CENTERS OF MASS FOR OPERATOR-FAMILIES

by MAKOTO TAKAGUCHI

(Received 5 November, 1990)

1. Introduction. Let H be a complex Hilbert space and let $B(H)$ be the algebra of (bounded) operators on H. Let $A = (A_1, \ldots, A_n)$ be an n-tuple of operators on H. The *joint numerical range* of A is the subset $W(A)$ of $\mathbb{C}^n$ such that

$$W(A) = \{((A_1x, x), \ldots, (A_nx, x)) : x \in H, \|x\| = 1\}.$$

We now describe several definitions of joint spectra of a commuting n-tuple A of operators (see [**3**]): the *approximate point spectrum*

$$\sigma_\pi(A) = \{\lambda \in \mathbb{C}^n : \text{there exists a sequence } \{x_i\} \text{ of unit vectors in } H \text{ such that } \|(A_k - \lambda_k)x_i\| \to 0 \text{ as } i \to \infty,\ k = 1, \ldots, n\},$$

the *left spectrum*

$$\sigma_l(A) = \{\lambda \in \mathbb{C}^n : A - \lambda \text{ generates a proper left ideal in } B(H)\},$$

the *right spectrum*

$$\sigma_r(A) = \{\lambda \in \mathbb{C}^n : A - \lambda \text{ generates a proper right ideal in } B(H)\},$$

the *Harte spectrum*

$$\sigma_H(A) = \sigma_l(A) \cup \sigma_r(A),$$

the *commutant spectrum*

$$\sigma'(A) = \{\lambda \in \mathbb{C}^n : A - \lambda \text{ generates a proper ideal in } A'\},$$

the *double commutant spectrum*

$$\sigma''(A) = \{\lambda \in \mathbb{C}^n : A - \lambda \text{ generates a proper ideal in } A''\}$$

(where A' and A'' are the commutant and double commutant of A in $B(H)$, respectively), the *Taylor spectrum*

$$\sigma_T(A) = \{\lambda \in \mathbb{C}^n : \text{the Koszul complex } E(A - \lambda, H) \text{ on } H \text{ associated with } A - \lambda \text{ is not exact}\}$$

(see [**7**] or [**8**]) and the *polynomial spectrum*

$$\sigma_P(A) = \{\lambda \in \mathbb{C}^n : p(\lambda) \in \sigma(p(A)) \text{ for all } n\text{-variate polynomials } p\}$$

(of course, $\sigma(p(A))$ is the usual spectrum of $p(A) \in B(H)$). Note that σ_π, σ_l, σ_r and σ_H can be defined even if A is not commuting.

The *joint operator norm*, *joint numerical radius* and *joint spectral radii* of A, denoted by $\|A\|$, $w(A)$ and $r.(A)$ respectively, are defined by

$$\|A\| = \sup\left\{\left(\sum_{k=1}^{n} \|A_kx\|^2\right)^{1/2} : \|x\| = 1\right\},$$

$$w(A) = \sup\left\{\left(\sum_{k=1}^{n} |(A_kx, x)|^2\right)^{1/2} : \|x\| = 1\right\}$$

and

$$r.(A) = \sup\{|\lambda| : \lambda \in \sigma.(A)\}$$

respectively, where $\sigma. = \sigma_\pi$, σ_l, σ_H, σ', σ'', σ_T or σ_P and $r. = r_\pi$, r_l, r_H, r', r'', r_T or r_P. Note that we define the joint operator norm, joint numerical radius, r_π, r_l and r_H for all (not necessarily commuting) n-tuples of operators, but we define r', r'', r_T and r_P only for commuting n-tuples of operators.

We shall call A *jointly normaloid* if $\|A\| = w(A)$, and call A *jointly transloid* if $A - z$ is jointly normaloid for any point $z \in \mathbb{C}^n$.

In this note we shall define the center of mass for an n-tuple of operators and state that the center of mass of A is coincident with the center of the smallest sphere containing the joint spectrum of A in case of a jointly transloid n-tuple $A = (A_1, \dots, A_n)$ of operators. The center of mass of a single operator has been defined by Stampfli **[5]**.

