\mathbb{R}$ injects into the other.

[21]The problem of whether, given two measurable sets, one can be partitioned into finitely many measurable pieces and rearranged so as to obtain the other.

[22]Larson has shown that Mokobodzki's theorem 9 requires more than just ZFC [49], as the *Filter Dichotomy* (the statement that every uniform non-meager filter on the integers is mapped to an ultrafilter by a finite-to-one function) implies there are no universally measurable shift invariant means. Normann showed that Martin's Axiom suffices to obtain such means [61].

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3.1.4. *Stern's absoluteness.* Stern [76] showed that if ω_1 is inaccessible to reals, then there is no thin Σ^1_1 equivalence relation having $\aleph_1$-many equivalence classes with bounded Borel rank. To do so, he proved a form of generic absoluteness for Borel sets which says that if a Borel set of rank α can be created by forcing (such a set is described by a *virtual Borel set*), then it can be approximated in the ground model by $\beth_\alpha$-Borel sets.

Theorem 12 (Stern; [43, Corollary 1.8])**.** *Let $M, N \models$ ZFC be well-founded and suppose that $M \subseteq N$ and $\beth^M_{-1+\alpha} < \omega_1^N$. Suppose $(\mathbb{P}, p, \tau) \in M$ is a virtual Borel set, $\alpha < \omega_1$, and*

$$p \Vdash_{\mathbb{P}} \text{ “}\tau \text{ is a Borel code of rank } \alpha\text{,” and}$$

$$(p, p) \Vdash_{\mathbb{P}\times\mathbb{P}} \text{ “}\tau[g_1] \text{ codes the same Borel set as } \tau[g]\text{.”}$$ [23]

Then there is a Borel code $(\dot{T}, \dot{f}) \in N$ of rank α such that

$$p \Vdash_{\mathbb{P}} \text{“}B(\varnothing, \dot{T}, \dot{f}) \text{ is the Borel set coded by } \tau[g]\text{.”}$$

Hjorth, describing Stern's absoluteness and his own use of it, writes

> Unlike, say, Shoenfield absoluteness, Stern's absoluteness can only be made understood in the terminology of forcing. Since forcing is typically associated with the pursuit of independence results, we could easily assume that Stern's work has little relevance in proving positive theorems about the Borel hierarchy.
>
> However, this would be untrue. Using abstract and indirect metamathematical arguments, and availing ourselves of Stern's absoluteness principle, we will prove a string of ZFC theorems for which no direct proof is known. [43, p. 663]

Among the items in this string are

Theorem 13 (Kanovei; [43, Theorem 2.2])**.** *If A is a non-Borel Σ^1_1 set, then the constituents of A do not have bounded Borel rank.*

Hjorth's proof appeals to Π^1_2-absoluteness and derives a contradiction with Stern's absoluteness Theorem 12. The proof of the next theorem appeals to Shoenfield absoluteness.

Theorem 14 ([43, Corollary 4.5])**.** *For $\alpha < \beta < \omega_1$, there is no injection $f : \Pi^0_\beta \to \Pi^0_\alpha$ which is Borel (in the codes).*

In fact this generalizes to the nonexistence of such an injection in $L(\mathbb{R})$ assuming $\mathsf{AD}^{L(\mathbb{R})}$ using essentially the same argument [43, Theorem 4.7].

Theorem 15 (Harrington; [43, Theorem 5.5])**.** *The continuous actions of S_∞ on Polish spaces induce a $\leq_B$-unbounded, length ω_1 sequence of Borel equivalence relations.*

Both Shoenfield and Stern absoluteness are appealed to in the proof.

Theorem 16 ([43, Theorem 5.19])**.** *Isomorphism on countable sets of reals is not Borel reducible to the orbit equivalence relation induced by continuous actions of a Polish abelian group on any Polish space.*

[23] Let g be V-generic for $\mathbb{P} \times \mathbb{Q}$. Then g_1 is the projection of g onto $\mathbb{P}$.

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The proof passes to an extension of MA_{ω_1} to utilize the fact that Σ^1_2 sets have the Baire property.[24]

A result going beyond ZFC but still about the concrete is also shown and justified by Hjorth as follows:

> I should stress that this does not mean that the argument has no relevance assuming the axiom of choice. In fact, Harrington's argument directly implies that in the presence of large cardinals there is no simply definable uncountable sequence of Borel sets with bounded rank. The method of proof is also fertile, in the sense that it produces useful theorems just in ZFC.[25]

Theorem 17 (Harrington; [43, Theorem 3.1]). *Assume $\mathsf{AD}^{L(\mathbb{R})}$. Then there is no injection $f : \omega_1 \to \Pi^0_\alpha$, for $\alpha < \omega_1$, in $L(\mathbb{R})$.*

3.1.5. *Forcing constructions and countable Borel equivalence relations.* [33] contains many theorems about Borel complete sections of orbit equivalence relations whose forcing-and-absoluteness proofs use forcings constructed specifically for their proofs: orbit forcing, Cohen forcing on a countable group, minimal 2-coloring forcing, and grid periodicity forcing. Most of the theorems proved in the paper using forcing are positive existence results, and we restrict to just two.

Theorem 18 (Gao-Jackson-Krohne-Seward [33]). *Let $\Gamma \curvearrowright X$ be a continuous action of a countable group Γ on a compact Polish space X, and let E^X_Γ be the associated orbit equivalence relation. Let $(A_n : n < \omega)$ be a sequence of finite subsets of Γ such that every finite subset of Γ is contained in an A_n. Let $(S_n : n < \omega)$ be a sequence of Borel complete sections of E^X_Γ. Then there is an $x \in X$ such that for infinitely many n, $A_n \cdot x \cap S_n \neq \varnothing$.*

The proof appeals to both Π^1_1 and Σ^1_1-absoluteness at distinct stages of the argument. It generalizes to give the analogous result when the S_n's are absolutely Δ^1_2, in which case Π^1_2 and Σ^1_2-absoluteness are required. The authors give a pure topological proof of the theorem using strong Choquet games *to justify why they proceed to use forcing and absoluteness in the rest of the paper*. They see value in the technique beyond its being an instrument for proof discovery. For one thing, the forcing-and-absoluteness argument itself gives corollaries on when recurrent points exist in the range of Borel equivariant maps ([33, Theorem 3.7]).

Theorem 19 (Gao-Jackson-Krohne-Seward [33]). *Suppose $f : F(2^{\mathbb{Z}^2}) \to \{0, 1, \ldots, n-1\}$ is a Borel function. Then there is $x \in F(2^{\mathbb{Z}^2})$ and $N \in \mathbb{N}$ such that for any chromatic 2-coloring $t \mapsto f(t \cdot x)$ on $[a, b] \times [c, d]$, $b - a, d - c \leq N$.*

The authors note [33, Remark 6.10] that the usual measure and category methods used to prove similar results cannot work to prove this theorem. We will return to this below.

---

[24] Recall that MA_κ is the assertion that for any forcing notion $\mathbb{P}$ whose antichains are countable, and any family D of dense sets in $\mathbb{P}$ of size at most κ, there is a filter $G \subseteq \mathbb{P}$ that meets every dense set in D. Martin's Axiom (MA) is the assertion that MA_κ holds for all κ less than the continuum, and it is most interesting when it is assumed that CH is false.

