P 116. Where is the centre of mass of the Cantor set bent into a ring.

J. Wilker, University of Toronto

Solution by M. Shiffman and S. Spital, California State College at Hayward

It is known that in its n'th stage of symmetrical construction the Cantor structure, on the interval $[0, 2\pi]$, contains 2^n subintervals of equal lengths $2\pi/3^n$ with initial points at

$$(*) \qquad \theta_k = 2\pi \sum_{r=1}^{n} d_r/3^r \ , \quad \text{all } d_r = 0 \text{ or } 2 (k = 1, 2, \ldots, 2^n).$$

Therefore the corresponding unit-circle structure, when placed in the complex plane (symmetrically about the real axis with centre at the origin), yields the centre of gravity (c.g.)

$$Z_n' = \sum_{k=1}^{2^n} \int_{\theta_k}^{\theta_k + (2\pi/3^n)} (\exp i\,\theta)\, d\theta / 2^n (2\pi/3^n).$$

However, an application of the mean value theorem for integrals, shows that this is asymptotically equal to the simpler c.g., Z_n due only to the initial points $\theta_1, \ldots, \theta_{2^n}$ - more specifically

$$\frac{1}{2^n} \sum_{k=1}^{2^n} \exp i\,\theta_k = Z_n = Z_n' + 0\left(\frac{2\pi}{3^n}\right) .$$

The use of (*) now enables a recursive formulation of Z_n:

$$Z_n = \frac{1}{2^n} \sum_{d_r = 0, 2} \exp\left(2\pi i \sum_{r=1}^{n} \frac{d_r}{3^r}\right)$$

$$Z_n = \frac{1}{2^n} \sum_{d_r = 0, 2} \left[\exp\left(2\pi i \sum_{r=1}^{n-1} \frac{d_r}{3^r}\right)\right] \left[\exp \frac{0}{3^n} + \exp \frac{4\pi i}{3^n}\right]$$

$$Z_n = \frac{1}{2} \left(1 + \exp \frac{4\pi i}{3^n}\right) Z_{n-1} .$$

https://doi.org/10.1017/S0008439500029027 Published online by Cambridge University Press

Since $Z_o = 1$ (c.g. of the single point at $\theta_1 = 0$),

$$Z_n = \prod_{p=1}^{n} \left(\frac{1}{2} + \frac{1}{2} \exp \frac{4\pi i}{3^p}\right) = \prod_{p=1}^{n} \left(\cos \frac{2\pi}{3^p}\right)\left(\exp \frac{2\pi i}{3^p}\right).$$

Hence the c.g. of the completed Cantor set is given by

$$\lim_{n\to\infty} Z_n = \left[\prod_{p=1}^{\infty} \cos \frac{2\pi}{3^p}\right]\left[\exp \sum_{p=1}^{\infty} \frac{2\pi i}{3^p}\right]$$

$$= - \prod_{p=1}^{\infty} \cos \frac{2\pi}{3^p} \doteq 0.37143736.$$

Also solved by the proposer.

P 128. Let $\mathfrak{M}$ be the set of square matrices of order n whose entries are real numbers in the interval $a \le x \le b$. Show that the maximum value of a determinant of matrices in the set $\mathfrak{M}$ is attained by a matrix M whose entries are exclusively a and b.

N.S. Mendelsohn, University of Manitoba

Solution by A.R. Rhemtulla, University of Alberta

Let $X = (x_{ij})$ be an $n \times n$ matrix with $a \le x_{ij} \le b$. Det $X = x_{i1}X_{i1} + x_{i2}X_{i2} + \ldots + x_{in}X_{in}$ where X_{ij} is the cofactor of x_{ij}. If $a < x_{ij} < b$, then by replacing x_{ij} either by a or by b, and leaving all the other entries intact we obtain X' such that det $X' \ge$ det X. Carry on the process until each x_{ij} is replaced either by a or by b and the resulting matrix at each stage has determinant not less than the one before.

Of course it does not mean that the determinant of a matrix not all of whose entries are exclusively a and b must always be strictly less than the maximum. This can be seen easily with $a = 0$, $b = 1$, and $n = 2$:

$$\det\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = 1 \text{ for all } 0 \le x \le 1.$$

Also solved by L. Cummings, D.Ž. Djoković, M. Edelstein, D.G. Kabe, J. Schaer, K.W. Schmidt, S. Spital and J. Zelver jointly, and the proposer.

https://doi.org/10.1017/S0008439500029027 Published online by Cambridge University Press

P 129. Characterize all finite groups such that exactly half of their elements are of order 2 (the identity is not counted).

N.S. Mendelsohn, University of Manitoba

Solution by the proposer.

All such groups are obtained as follows. Let H be an abelian group of odd order. Let G be a normal extension by an involution t which maps every element of H into its inverse, i.e., $t^2 = 1$, $t^{-1}at = a^{-1}$ for all $a \varepsilon H$. It is clear that a group G constructed this way satisfies the conditions.

Conversely, let $G = \{b_1 = 1, b_2, \ldots, b_k, a_1, \ldots, a_k\}$ be a group such that half of its elements, say $a_1, \ldots, a_k$, are of order 2. By pairing each b_i, $i > 1$, with its inverse we see that k must be odd. The product of two a's must be one of the b's: otherwise there would be a subgroup of order 4, but 4 does not divide 2k. Hence $a_1a_1, a_1a_2, \ldots, a_1a_k$ are the b_i in some order. Now the product of two b's is again a b since $(a_1a_i)a_1a_j = (a_1a_ia_1^{-1})a_j$ is a product of two a's. Hence $H = \{b_1, \ldots, b_k\}$ is a subgroup and $G = H + Ha_1$. Finally, since $b_ia_1b_ia_1 = 1$ or $a_1^{-1}b_ia_1 = b_i^{-1}$, the inverse mapping is an automorphism of H so H is abelian.

Also solved by C. Ayoub, B. Chang, D.Ž. Djoković, and A.R. Rhemtulla. Several solvers pointed out that this problem has also appeared in the American Math. Monthly [1967, p.871].

P 130. Show that the system $x^n + y^n = u^n + v^n$, $x + y = u + v$ where n is an integer ≥ 2 has only trivial solutions in the real field.

D.R. Rao, Secunderabad, India

Solution by D.Ž. Djoković, University of Waterloo

Let $s = x + y = u + v$. The trivial solutions are (1) $u = x$, $v = y$; (2) $u = y$, $v = x$; (3) $s = 0$, n odd, so we may assume $s \neq 0$ when n is odd. We have (4) $x^n + (s-x)^n = u^n + (s-u)^n$. Let $f(t) = t^n + (s-t)^n$, $f'(t) = n(t^{n-1} - (s-t)^{n-1})$. Since $f'(t) = 0$ implies $t = s/2$ and $f(t) = f(s-t)$ for all t, we conclude that f is strictly monotonic in $(-\infty, s/2)$ and its graph is symmetric with respect to the line $t = s/2$.

https://doi.org/10.1017/S0008439500029027 Published online by Cambridge University Press

Hence, $f(x) = f(u)$ implies that $u = x$ or $u = s - x$ which together with (4) leads to the trivial solutions (1) and (2). So there is no other solution.

Also solved by the proposer.