NONTRIVIAL SOLUTIONS FOR A MULTIVALUED PROBLEM WITH STRONG RESONANCE

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The Mountain-Pass Theorem of Ambrosetti and Rabinowitz (see [**1**]) and the Saddle Point Theorem of Rabinowitz (see [**21**]) are very important tools in the critical point theory of C^1-functionals. That is why it is natural to ask us what happens if the functional fails to be differentiable. The first who considered such a case were Aubin and Clarke (see [**6**]) and Chang (see [**12**]), who gave suitable variants of the Mountain-Pass Theorem for locally Lipschitz functionals which are defined on reflexive Banach spaces. For this aim they replaced the usual gradient with a generalized one, which was firstly defined by Clarke (see [**13**], [**14**]). As observed by Brezis (see [**12**, p. 114]), these abstract critical point theorems remain valid in non-reflexive Banach spaces.

We apply some of these results to solve a multivalued problem with strong resonance at infinity. We remark that it is not usual to consider nonlinearities which are strongly resonant at $+\infty$ unless they are also strongly resonant at $-\infty$. The literature is very rich in resonant problems; the first who studied such problems (in the smooth case) were Landesman and Lazer (see [**18**]). They found sufficient conditions for the existence of solutions for some single-valued equations with Dirichlet conditions. These problems, which arise frequently in mechanics, were thereafter intensively studied and many applications to concrete situations were given.

1. Abstract framework. Let X be a real Banach space and let $f: X \to \mathbf{R}$ be a locally Lipschitz function. For each $x, v \in X$, we define the *generalized directional derivative* of f at x in the direction v as

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

The *generalized gradient* (the *Clarke subdifferential*) of f at the point x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^*; f^0(x, v) \geq \langle x^*, v \rangle, \text{ for all } v \in X\}.$$

We also define the lower semi-continuous function

$$\lambda(x) = \min\{\|x^*\|; x^* \in \partial f(x)\}.$$

For further properties of these notions we refer to [**12**], [**13**], [**14**].

We say that a point $x \in X$ is a *critical point* of f provided that $0 \in \partial f(x)$, that is $f^0(x, v) \geq 0$ for every $v \in X$. If c is a real number, we say that f satisfies the *Palais-Smale condition* at the level c (in short, $(PS)_c$) if any sequence $(x_n)_n$ in X with the properties $\lim_{n\to\infty} f(x_n) = c$ and $\lim_{n\to\infty} \lambda(x_n) = 0$ is relatively compact.

We shall use in this paper the following result, which is an immediate consequence of the Mountain-Pass Theorem proved in [**12**].

THEOREM 1. *Let* $f: X \to \mathbf{R}$ *be a locally Lipschitzian function. Suppose that* $f(0) = 0$ *and there is some* $v \in X \setminus \{0\}$ *such that* $f(v) \leq 0$. *Moreover, assume that* f *satisfies the following geometric hypothesis: there exist* $R > 0$ *and* $\alpha > 0$ *such that* $R < \|v\|$ *and, for each* $u \in X$ *with* $\|u\| = R$, *we have* $f(u) \geq \alpha$. *Let* $\mathcal{P}$ *be the family of all continuous paths* $p: [0, 1] \to X$ *that join* 0 *to* v *and*

$$c = \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} f(p(t)).$$

Then there exists a sequence (x_n) *in* X *such that*:

(i) $\lim_{n \to \infty} f(x_n) = c$;

(ii) $\lim_{n \to \infty} \lambda(x_n) = 0$.

Moreover, if f *satisfies* $(\mathrm{PS})_c$ *then* c *is a critical value of* f.

The following saddle point type result generalizes the Rabinowitz's theorem (see [**21**]). Its proof is an easy exercise and is left to the reader.

THEOREM 2. *Let* $f: X \to \mathbf{R}$ *be a locally Lipschitzian function. Assume that* $X = Y \oplus Z$, *where* Z *is a finite dimensional subspace of* X *and for some* $z_0 \in Z$ *there exists* $R > \|z_0\|$ *such that*

$$\inf_{y \in Y} f(y + z_0) > \max\{f(z); z \in Z, \|z\| = R\},$$

Let

$$K = \{z \in Z; \|z\| \leq R\}$$

and

$$\mathcal{P} = \{p \in C(K, X); p(x) = x \text{ if } \|x\| = R\}.$$

If c *is defined as in Theorem* 1 *and* f *satisfies* $(\mathrm{PS})_c$, *then* c *is a critical value of* f.