[25] One of the ZFC theorems derivable from this is that there is no provably Δ^1_2 injection from countable ordinals to Borel sets of bounded rank.

3.1.6. *Turbulence.* One of the central themes of the theory of countable Borel equivalence relations is classification by complete sets of invariants. A Borel reducibility $E \leq_B F$ tells us that E is no harder to classify by a complete set of invariants than F is. A liberal class of invariants is the class of countable structures. An equivalence relation E is classifiable by countable structures if it is Borel reducible to the isomorphism relation on a class of countable structures. Hjorth identified the notion of a *turbulent* group action and showed that orbit equivalence relations of turbulent group actions cannot be classified by countable structures.[26]

Theorem 20 (Hjorth)**.** *Let Γ be a Polish group and X a Polish space. Suppose $\Gamma \curvearrowright X$ is a turbulent action, and let E^X_Γ be the associated orbit equivalence relation. Let L be a countable language, let X_L be the space of L-structures with domain $\mathbb{N}$, and let $f : X \to X_L$ be a Borel function such that if $x\, E^X_\Gamma\, y$ then $f(x) \sim f(y)$. Then f is not a classification of E^X_Γ by countable structures.*

Larson and Zapletal reproved Hjorth's Turbulence Theorem using forcing and absoluteness together with a new characterization of turbulence implying that if a group action $\Gamma \curvearrowright X$ is turbulent, and $\mathbb{P}_\Gamma$ is Cohen forcing on Γ and $\mathbb{P}_X$ is Cohen forcing on X, then $W[x]$ and $W[g\cdot x]$ are mutually generic (i.e. $W[x]\cap W[g\cdot x] = V$) [50, Theorem 3.2.2]. Their proof of Theorem 20 is notable for using two forcing extensions: It is shown that, since canonical Scott sentences are Σ_1-absolute, the interpretation of a particular canonical Scott sentence in $W[g \cdot x]$ and $W[x]$ agree, and hence (by the characterization) that interpretation exists already in W.[27]

Larson and Zapletal write that their work

> restates and greatly generalizes Hjorth's notion of turbulence in forcing terms. This development shows that nonturbulent equivalence relations are in fact parallel to pinned equivalence relations in a very precise sense. The forcing relation encapsulates many distracting estimates needed in the traditional treatment of turbulence, resulting in a clean and efficient general calculus. [50, p. 17]

Because it gives them new language with which to reformulate turbulence, the theory developed in [50] leads to arguably the "right" statement—in the sense that it is easily applicable in the wild—and proof of Hjorth's theorem. Moreover, their version generalizes to give a notion of turbulence for measure.

3.1.7. *Baire-1 functions.* The following theorems of Todorčević involve the class of Baire-1 functions. For the first, Todorčević proves an absoluteness lemma ensuring that a relatively compact subset of the Baire-1 functions over Baire space remains relatively compact in generic extensions. Using this lemma, Todorčević first gives a forcing-and-absoluteness proof of a theorem of Bourgain by forcing CH without adding reals.

Theorem 21 (Bourgain; Todorčević [78])**.** *Every compact subset of the first Baire class contains a dense set of G_δ-points.*

To establish the next theorem, Todorčević uses a forcing notion which Fremlin [29] showed exists under the assumption MA + ¬CH. This is an example which can be construed as using a forcing extension of a forcing extension; that is, two

[26] Hjorth's Turbulence Dichotomy Theorem says that in fact turbulence is the obstruction to being classifiable by countable structures.

[27] We found Marks' exposition [55] helpful in understanding the structure of the proof.

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hops (one to the MA + ¬CH world, another to the Fremlin extension) are taken in proving the theorem.

Theorem 22 (Bourgain; Todorčević [78]). *Every Radon measure on some compact set of Baire class-1 functions has separable L_1-space.*

Finally, Todorčević uses the same proof technique to solve the problem whether there exists a non-separable Rosenthal compactum satisfying the countable chain condition.

Theorem 23 (Todorčević [78]). *Every compact space that can be represented inside the class of Baire-1 functions on a Polish space has a dense metrizable subspace.*

The forcing notion Todorčević uses is the algebra of all regular open subsets of a given set of the first Baire class. In addition to the technical lemma, the proof uses Theorem 21.

3.1.8. *A K_4-free graph.* We close with a particularly nice example from graph theory which answers a question of Erdős and Hajnal. The graph K_4 is the complete graph with four vertices, and a graph is K_4-*free* if it does not contain K_4 as a subgraph.

Theorem 24 (Shelah). *There is a K_4-free graph (G, E) such that for every coloring $c : E \to 2$, there is a monochromatic triangle contained in G.*

Shelah forces the existence of an infinite graph with the monochromatic triangle coloring property for ω many colors. By compactness, this graph contains a finite subgraph with the triangle coloring property for 2 colors. Since the subgraph is a finite object, it exists in the ground model.[28] Thus a simple, finite object's existence is proved using complicated uncountable objects that are *prima facie* irrelevant.

4. Developing theories

The proofs in §3 demonstrate that theories which can be forced are interesting and useful, even if they are not true. Consequently, and this is perhaps the ultimate moral of the paper, *it is important to develop set theories.* Without Mokobodzki having developed the theory of ZFC + CH, would Kechris have proved his theorem, or Cieśla and Sabok theirs? Without the MA + ¬CH result of Fremlin, would Todorčević have proved Theorem 22? The theory of selective ultrafilters enables the methodology of forcing CH without adding reals to be useful for proving theorems

[28] Shelah uses forcing and absoluteness to show the following: (1) If all formulas $\varphi(\bar{x}, \bar{y}) \in L$, $|y| = 1$, have the NIP, then T has the NIP [65] and (2) any abstract elementary class admits ordered graph blueprints. These results are not "concrete" mathematics in the sense we intend, but they show the utility of the method. A direct proof of the first result by Laskowski led to the study of the Vapnik-Chervonenkis property, with applications to real semi-algebraic geometry [51].

With collaborators, Shelah has also used forcing and absoluteness at least twice more. There is a step in the proof of a theorem of Harrington-Marker-Shelah [41]—that if $\langle X, \leq \rangle$ is a thin Δ^1_1 order, then there is an $\alpha < \omega_1^{CK}$ and a Δ^1_1 order-preserving embedding $f : X \to 2^\alpha$—which uses forcing to add a perfect set of elements which are incomparable in the order, and pulls this fact back via absoluteness to derive a contradiction. Shelah and Gurevich use forcing and the absoluteness of a decision procedure to solve Rabin's uniformization problem for monadic second order logic negatively [35]. These proofs do not use a higher set theory, but rather force to bring useful affairs into focus. The Harrington-Marker-Shelah proof is an example of a pure(r) proof in which the generic multiverse is useful.

in analysis. These theories having been sufficiently developed made the set theorists' work easier.

Two natural theories recur in the above examples: ZFC + CH and ZFC + MA + ¬CH. This is likely because of how developed they have become through intense investigation into independence spanning decades, how familiar they are to set theorists, and the role of the continuum in many of the theorems. But the proofs of Theorems 18, 19, 22 and 23 show that it is useful to go to universes which one is, so to speak, the first to inhabit, ones whose theories have never been studied before.