2. Results.

THEOREM 1. *For an n-tuple $A = (A_1, \dots, A_n)$ of operators the following conditions are equivalent*:

(i) $\|A\|^2 + |\lambda|^2 \leqq \|A - \lambda\|^2$ *for all* $\lambda \in \mathbb{C}^n$;

(ii) $\|A\| \leqq \|A + \lambda\|$ *for all* $\lambda \in \mathbb{C}^n$.

Proof. It is clear that (i) implies (ii). So we shall show that (ii) implies (i). For any natural number m, it follows that

$$\begin{aligned}
\|A - \lambda\|^2 - \|A\|^2 &= \sup\left\{\sum_{k=1}^n \|(A_k - \lambda_k)x\|^2 : \|x\| = 1\right\} + (m-1)\sup\left\{\sum_{k=1}^n \|A_k x\|^2 : \|x\| = 1\right\} \\
&\quad - m \sup\left\{\sum_{k=1}^n \|A_k x\|^2 : \|x\| = 1\right\} \\
&\geqq \sup\left\{\sum_{k=1}^n (((A_k - \lambda_k)^*(A_k - \lambda_k) + (m-1)A_k^* A_k)x, x) : \|x\| = 1\right\} \\
&\quad - m \sup\left\{\sum_{k=1}^n \left\|\left(A_k - \frac{\lambda_k}{m}\right)x\right\|^2 : \|x\| = 1\right\} = \sup\left\{\sum_{k=1}^n \left(\left(m\left(A_k - \frac{\lambda_k}{m}\right)^*\left(A_k - \frac{\lambda_k}{m}\right)\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{m-1}{m}|\lambda_k|^2\right)x, x\right) : \|x\| = 1\right\} - m \sup\left\{\sum_{k=1}^n \left\|\left(A_k - \frac{\lambda_k}{m}\right)x\right\|^2 : \|x\| = 1\right\} \\
&= \sup\left\{\sum_{k=1}^n m\left(\left(A_k - \frac{\lambda_k}{m}\right)^*\left(A_k - \frac{\lambda_k}{m}\right)x, x\right) : \|x\| = 1\right\} \\
&\quad + \frac{m-1}{m}\sum |\lambda_k|^2 - m \sup\left\{\sum \left\|\left(A_k - \frac{\lambda_k}{m}\right)x\right\|^2 : \|x\| = 1\right\} \\
&= \frac{m-1}{m}\sum |\lambda_k|^2 = \frac{m-1}{m}|\lambda|^2.
\end{aligned}$$

Thus the proof is complete.

THEOREM 2. *Given $A = (A_1, \dots, A_n)$, there exists a unique $z_0 \in \mathbb{C}^n$, such that*

$$\|A - z_0\| \leqq \|A - \lambda\| \quad \textit{for all} \quad \lambda \in \mathbb{C}^n.$$

Proof. Since $\|A-\lambda\|$ is large for λ large, $\inf\{\|A-\lambda\| : \lambda \in \mathbb{C}^n\}$ must be attained at some point, say z_0. The uniqueness of z_0 is deduced from the above theorem.

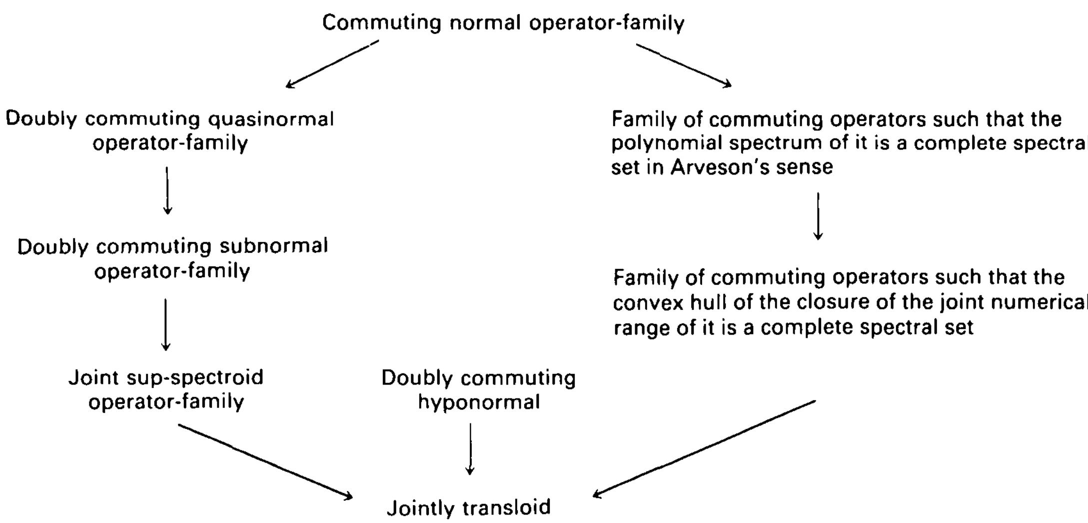
DEFINITION 3. For $A=(A_1,\ldots,A_n)$, we define the *center of mass* of A to be the point z_0 specified in Theorem 2, and designate it by c_A.