2. Main results. Let M be a m-dimensional smooth compact Riemann manifold, possibly with smooth boundary ∂M. Particularly, M can be any open bounded smooth subset of $\mathbf{R}^m$. We shall consider the following multivalued elliptic problem

$$\begin{cases} -\Delta_M u(x) - \lambda_1 u(x) \in [\underline{f}(u(x)), \overline{f}(u(x))] & \text{a.e. } x \in M, \\ u = 0 & \text{on } \partial M, \\ u \not\equiv 0, & \end{cases}$$

where:

(i) Δ_M is the Laplace–Beltrami operator on M;
(ii) λ_1 is the first eigenvalue of $-\Delta_M$ in $H_0^1(M)$;
(iii) $f \in L^\infty(\mathbf{R})$;
(iv) $\underline{f}(t) = \lim_{\varepsilon \searrow 0} \operatorname{essinf}\{f(s); |t - s| < \varepsilon\}$, $\overline{f}(t) = \lim_{\varepsilon \searrow 0} \operatorname{esssup}\{f(s); |t - s| < \varepsilon\}$.

As proved in [**12**], the functions $\underline{f}$ and $\bar{f}$ are measurable on **R** and, if

$$F(t) = \int_0^t f(s)\,ds,$$

then the Clarke subdifferential of F is given by

$$\partial F(t) = [\underline{f}(t), \bar{f}(t)] \quad \text{a.e. } t \in \mathbf{R}.$$

Let $(g_{ij}(x))_{i,j}$ define the metric on M. We consider on $H_0^1(M)$ the locally Lipschitz functional $\varphi = \varphi_1 - \varphi_2$, where

$$\varphi_1(u) = \frac{1}{2}\int_M \left(\sum_{i,j} g_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \lambda_1 u^2\right) dx \quad \text{and} \quad \varphi_2(u) = \int_M F(u)\,dx.$$

By a *solution of the problem* (P) we shall mean any critical point of the energetic functional φ. Denote

$$f(\pm\infty) = \operatorname{ess}\lim_{t \to \pm\infty} f(t) \quad \text{and} \quad F(\pm\infty) = \lim_{t \to \pm\infty} F(t).$$

Our basic hypothesis on f will be

$$f(+\infty) = F(+\infty) = 0, \tag{f1}$$

which makes the problem (P) a Landesman–Lazer type one, with strong resonance at $+\infty$.

The following formulates a sufficient condition for the existence of solutions of our problem.

THEOREM A. *Assume that f satisfies* (f1) *and either*

$$F(-\infty) = -\infty \tag{F1}$$

or $-\infty < F(-\infty) \leq 0$ *and there exists* $\eta > 0$ *such that*

$$F \text{ is non-negative on } (0, \eta) \text{ or } (-\eta, 0). \tag{F2}$$

Then the problem (P) *has at least one solution.*

For positive values of $F(-\infty)$ it is necessary to impose additional restrictions on f. Our variant for this case is the following theorem.

THEOREM B. *Assume* (f1) *and* $0 < F(-\infty) < +\infty$. *Then the problem* (P) *has at least one solution provided the following conditions are satisfied*:

$$f(-\infty) = 0$$

and

$$F(t) \leq \frac{\lambda_2 - \lambda_1}{2} t^2 \qquad \textit{for each } t \in \mathbf{R}.$$

For the proof of Theorem A we shall make use of the following non-smooth variants of Lemmas 6 and 7 in **[15]** (see also **[3]** for Lemma 1) which can be obtained in the same manner.

LEMMA 1. *Assume* $f \in L^\infty(\mathbf{R})$ *and there exist* $F(\pm\infty) \in \bar{\mathbf{R}}$. *Moreover, suppose that*

(i) $f(+\infty)=0$ *if* $F(+\infty)$ *is finite*;

and

(ii) $f(-\infty)=0$ *if* $F(-\infty)$ *is finite.*

Then

$$\mathbf{R}\setminus\{a\,.\,\mathrm{meas}(M);\, a=-F(\pm\infty)\}\subset\{c\in\mathbf{R};\ \varphi \textit{ satisfies } (\mathrm{PS})_c\}.$$

LEMMA 2. *Assume* f *satisfies* (f1). *Then* φ *satisfies* $(\mathrm{PS})_c$, *whenever* $c\neq 0$ *and* $c<-F(-\infty)\,.\,\mathrm{meas}(M)$.

Here meas(M) denotes the Riemannian measure of M.

Proof of Theorem A. We shall develop some of the ideas used in **[26]**. There are two distinct situations.

Case 1. $F(-\infty)$ is finite, that is $-\infty<F(-\infty)\le 0$. In this case, φ is bounded from below since

$$\varphi(u)=\frac{1}{2}\int_M\left(\sum_{i,j} g_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}-\lambda_1 u^2\right)dx-\int_M F(u)\,dx$$

and, by our hypothesis on $F(-\infty)$,

$$\sup_{u\in H_0^1(M)}\int_M F(u)\,dx<+\infty.$$

Therefore,

$$-\infty<a:=\inf_{u\in H_0^1(M)}\varphi(u)\le 0=\varphi(0).$$

Choose c small enough in order to have $F(ce_1)<0$ (note that c may be taken positive if $F>0$ in $(0,\eta)$ and negative if $F<0$ in $(-\eta,0)$). Here $e_1>0$ denotes the first eigenfunction of $-\Delta_M$ in $H_0^1(M)$. Hence $\varphi(ce_1)<0$, so $a<0$. It follows now from Lemma 2 that φ satisfies $(\mathrm{PS})_a$. The proof ends in this case by applying Theorem 1.