This collection of proofs also suggests an explanation for why set theorists do not e.g. reject papers proving theorems using hypotheses they personally do not believe in. A theorem is a theorem, and a theorem proved from hypotheses one does not believe, or for which one might think it does not make sense to ask about their truth value at all, might still come in handy in proving something one *does* believe. This suggests in turn that, taken too literally, views like Strong Absolutism might inhibit the development of mathematics. If one rejected or tried to prevent research using a "false" theory, one could miss out on ZFC theorems. Of course, no set theorist holds such an extreme position. No one was claiming that accepting, e.g. $V =$ Ultimate L, requires foregoing forcing-and-absoluteness as a proof technique or studying forcing extensions. But this fact seems to get lost in the debate between pluralists and non-pluralists. The staunchest nonpluralist does not dismiss research done using hypotheses they don't believe hold in V. We learn a lot about how things are by studying the ways things could be. Alternative theories can carry instrumental value for proving theorems even in the nonpluralist's theory.[29]

4.1. **Absoluteness theorems and other multiverse conceptions.** There are more inclusive superstructures than the generic multiverse in the literature. Hamkins describes some possibilities:

> The generic multiverse [of a world W] can be viewed as a small part, a local neighborhood, of any of the much larger collections of models that express fuller multiverse conceptions. For example, one could look at the class-forcing multiverse, arising by closing W under class forcing extensions and grounds, or the pseudo-ground multiverse, obtained by closing under pseudo-grounds, or the multiverse arising by closing under arbitrary extensions and inner models, and so on [39, p. 1].

Just as the generic multiverse formalizes an intrinsic feature of the ground universe—namely what can be forced over it with set-sized forcings—these other

[29]Number theorists study the way things could be by proving theorems conditioned on the Riemann Hypothesis ("If the RH is true, then...") and theorems conditioned on its negation. The phenomenon I am describing seems to differ from this practice in ways having to do with truth and belief. Since the RH is expected to be solvable in ZFC, theorems conditioned on RH are relevant to all number theorists. They bear on solving the RH, and once the RH is solved, some of them will become non-hypothetical. (See [11] for more discussion of this circumstance.) A rational basis for belief in the RH is evidence as to its provability in our common mathematical framework.

There is of course no such evidence for independent-from-ZFC propositions; the basis for belief in their truth lies elsewhere. Using such propositions goes beyond our common framework, changing the rules of the game, so to speak. In forcing-and-absoluteness proofs, they serve as tools for unconditional ZFC theorems. On the face of it, it is surprising that theorems proved using tools like MA + ¬CH are relevant to all who use the standard axioms.

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multiverse conceptions strive to articulate e.g. what can be class-forced over the ground universe. A class-forcing and absoluteness proof of a ZFC theorem about concrete mathematics would be of great interest. Perhaps there will be such a proof using coding into the continuum function, or making use of the Ground Axiom.[30] Theories of class forcing extensions could conceivably contribute to our understanding of V similarly to those of set forcing extensions.[31] So developing these broader multiverse conceptions may be similarly important for multiverse proponents and nonpluralists alike.

Question 25. *Is there a useful formal theory of a class generic multiverse?*[32]

To get such proofs as those just speculated—and further generic multiverse proofs—to work, generic absoluteness theorems tailored to particular extensions, like in the proofs of Lemma 10 and Theorem 21, will presumably be required. Generic absoluteness is intrinsically interesting, but the potential for forcing-and-absoluteness proofs provides practical impetus for studying aspects of it that would likely otherwise go unresearched.

4.2. **Extent.** Of course, most of the theorems surveyed were proved prior to the generic multiverse being formalized in [74], so saying that the generic multiverse is brought to bear in the proofs is admittedly a charitable construal. Likewise, the extent to which the multiverse proofs show that the whole generic multiverse is not going away is a matter of perspective. It seems enough to know that some forcing extensions are useful in proving theorems in concrete mathematics, a significant part of mathematics accepted by the weak relativist, the weak absolutist, and the strong absolutist. We cannot predict exactly which extensions will prove useful in future proofs, and it seems arbitrary to argue that multiverse proofs only show the utility of a mere "submultiverse" consisting of the specific generic extensions used in §3.1, or e.g. the c.c.c. multiverse.

There is a lacuna: The amalgamability asserted in condition (4) of Definition 2 has not factored into the proofs surveyed. A multiverse proof using amalgamation would be of great interest, as Woodin's generic multiverse does not satisfy amalgamability and [53] questions whether there is a principled reason to accept it.[33]

---

[30]The statement that the universe is not a (non-trivial) set-generic extension of an inner model.

[31]Not every multiverse conception promises to bear on V. Closing the multiverse under ultrapowers will not help prove theorems about concrete mathematics, since an ultrapower is elementarily equivalent to the structure which is ultrapowered. It does not get you any new theory to work with.

[32]This seems like a hard problem. As Steel writes, "...our multiverse does not include class-generic extensions of the worlds. There seems to be no way to do this without losing track of the information in what we are now regarding as the multiverse, no expanded multiverse whose theory might serve as a foundation. [Footnote: Amalgamation will fail if we start counting sets, definable inner models, or class-generic extensions as worlds.] We seem to lose interpretative power" [74, p. 167]. Indeed, some natural "submultiverses" of the class-generic multiverse fail to include definable inner models with Woodin cardinals.

I use the indefinite article in Question 25 because it seems that restriction to certain class forcings is necessary.

[33]Amalgamability is needed to give a formal axiomatization of Steel's version of the generic multiverse and to ensure the existence of the translation function central to the discussion of the generic multiverse in [74]. Woodin's multiverse is not formally axiomatizable.

4.3. **The content of the Absolutist-Relativist debate.** The generic multiverse must be kept distinct from the views about it like Weak Relativism or the Generic Multiverse conception of truth.[34] The former is an object, the object given by the multiverse language and axioms. It is a category mistake to assent to or argue against an object as opposed to a claim.[35] The Absolutist is committed to the generic multiverse just as the Relativist is, since the first order theory of the forcing extensions of V is part of what holds in V, and this theory is what the generic multiverse theory captures. Forcing-and-absoluteness proofs indicate that there is practical benefit to studying this theory: One can use worlds in the generic multiverse to prove ZFC theorems about concrete objects relevant to wider mathematics, like definable equivalence relations.

The proofs embody the idea that, for proving ZFC theorems, any universe will do. This is a practical maxim in the neighborhood of the Weak Relativist's idea that all worlds in the multiverse are on equal footing, although it is weaker. The proofs do not substantiate other tenets of Weak Relativism or Generic Multiversism, namely that forcing extensions "actually" exist as full-blooded universes (whatever that is taken to mean), or that the only meaningful sentence in the language of set theory are those that hold in all universes. (This is part of why they are *proofs*. Their validity does not depend on metaphysics. Forcing allows the simulation of the universe's forcing extensions within the universe, and its legitimacy as a proof technique turns on being able to treat forcing extensions entirely virtually.) In these "best case" uses of the generic multiverse conception, the Absolutist is not compelled to see the generic multiverse as anything more than the behavior of the forcing relation, and the Absolutist sees the auxiliary theories used, which are indeterminate for the Weak Relativist, as meaningful.