THEOREM 4. *Let $A=(A_1,\ldots,A_n)$ be commuting and jointly transloid, and let $\sigma.(A)$ be $\sigma_\pi(A)$, $\sigma_l(A)$, $\sigma_H(A)$, $\sigma_T(A)$, $\sigma'(A)$, $\sigma''(A)$ or $\sigma_P(A)$. Then c_A is the center of the smallest closed ball containing $\sigma.(A)$.*

Proof. It is well known that $\sigma_\pi(A)=\sigma_l(A)\subset\sigma_H(A)\subset\sigma_T(A)\subset\sigma'(A)\subset\sigma''(A)\subset\sigma_P(A)$ (see [**8**], [**7**], [**1**]). Moreover, from Theorem 2.5.4 of [**4**] and Corollary 2 of Proposition 1.1.2 of [**1**], it follows that $\sigma_P(A)\subset \operatorname{co}\overline{W(A)}$, the convex hull of the closure of $W(A)$. Consequently, $r_\pi(A)=r_l(A)\leqq r_H(A)\leqq r_T(A)\leqq r'(A)\leqq r''(A)\leqq r_P(A)\leqq w(A)\leqq \|A\|$. On the other hand, if A is jointly normaloid, then $\|A\|=r_\pi(A)$ (see [**6**]). Consequently, if A is jointly transloid, then $r_\pi(A-z)=r_l(A-z)=r_H(A-z)=r_T(A-z)=r'(A-z)=r''(A-z)=r_P(A-z)=\|A-z\|$ for every $z\in\mathbb{C}^n$. Thus we have $\sup\{|z-c_A| : z\in\sigma.(A)\}=\|A-c_A\|\leqq\|A-\lambda\|=\sup\{|z-\lambda| : z\in\sigma.(A)\}$ for each $\lambda\in\mathbb{C}^n$. Thus the proof is complete.

REMARKS. (i) In Theorem 4, the condition that A is commuting is not necessary in respect of $\sigma_\pi(A)$, $\sigma_l(A)$ and $\sigma_H(A)$.

(ii) The class of jointly transloid operator-families includes the following kinds of classes of operator-families (see [**2**], [**6**]).



ACKNOWLEDGEMENT. I would like to express my gratitude to Professor Yoshihiro Nakamura for his help.

REFERENCES

1. W. Arveson, Subalgebras of C^*-algebras II, *Acta Math.* **128** (1972), 271–308.

2. M. Chō and M. Takaguchi, Some classes of commuting n-tuples of operators, *Studia Math.* **80** (1984), 245–259.

https://doi.org/10.1017/S0017089500008612 Published online by Cambridge University Press

3. R. E. Curto, Connections between Harte and Taylor spectra, *Rev. Roumaine Math. Pures Appl.* **31** (1986), 203–215.

4. N. P. Dekker, Joint numerical range and joint spectrum of Hilbert space operators (Ph.D. thesis, University of Amsterdam, 1969).

5. J. G. Stampfli, the norm of a derivation, *Pacific J. Math.* **33** (1970), 737–747.

6. M. Takaguchi and M. Chō, Joint normaloid operator-family, *Sci. Rep. Hirosaki Univ.* **27** (1980), 50–59.

7. J. L. Taylor, A joint spectrum for several commuting operators, *J. Funct. Anal.* **6** (1970), 172–191.

8. F.-H. Vasilescu, A characterization of the joint spectrum in Hilbert spaces, *Rev. Roumaine Math. Pures Appl.* **22** (1977), 1003–1009.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HIROSAKI UNIVERSITY
HIROSAKI
036 JAPAN