Case 2. $F(-\infty)=-\infty$. Then, by Lemma 1, φ satisfies $(\mathrm{PS})_c$ for each $c\neq 0$. Let V be the orthogonal complement of the space spanned by e_1 with respect to $H_0^1(M)$, that is

$$H_0^1(M)=\mathrm{Sp}\{e_1\}\oplus V.$$

For fixed $t_0>0$, denote

$$V_0=\{t_0e_1+v;\, v\in V\}\quad\text{and}\quad a_0=\inf_{v\in V_0}\varphi(v).$$

Note that φ is coercive on V. Indeed, if $v\in V$, then

$$\varphi(v)\ge\frac{1}{2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\|v\|_{H_0^1}^2-\int_M F(v)\to+\infty\quad\text{as}\quad\|v\|_{H_0^1}\to+\infty,$$

because the first term has a quadratic growth at infinity (t_0 being fixed), while $\int_M F(v)$ is uniformly bounded (in v), in view of the behaviour of F near $\pm\infty$. Thus, a_0 is attained, because of the coercivity of φ on V. From the boundedness of φ on $H_0^1(M)$ it follows that $-\infty < a \leq 0 = \varphi(0)$ and $a \leq a_0$.

Again, there are two possibilities.

(i) $a < 0$. In this case, by Lemma 2, φ satisfies $(PS)_a$. Hence $a < 0$ is a critical value of φ.

(ii) $a = 0 \leq a_0$. Then, either $a_0 = 0$ or $a_0 > 0$. In the first case, as we have already remarked, a_0 is attained. Thus, there is some $v \in V$ such that

$$0 = a_0 = \varphi(t_0 e_1 + v).$$

Hence, $u = t_0 e_1 + v \in H_0^1(M)\setminus\{0\}$ is a critical point of φ, that is a solution of (P).

If $a_0 > 0$, notice that φ satisfies $(PS)_b$ for each $b \neq 0$. Since $\lim_{t\to+\infty} \varphi(te_1) = 0$, we may apply Theorem 2 to conclude that φ has a critical value $c \geq a_0 > 0$.

Proof of Theorem B. Assume V has the same definition as above, and let

$$V_+ = \{te_1 + v\,; t > 0, v \in V\}.$$

It will be sufficient to show that the functional φ has a non-zero critical point. To do this, we shall make use of two different arguments. If $u = te_1 + v \in V_+$ then

$$\varphi(u) = \frac{1}{2}\int_M (|\nabla v|^2 - \lambda_1 v^2) - \int_M F(te_1 + v).$$

In view of the boundedness of F it follows that

$$-\infty < a_+ := \inf_{u \in V_+} \varphi(u) \leq 0.$$

We analyse two distinct situations.

Case 1. $a_+ = 0$. To prove that φ has a critical point, we use the same arguments as in the proof of Theorem A (the second case). More precisely, for some fixed $t_0 > 0$ we define in the same way V_0 and a_0. Obviously, $a_0 \geq 0 = a_+$, since $V_0 \subset V_+$. The proof follows from now on as in Case 2 of Theorem A, by reconsidering the two distinct situations $a_0 > 0$ and $a_0 = 0$.

Case 2. $a_+ < 0$. Let $u_n = t_n e_1 + v_n$ be a minimizing sequence of φ in V_+. We observe that the sequences $(u_n)_n$ and $(v_n)_n$ are bounded. Indeed, this is essentially a compactness condition and may be proved in a similar way to Lemma 1. It follows that there exists $w \in \bar{V}_+$, such that, going eventually to a subsequence,

$$u_n \to w \quad \text{weakly in } H_0^1(M),$$
$$u_n \to w \quad \text{strongly in } L^2(M),$$
$$u_n \to w \quad \text{a.e.}$$

Applying the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\leftarrow\infty} \varphi_2(u_n) = \varphi_2(u).$$

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On the other hand,

$$\varphi(w) \leq \liminf_{n\to\infty} \varphi_1(u_n) - \lim_{n\to\infty} \varphi_2(u_n) = \liminf_{n\to\infty} \varphi(u_n) = a_+.$$

It follows that, necessarily, $\varphi(w) = a_+ < 0$. Since the boundary of V_+ is V and

$$\inf_{u\in V} \varphi(u) = 0,$$

we conclude that w is a local minimum of φ on V_+ and $w \in V_+$.

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