So Weak Relativist and Strong Absolutist alike are committed to the object, but they disagree about its significance. On my reading, Steel put forth the Weak Absolutist Thesis as a way of making this disagreement mathematically tractable in such a way that a confirmation of it would bring the parties closer together. (A perhaps silly way of putting this reading is that if the Weak Absolutist Thesis is true, then we should all just be Weak Absolutists because the content of Strong Absolutism and Weak Relativism that goes beyond the Weak Absolutist Thesis does not seem to have strictly mathematical content or justification.) Usuba's theorem confirms the Thesis, assuming an extendible cardinal. This is satisfying to the Absolutist in inverse proportion to how disappointing it would have been for there to provably not be a core. The Strong Absolutist may believe the core is V. But the theorem by itself does not refute the rival philosophical position. Weak Relativism is not the negation of Weak Absolutism.

Usuba's theorem does, however, put the Weak Relativist on their back foot. They have to give a philosophically coherent answer to (something like) Question 26.

Question 26. *Granting large cardinals, does there being a uniquely definable universe in the generic multiverse undermine the Weak Relativist tenet that all universes in the generic multiverse are on an equal footing? That the only meaningful statements in the language of set theory are those that are generically invariant?*

[34] [58] conflates the two.
[35] Perhaps this is an easier way to argue for the titular claim!

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For the Weak Relativist, the core is at minimum a universe with an interesting metamathematical property. Simply saying that that's all there is to it, that all universes nonetheless remain on an equal footing, seems unsatisfactory as a response to Question 26. That large cardinals imply exactly one world in the multiverse enjoys a property—let alone the property of being the world from which all others arise by generic extension and refinement—cannot be hand waved away. A complete response has to make explicit what it means when the Weak Relativist claims all universes are on an equal footing and how that equal footing is compatible with there being a uniquely definable universe in the multiverse.[36]

Apart from Question 26, or the significance of the core generally, what is there to give traction to the disagreement between the Weak Relativist and the Strong Absolutist? Each view can interpret all of the relevant mathematics, at least if Question 26 has an answer. So insofar as Absolutism and Relativism countenance the same proof techniques (like forcing-and-absoluteness) and are committed to the generic multiverse, and assuming the Weak Relativist has a plausible answer to Question 26, *is there content to the Relativist-Absolutist debate?* If so, can that content be made mathematically tractable? If either of these questions has a negative answer, it seems to me that Weak Absolutism represents the limits of what can be said with mathematical justification on the topic.

5. Purity of proof

In the introduction to the paper "Lusin's restricted continuum problem," the only of the forcing-and-absoluteness papers discussed which was published in a premier non-specialist mathematics journal, Stern assures the general mathematical audience that the technique is rigorous:

> It should be clear from the introduction that the present paper is of metamathematical character... We also use throughout the paper Cohen's method of forcing, not only when generic subsets over inner models are known to exist but also when we form generic extensions of the universe of sets to derive various properties of the forcing relation. This is known to be a valid method of proof. [76, p. 10]

The proof technique is so *impure* that a nonexpert could wonder whether it is fallacious more than 20 years after the discovery of forcing.

There is a serious objection to this paper's titular claim arising from considerations of purity of proof. To state it, we must first clarify the sense in which forcing-and-absoluteness proofs are impure.

Philosophical accounts of purity of method seek to identify what purity consists of in practice and how it has historically constrained mathematicians' sense of what the "right" proof of a theorem is. We will draw on Arana's account:

> A purity constraint, restricting proofs of theorems to what is "close" or "intrinsic" to that theorem, requires an account of how the distance between proof and theorem is to be measured. Two such

[36]Do more expansive multiverse views have to answer Question 26? If (1) the generic multiverse is the only philosophically appealing multiverse conception which is formalizable, and (2) on that conception (and assuming there is an extendible cardinal), the multiverse has a core, then a more general Relativist position may stand or fall with the Weak Relativist's answer to Question 26. Consider this more motivation for formalizing more expansive multiverse views.

> measures of distance are what we have called "elemental" and "topical" distance. A proof is *elementally close* to a theorem if the proof draws only on what is more elementary or simpler than the theorem. A proof is *topically close* to a theorem if the proof draws only on what belongs to the content of the theorem, or what we have called the *topic* of the theorem. Each of these distance metrics induces a purity constraint, viz. elemental purity and topical purity.[3]

The proofs in §3 are paradigmatically impure elementally (the forcing constructions are like using a nuclear bomb as a flyswatter) and topically (none of the theorems in §3 talk about generic extension or e.g. CH). In fact, I claim that given a purity metric, if they do not qualify as impure, the metric is defective.[37] The proofs should be as far from the theorems they establish as is allowable by the metric. The reason is that they appeal essentially to content expressed in independent-from-ZFC-propositions: They answer whether $\mathrm{ZFC} \vdash \varphi$ by showing $\mathrm{ZFC} + A \vdash \varphi$, where A is independent of ZFC. This is unlike the standard examples of impure proof, which tend to involve "detours" via higher type objects in the same universe.[38] In contrast, the detours in forcing-and-absoluteness proofs involve new domains of mathematical discourse.

Not only is there a sense that this content must be irrelevant to the conclusion proved, in many cases it is formally provable that there is a ZFC proof of the conclusion which eliminates A. So A is provably extraneous.[39] One way of bringing out the proofs' impurity is to consider that an agent can consistently maintain that the A appealed to is in fact false without holding a "limitative" position in the philosophy of mathematics.[40] For such an agent, these proofs derive true conclusions from false premises. The knowledge gained from forcing-and-absoluteness proofs is highly *unstable* in the sense of [5].[41] Due to their logic, the proofs work whether or

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[37] Arana has considered other natural ways of spelling purity out, and again the proofs in §3 are impure according to them. A proof is *syntactically* pure if all formulas appearing in it are subformulas of its conclusion. This is motivated by Gentzen's Cut Elimination Theorem. [1] argues that this notion is unsatisfactory on many grounds, starting with the fact that Cut Elimination fails for formal theories of arithmetic. There are two notions from [2]: First, a proof is *logically* pure if it uses the minimally sufficient set of axioms required to prove the theorem. Second, a proof is *semantically* pure if it draws only on what must be understood in order understand the theorem (this is a prototype of topical purity).

[38] Feferman observes:

> Abstraction and generalization are constantly pursued [in mathematics] as the means to reach really satisfactory explanations which account for scattered individual results. In particular, extensive developments in algebra and analysis seem necessary to give us real insight into the behavior of natural numbers. Thus we are able to realize certain results, whose instances can be finitistically checked, only by a detour via objects (such as ideals, analytic functions) which are much more "abstract" than those with which we are finally concerned [25, p. 3]

[39] A formal characterization of when a hypothesis or inference is extraneous has not been obtained in generality. [1] shows that using cut elimination in Gentzen proof systems as a criterion does not work, and casts doubt on purity being a syntactic notion.

[40] Where "limitative" means "requires truncating the interpretability hierarchy below some point widely accepted by the mathematical community."

[41] "The epistemic significance of topical purity derives from the stability it brings to problem solutions. Every topically pure solution $\mathcal{E}$ to a problem $\mathcal{P}$ is stable in the sense that were α to retract a premise or inference from $\mathcal{E}$, the content of $\mathcal{P}$ would change for her. In other words,

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not one accepts all of the auxiliary hypotheses that function in the argument. One only has to recognize that the theories of the forcing extensions are consistent.[42]

Purity of proof involves a contrast between proofs, which implies that there are distinct proofs of the same theorem. When is a proof of a known theorem genuinely new? Dawson [17] describes the problem of formally differentiating between different proofs conceptually. In brief, formal characterizations of proof differentiation face challenges arising from what gets washed away by formalizing informal arguments. But there certainly seems to be something mathematicians are talking about when they talk about an argument using new ideas, or when claiming that an argument is "really" the same as another.[43] The forcing-and-absoluteness proofs are clearly conceptually distinct from the direct proofs of the same theorems.

Finally, many of the results in §3, or the work they figure into, are central to the fields they arise from. Silver's theorem is foundational in the study of equivalence relations, and whether the dichotomy it expresses holds for orbit equivalence relations of Polish group actions is the *Topological Vaught Conjecture* (Burgess' Theorem 7 is part of this story as well). Kechris' work introduced the concept of amenable equivalence relation, which has since been studied in the theory of countable Borel equivalence relations. Todorčević's work is fundamental to the theory of Rosenthal compacta. Theorems 6, 35, 8, and 24 answered questions raised in the literature. These results are representatives of sometimes disparate fields. One property they have in common is that the disparate fields do not usually depend on higher set theory.

To summarize, the proofs from §3 are striking examples of impurity because 1) they are of sometimes fundamental results upon which much theory has been built, 2) it is uncontroversial that they are conceptually distinct from their pure counterparts, 3) they use provably extraneous hypotheses, and 4) they are sufficiently impure that any purity metric which does not register them as such is inadequate.

Miller's program. Bolzano's efforts to remove geometric intuition from proofs in analysis constituted a research program explicitly aimed at achieving topical purity. There is an analogous program to find elementally pure proofs of some of the theorems in §3: B. Miller [59] pursues classical, forcing-free proofs of classical descriptive set theoretic dichotomies like Silver's and Burgess' theorems using chromatic numbers. Miller's stated motivation is that techniques like Gandy-Harrington forcing—and certainly forcing-and-absoluteness—should be unnecessary to establish these dichotomies. After all, Cantor, Hausdorff, and Suslin proved the perfect set property for closed, Borel, and analytic sets with elementary techniques. Whether Silver's Theorem is true could have been asked in Cantor's time, whereas forcing constituted a paradigm shift in set theory and emerged over half a century later. Neither Silver's nor Harrington's proof is "classical" in the sense that it is a proof that Cantor could have arrived at. While Harrington's eliminates passing to a forcing extension and does not need GCH, there's a sense that *any* forcing should

her retraction would contentually dissolve $\mathcal{P}$ for her (i.e. $\mathcal{E}$ would be cofinal with $\mathcal{P}$ for her). By contrast, if $\mathcal{E}$ were a topically impure solution to $\mathcal{P}$, there would be premises or inferences in $\mathcal{E}$ that the investigator could retract without contentually dissolving $\mathcal{P}$. In that case, $\mathcal{E}$ would not be stable in the aforementioned sense" [5, p. 13].

[42]And in the examples we've discussed, denying the consistency of those theories will induce a change in the content of the theorem, as the theories are usually equiconsistent with ZFC.

[43][69] bears this out, identifying a fundamental idea common to the over 40 proofs of the Cantor-Bernstein theorem in [42] and concluding that there is ultimately just *one* proof in different guises.

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be unnecessary for establishing ZFC theorems about concrete sets of reals. Miller describes the situation:

> The proofs of the results established in the first half-century after Souslin's theorem followed the same basic outline. Using the tree structure afforded by analyticity, one defines an appropriately chosen Cantor-Bendixson-style derivative, reducing the problem to a combinatorially and topologically simple special case with a straightforward solution.
>
> This basic outline was cast aside, however, beginning with Silver's generalization of the perfect set theorem. His proof was a technical tour de force, relying on sophisticated techniques from mathematical logic in addition to a much larger fragment of the standard set-theoretic axioms than typical. Although Harrington later found a simpler proof, his argument still relied on a detailed recursion-theoretic analysis of the real numbers, as well as the method of forcing distilled from Cohen's proof of the independence of the Continuum Hypothesis.
>
> Over the next thirty years, Harrington's techniques unearthed an astonishing number of structural properties of Borel sets. While the proofs of many results closely mirrored that of Silver's theorem, others used progressively more elaborate refinements of Harrington's ideas. [59, p. 555]

The epistemic payoffs of Miller's program corroborate those which are thought to be associated with purity: "In addition to eliminating the need for sophisticated machinery from mathematical logic, this approach illuminates new connections between seemingly unrelated theorems, leads to a global view of dichotomy theorems from which new results emerge, and readily generalizes to broader classes of definable sets" [59, p. 555].

The objection. The objection to our main claim draws on two facts. First, theorems like Platek's Theorem 3 tell us that if forcing-and-absoluteness can be used to prove a theorem of a certain logical complexity from GCH, then there is a proof of that theorem which does not use GCH. GCH is provably extraneous to the theorem. Interpolation theorems suggest that there should be proofs which avoid forcing extensions. Second, the mathematical community esteems pure proofs, even when they are not the first proof of a theorem. Finding a pure or explanatory proof, or one that eliminates extraneous or controversial hypotheses, is one of the primary reasons that mathematicians re-prove theorems [17, 18]. Forcing-and-absoluteness proofs could not be more prime candidates for re-proof.

Together, these facts suggest that the generic multiverse *is* eliminable because set theorists will seek direct ZFC proofs which use resources local to the theorems they establish. Miller's program §5 exemplifies this tendency.

5.1. **Explanation.** While hops in the generic multiverse are ultimately unnecessary to prove some of these results, one wonders whether and when the "pure" proofs would have been discovered without their impure counterparts to guide them. Todorčević, in regards to his theorems in §3.1.7, describes his use of forcing as "the guiding force behind the discovery of these results" [78, p. 1183]. These

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proofs show that extreme impurity can be fruitful in the context of discovery. That Harrington and Shelah have employed the technique so many times attests to this.

Aristotle's distinction between deduction and demonstration—and the distinctions of Leibniz between sequences of discoveries and orderings of truths and of Bolzano between *Gewissmachung* and *objektive Begründung*—track the contexts of discovery and justification. For Aristotle, a *deduction* is simply a valid argument, whereas a *demonstration* is a deduction which is in addition explanatory in the sense that it reveals the true grounds for a claim. More broadly, an explanatory proof of a theorem is an argument that gives the reason *why* the theorem is true as opposed to merely showing *that* the theorem is true. Deductions—which include impure proofs—are permissible, for Aristotle, in the context of discovery, but demonstrations are required in the context of justification. The proofs discussed in §3 seem to fall in the former category, as one might wonder *why* the result is true even after digesting its proof, and the subsequent forcing-free proofs function as demonstrations.

Harrington's proof of the Halpern-Läuchli theorem clashes with the Aristotelian account in a strong sense. It is impure and metamathematical. It was not the first proof of the result. Yet practitioners report that Harrington's proof provides the true grounds of the theorem. It is the proof that allows the generalization of the theorem to higher cardinals. Todorčević's proofs give, at least to him, a proper understanding of the results they establish. Similar for Gao et al.'s proofs. The Larson-Zapletal proof of Hjorth's turbulence theorem arguably give the proper understanding of what the theorem is "really" saying, what its topic is. These are examples of impure proofs which function in the context of justification better than their pure counterparts—the impure proof is the demonstration, the pure proof a deduction.

There is a philosophical literature on explanatory proof (see e.g., [75, 7, 8, 62, 48]), and how purity of method and explanation relate [48, 4, 64], much of which seeks to identify just what it is that makes a proof explanatory. To be clear, I am not sure of the significance of the distinction between explanatory proofs and those which are supposedly not. A proof is why a theorem is true. Even the most unwieldy, computer-assisted enumerative proof contains why the claim is shows is true. It is immaterial if all of its cases cannot be grasped all at once by a human mathematician. Dissatisfaction with a particular proof need not reflect epistemic virtues inherent to the proof but rather one's understanding of it. Explanations are answers to "Why?" questions, after all, and the kinds of "Why?" questions one can ask after being presented a proof reflect a lack of comprehension, not shortcomings of the proof. Whether one gains understanding from this proof or that proof is too autobiographical, too much a matter of an individual's psychology, to be about the proofs themselves.

By saying that some forcing and absoluteness proofs are explanatory, I do not mean that they fall under any of the philosophical rubrics of explanatory proof. (I think they do not, and I think no analysis of proofs will be successful in characterizing explanatory proofs.) I just mean that among mathematicians in the relevant research area, the proofs seem to be the "right" ones, probably by virtue of exhibiting informative connections with other areas of mathematics or casting the problem in a domain familiar to those mathematicians and in which their methods are applicable. There is nothing wrong with the non-forcing and absoluteness proofs, nor anything philosophically deep about the judgments that the forcing and absoluteness proofs are "right."

5.2. **Essential uses.** I read Gao et al's inclusion of the purely topological proof of Theorem 18 as a confirmation of the theorem justifying further use of the forcing-and-absoluteness method in their paper. Together with their observation that from the forcing-and-absoluteness proof of Theorem 18 further corollaries can be drawn, this suggests that the authors see their proofs as being the "right" ones of their theorems. Another reason is that the limitations of measure techniques for establishing Borel nonreducibilities and the fruitfulness of forcing-and-absoluteness suggest to Gao et al. that the latter method may become essential:

> It is our opinion that this is only the beginning of nontrivial results about countable Borel equivalence relations that can be proved using forcing. It is curious to note the tension that the first five "positive" results all state the existence of points with certain regularity properties, whereas the last three "negative" results state the nonexistence of regular structure on orbits. Of course, the positive results are all obtained by generically building such elements in the generic extension (and then asserting their existence in the ground model by absoluteness), and it is known that the results do not hold for comeager or conull sets of reals. Thus what we are using is some method that goes beyond the usual measure and category arguments. [33, p. 4]

They predict that forcing proofs will be crucial to progress in definable equivalence relations theory:

> One of the central notions of the theory of countable Borel equivalence relations is that of Borel reducibility... All known methods to prove nonreducibility results for countable Borel equivalence relations have been measure-theoretic (it is well known that category arguments would not work). But measure-theoretic arguments have their limitations. There have been persistent attempts by researchers to invent new methods that are not measure-theoretic. For instance, recent work of S. Thomas and A. Marks explore the use of Martin's ultrafilter and its generalizations as a largeness notion... The forcing methods presented in this paper can also be viewed as an attempt in this direction. [33, p. 4]

Similarly, Larson and Zapletal's forcing formulation of turbulence provides methodological advantages over Hjorth's original. Their proof of Hjorth's turbulence theorem is an example of a proof which arguably gives insight into the content of the theorem that is not revealed by the original proof. Forcing-and-absoluteness may be a necessary tool in practice.

At least for human mathematicians. These forcing and absoluteness proofs are natural for set theorists to find. They cast the problems in language the set theorist is at home in, and human mathematicians—as opposed to computers—need heuristics, intuitions, and creative insight to guide their proof searches. If carried out by human mathematicians, then, developing theories requires forcing and absoluteness proofs. But if the task of completing theories were left to computers programmed to find the proofs guaranteed to exist by e.g. Platek's theorem and interpolation theorems, and if human understanding were no longer as central a goal of the pursuit of mathematics, forcing and absoluteness could disappear from deductive mathematical activity.[44] What about non-deductive mathematical activity? If forcing and absoluteness, as a technique, does not lead set theorists to useful new axioms, will it remain a useful tool?

Question 27. *Is there a scenario in which forcing and absoluteness could be used to find new axioms?*

5.3. **An inversion.** Arana and Detlefsen sought to formally characterize "a widespread, persistent tendency to think that there are ways in which the resources used in a proof/solution can match or fail to match the theorem proved/problem solved, and that when proper match is achieved it in some way(s) adds to the value of the proof/solution produced" [5, p. 1]. Kreisel was suspicious of the value of this tendency.

> Defects of ideals are generally seen most clearly in areas where they have been realized, and so the results can be compared both with earlier expectations and with alternatives (which violate the ideal in question). In the cases under discussion, algebraic and number-theoretic purity, plenty of comparisons are available since, with time, impure proofs have become more common in practice, not less. Moreover—and this is often neglected—(i) their actual reliability or 'security' is obviously unaffected by the possibility of pure proofs if that possibility has not been realized, and (ii) impure methods are not only used heuristically, for discovering conjectures and proofs, but have turned out to be essential for checking proofs. . . . But also there is the void created by simply not saying out loud what (knowledge) is gained by impure proofs, for example by analytic proofs in number theory: knowledge of *relations between the natural numbers and the complex plane* or, more fully, between arithmetic and geometric properties. It is precisely this knowledge which provides effective new means of checking proofs: if this conflicts with some ideal of rigour, so much the worse for the ideal (which is being tested). [47, p. 167]

With regards to forcing-and-absoluteness proofs, the rigor of the method and the fact that many of the theorems in §3 have *not* been purely reproved bear out (i). The potential Gao et al. see (§5.2) for the method to establish Borel nonreducibilities would bear out (ii). Through the proofs we gain knowledge of how the structure of V and the structure of its forcing extensions relate and bear on the concrete. These relations are variegated and informative, and they give us *confirmation* of the coherence of set theory in a sense reminiscent of the following passage by Dawson:

> . . . the existence of multiple proofs of theorems serves an overarching purpose that is often overlooked, one that is analogous to the role of confirmation in the natural sciences. For just as agreement among the results of different experiments heightens credence in scientific hypotheses (and so also in the larger theories within which

[44] Although, as a reviewer speculates, perhaps the forcing and absoluteness heuristic could lead to more efficient proof search algorithms.

> those hypotheses are framed), different proofs of theorems bolster confidence not only in the particular results that are proved, but in the overall structure and coherence of mathematics itself. To paraphrase a remark C. S. Peirce once made with regard to philosophy, trust in mathematical results is based rather on 'the multitude and variety' of the deductions that lead to them than on 'the conclusiveness of any one' of those deductions. Mathematical reasoning forms not 'a chain, which is no stronger than its weakest link, but a cable', whose fibers, though 'ever so slender', are 'numerous and intimately connected'. [17, p. 281]

That we can bring forcing extensions to bear on concrete problems in number theory, analysis, or computability theory shouldn't be possible, the thought goes, unless the generic multiverse and set theory itself are coherent.[45]

Confirmations of set theory's coherence are ultimately why the generic multiverse will not go away. Pure reproofs of the theorems in §3 will not have the same confidence-bolstering effects. They will not reveal far-reaching connections between generic extensions. Miller's program has mathematical benefits that justify its pursuit—any directive that leads to finding more proofs is worthwhile—but confirmations of the coherence of set theory are not among them. Following Kreisel, if an ideal of purity precludes gaining knowledge of how generic extensions bear on one another in productive ways, so much for that ideal—which is not to say that seeking pure proofs should be discouraged. There are many different values a proof can have. The variety of proofs of a theorem, with all the different values they enjoy—like elemental purity or coherence-confirmation—collectively contribute to our understanding of set theory.[46]

5.4. **Conclusion.** The generic multiverse is not going away, then, because—through the forcing-and-absoluteness method—it aids in proof discovery; some of the impure proofs discovered are the explanatory or "right" proofs of the theorem they establish; and the method may be essential for certain purposes, like proving Borel nonreducibilities. (Should the method somehow lead to new axioms, so much the better.) Forcing-and-absoluteness proofs give a non-local kind of knowledge about—a confirmation of—the coherence of set theory, knowledge which cannot be gotten from pure, direct proofs of the same theorems. Bringing such machinery to bear on theorems about concrete mathematics shouldn't be possible unless we are onto something.

The proofs put a weak version of the central claim of generic multiversism—that every extension of set theory is as good as any other—to work in a way which is relevant and useful to Absolutists. The proofs have to be rigorously accounted for, and a rigorous setting for studying what forcing extensions V has is needed. The generic multiverse theory does this. Jettisoning the generic multiverse is too strong a reaction to Usuba's theorem.

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[45] I don't think coherence-confirmation is limited to proofs about concrete objects. Silver's proof that the Singular Cardinals Hypothesis cannot first fail at a singular cardinal of uncountable cofinality is no less coherence-confirming. The focus on the concrete in this paper is to emphasize connections with other areas of mathematics than set theory.

[46] I think this is a virtue of intramathematical applications generally—they contribute to our understanding of mathematics and how its disparate areas hang together.

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Appendix A. Forcing and absoluteness proofs

Automorphisms of the Turing degrees. In unpublished work combining recursion theory, forcing, and absoluteness, Slaman and Woodin have shown that any nontrivial automorphism of the Turing degrees $\mathcal{D}$ is definable. The following theorem is a consequence.

Theorem 28 (Slaman-Woodin)**.** *Let $\mathcal{I}$ be a Turing ideal containing $\mathbf{0}'$ and suppose $\rho : \mathcal{I} \to \mathcal{I}$ is a persistent automorphism.*[47] *Then ρ can be extended to a global automorphism $\pi : \mathcal{D} \to \mathcal{D}$.*

Slaman and Woodin show that an automorphism being persistent is absolute to forcing extensions. So after forcing to make $\mathcal{D}$ (and hence $\mathcal{I}$) countable, ρ is persistent in the extension $W[g]$. They show that ρ can be extended to any countable ideal extending $\mathcal{I}$, so it suffices to find in $W[g]$ an automorphism of $\mathcal{D}^V$ which is persistent and agrees with ρ on $\mathcal{I}^V$. Being persistent, that automorphism actually exists in $L(\mathbb{R}^V)$, hence in W.

As a corollary of Theorem 28, the statement "There is a nontrivial automorphism of the Turing degrees" is Σ^1_2, so forcing and absoluteness may provide the means to settle the question whether there is one.

Selective ultrafilters and analysis. CH implies structure useful for proving theorems in analysis, namely selective ultrafilters. [24] describes a general methodology: Force CH without adding reals, apply a selective ultrafilter to derive the theorem, and conclude that the theorem holds because the theorem is about reals and the forcing did not add any. Here's an example.

Theorem 29 ([24, Theorem 6.9])**.** *For every sequence $\{f_n\}$ of continuous functions from $[0,1]^\omega$ into $[0,1]$, there is a subsequence $\{g_n\}$ of $\{f_n\}$ and a sequence $\{P_n\}$ of perfect subsets of $[0,1]$ such that $\{g_n\}$ monotonically converges to a continuous function on $\prod_{i<\omega} P_i$.*

The proof uses Sacks forcing and the following equivalent of a version of the Halpern-Läuchli Theorem as an "upwards absoluteness" principle.

Theorem 30 ([24, Theorem 6.8])**.** *If U is a selective ultrafilter, then U generates a selective ultrafilter U^* in the ω-product Sacks forcing extension.*

Ergodicity under Martin's conjecture. Slaman and Steel asked the question Kechris answered in the context of their work on Martin's conjecture. Thomas has investigated the effects Martin's conjecture has on the structure of countable Borel equivalence relations. Many naturally occurring countable Borel equivalence relations are

[47] Intuitively, a persistent automorphism of a countable ideal $\mathcal{I}$ is one that can be extended to an automorphism of any countable $\mathcal{J} \supseteq \mathcal{I}$ which contains whatever Turing degree one wants.

weakly universal,[48] but it is unknown in general how extensive this phenomenon is. Martin's conjecture implies it is pervasive.

Theorem 31 (Thomas [77])**.** *Martin's conjecture implies that there are uncountably many weakly universal countable Borel equivalence relations up to Borel bireducibility.*

Thomas takes a Borel family of finitely generated groups $\{G_\alpha : \alpha \in 2^\omega\}$ which do not embed into one another and which do not have non-trivial finite normal subgroups,[49] and he takes the products $E_\alpha \times \equiv_T$ of their orbit equivalence relations E_α on the free part of the shift action of G_α with Turing equivalence. Each of the $E_\alpha \times \equiv_T$ is weakly universal. It suffices to show that none of them are Borel reducible to each other. The proof of this uses $\mathrm{MA} + \neg\mathrm{CH}$. For the absoluteness step, Thomas shows that Martin's conjecture is equivalent to a Π^1_2 statement and holds in the extension. The statement that the product relations $\equiv_T \times E_\alpha$ and $\equiv_T \times E_\beta$ are not Borel reducible to each other when $\alpha \neq \beta$ is Π^1_2 in α and β. Since this statement holds in the extension satisfying $\mathrm{MA} + \neg\mathrm{CH}$, it holds in the ground universe.

Analytic equivalence relations. Let $\cong_p$ be the isomorphism relation on countable abelian p-groups, and let $\equiv_{TA}$ be the bi-embeddability relation on countable torsion abelian groups.

Theorem 32 (Calderoni-Thomas [14])**.** $\cong_p$ *is not Borel reducible to* $\equiv_{TA}$.

Calderoni and Thomas use *pinned names*. They show that if an analytic equivalence relation E is Borel reducible to an analytic equivalence relation F, then $\lambda_{\mathbb{P}}(E) \leq \lambda_{\mathbb{P}}(F)$.[50] They calculate that $\lambda_{\mathbb{P}}(\cong_p) = 2^{\omega_1}$ and $\lambda_{\mathbb{P}}(\equiv_{TA}) = \omega_2^\omega$. Now suppose Theorem 32 were false and step into a forcing extension $V[g]$ in which $2^{\omega_1} > \omega_2^\omega$. By Theorem 2, the reduction lifts to a reduction from $(\cong_p)^{V[g]}$ to $(\equiv_{TA})^{V[g]}$. Then $\lambda_{\mathbb{P}}(\cong_p) = 2^{\omega_1} > \lambda_{\mathbb{P}}(\equiv_{TA})$, which is a contradiction.

Calderoni and Thomas also obtain a strong version of Theorem 32 which shows the utility of large cardinals in forcing-and-absoluteness proof.

Theorem 33 (Calderoni-Thomas [14])**.** $\cong_p$ *is not* Δ^1_2 *reducible to* $\equiv_{TA}$.

The proof structure is virtually the same except that, first, a Ramsey cardinal is used to get that if an analytic equivalence relation E is Δ^1_2 reducible to analytic equivalence relation F, then $\lambda_{\mathbb{P}}(E) \leq \lambda_{\mathbb{P}}(F)$. And, second, Σ^1_3-absoluteness as in [57, §5] is appealed to.

Subspaces of separable Banach spaces. There is a research program [34] aiming to classify Banach spaces up to isomorphism or, in light of the complexity of the class of Banach spaces,[51] up to their subspaces. The following theorem characterizes how many subspaces a *separable* Banach space contains up to isomorphism.

[48] Let E, F be countable Borel equivalence relations on Polish spaces X, Y, respectively. A *Borel reduction* of E to F is a Borel homomorphism $f : X \to Y$ such that xEy if and only if $f(x)Ff(y)$. A countable Borel equivalence relation E is *weakly universal* if for any countable Borel equivalence relation F there is a countable-to-one Borel homomorphism of F to E.

[49] Finite subgroups which are invariant under conjugation.

[50] If E is an equivalence relation and $\mathbb{P}$ is a forcing notion, $\lambda_{\mathbb{P}}(E)$ is the number of E-pinned $\mathbb{P}$-names up to E-equivalence.

[51] In [27] Ferenczi, Louveau, and Rosendal show that isomorphism between separable Banach spaces is a complete analytic equivalence relation.

Theorem 34 (Ferenczi-Rosendal [26])**.** *If X is a separable Banach space, then X contains either*

(1) *a perfect set of non-isomorphic subspaces, or*
(2) *a block-minimal subspace with an unconditional basis.*

Ferenczi and Rosendal show that the statement is Σ^1_2 via an ingenious coding, show that it holds in an extension satisfying $\mathrm{MA} + \neg\mathrm{CH}$, and apply Theorem 2.

The Baumgartner-Hajnal theorem. The following theorem of Baumgartner and Hajnal answered a question of Erdős and Hajnal [23]: Is it true that $\omega_1 \to (\alpha)^2_k$ for all finite k?[52]

Theorem 35 (Baumgartner-Hajnal [10])**.** *$\theta \to (\omega)^1_\omega$ implies $\theta \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and finite k.*

The higher set theory appealed to in the proof is Martin's Axiom, which is used to prove the following lemma.

Lemma 36. *Let θ be an ordinal such that $\theta \to (\omega)^1_\omega$. Let $|\theta| = \kappa$ and MA_κ holds. Then $\theta \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and $k < \omega$.*

Baumgartner and Hajnal prove upward (allowing the transfer of a statement from the ground universe to any extension) and downward absoluteness (allowing one to infer a same statement holds in the ground universe if it holds in an extension) lemmas for the statement "$\mathrm{otp}(C) \to (\omega)^1_\omega$". Then given an ordered set $\langle A, < \rangle$, let θ be the order type of A and let $\beta = |\theta|$. Suppose $\theta \to (\omega)^1_\omega$. Pass to a forcing extension which is correct about ω_1 and satisfies MA_β. By the upward absoluteness lemma, $\theta \to (\omega)^1_\omega$ holds. By Lemma 36, the extension satisfies $\theta \to (\alpha)^2_k$ for $\alpha < \omega_1$ and $k < \omega$. The downward absoluteness lemma implies that this holds in the ground universe.

Taking $\theta = \omega_1$, one needs to use MA_{ω_1}, so this is a proof using $\mathrm{MA}{+}\neg\mathrm{CH}$. Galvin removed the metamathematics in a subsequent proof [32].[53][54]

p-adic forms. The paper in which the following number theoretic theorem is proved does not explicitly use forcing, but it appeals to CH and absoluteness.

Theorem 37 (Ax-Kochen [6])**.** *For any $d > 0$ there is an $N > 0$ such that for any prime $p > N$, any homogeneous polynomial over the ring $\mathbb{Z}_p$ of p-adic integers with degree d and more than d^2 variables has a nontrivial zero over $\mathbb{Z}_p$.*

---

[52]We are using standard notation from the partition calculus. Let θ and γ be ordinals. We write $\theta \to (\omega)^n_\gamma$ for the assertion that for any coloring f of the n-element subsets of θ with γ-many colors, there is a subset $X \subseteq \theta$ such that the order type of X is ω and f is monochromatic on every n-element subset of X.

[53]See [36] for a detailed history.

[54][9, §3.1.1] discusses the Baugartner-Hajnal theorem as an example of forcing proving a theorem about the ground model, emphasizing that the theorem is not, to use our distinction, concrete. We have included it particularly to emphasize the specificity of the absoluteness lemmas Baumgartner and Hajnal proved.
The Malliaris-Shelah theorem—that two cardinal characteristics of the continuum, the tower number $\mathfrak{t}$ and the pseudointersection number $\mathfrak{p}$, are equal—is also discussed in [9, §3.1.1]. It is proved in [54] by assuming $\mathfrak{p} < \mathfrak{t}$ in the ground universe and deriving a contradiction in a forcing extension. This illustrates the utility of forcing extensions but does not use absoluteness.

A key step in the Ax-Kochen proof is showing that certain ultraproducts by nonprincipal ultrafilters over the primes are elementarily equivalent. This is implied by the *Ax-Kochen isomorphism theorem*:

Theorem 38 (Ax-Kochen)**.** *Assume* CH*. Then if U is a nonprincipal ultrafilter over prime numbers, then the ultraproduct $\prod_p \mathbb{F}_p((t))/U$ is isomorphic to the ultraproduct $\prod_p \mathbb{Q}_p/U$.*

Shelah has shown ZFC cannot prove the Ax-Kochen isomorphism theorem—something like CH is needed [67]. Ax and Kochen describe why CH can nevertheless be removed from their proof of Theorem 37:

> It is easily shown that the statement of [Theorem 37] is equivalent to an elementary number-theoretic statement; moreover the proof of this equivalence may be carried out in the set theory Σ described in [Gödel's "On the consistency of the continuum hypothesis"]. Also the proof we have given that the continuum hypothesis and the axiom of choice imply the statement of [Theorem 37] may be carried out in Σ. Now it is known (and follows easily from [Gödel's paper]) that a proof in Σ of an elementary number-theoretic statement which uses the continuum hypothesis and the axiom of choice may be transformed into a proof (in Σ) of the number-theoretic statement which does not use these assumptions. [6, p. 628]

Denef has given an algebraic geometric proof of the Ax-Kochen theorem [19].

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