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ON LOCALLY SOLUBLE PERIODIC GROUPS WITH CHERNIKOV CENTRALIZER OF A FOUR-SUBGROUP

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Let G be a locally soluble periodic group having a four-subgroup V. We show that if $C_G(V)$ is Chernikov then G is hyperabelian-by-Chernikov, if $C_G(V)$ is finite then G is hyperabelian.

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1. Introduction

Centralizers play a very important role in locally finite group theory. In some cases we can deduce information about a locally finite group G given only information about $C_G(V)$ for some finite subgroup V of G (see [**10**]). It is known now that if G contains an element v of prime power order with Chernikov centralizer then G is almost locally soluble. The proof of this result in [**11**] depends on the classification of finite simple groups. In [**2**] Asar has proved that if v above has order two then G is almost soluble. This result does not depend on the classification but uses some essential parts of it.

In this article we turn our attention to groups G containing a four-subgroup V such that $C_G(V)$ is Chernikov. These groups can be simple. Indeed, the example of an infinite simple locally finite group having a four-subgroup with Chernikov centralizer is provided by **PSL(2, F)** where **F** is an infinite locally finite field of odd characteristic. A locally soluble periodic group with Chernikov centralizer of the four-subgroup also can be non-soluble. To show this we use an example from [12].

Let p be an odd prime and let t denote the largest odd divisor of $p-1$. Let G_k be the group formed by the matrices

$$A=\begin{pmatrix} u+pa & pb \\ pc & v+pd \end{pmatrix}$$

of determinant 1, where a,b,c,d,u,v lie in the ring of residue classes $(\operatorname{mod} p^{k+1})$ and $uv=u^t=1 \pmod p$. Then G_k is of derived length m or $m+1$ where m is the least integer such that $2^m \geqq k+1$. Let α_k and β_k be the elements of $\operatorname{Aut} G_k$ such that $A^{\alpha_k}=(A^{-1})$ and $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{\beta_k}=\begin{pmatrix} a_1 & -a_2 \\ -a_3 & a_4 \end{pmatrix}$. It is not difficult to verify that $\langle \alpha_k, \beta_k \rangle$ is a four-group acting fixed-point-freely on G_k. Let G be the direct product of groups G_k; $k=1,2,\ldots$. Then G

admits a fixed-point-free four-group of automorphisms V. Clearly, the split extension of G by V is the required group.

In this paper we prove:

Theorem. *Let G be a periodic locally soluble group and V a four-subgroup of G.*
(i) If $C_G(V)$ is Chernikov then G is hyperabelian-by-Chernikov.
(ii) If $C_G(V)$ is finite then G is hyperabelian.

We recall that a group is said to be hyperabelian if it has an ascending invariant series with abelian quotients. In [14] the author proved that a locally finite group admitting a fixed-point-free four-group of automorphisms is hyperabelian. In the present paper we use some important technic and ideas of [14].

2. Lemmas

Lemma 1. *Let G be a locally soluble group containing a hyperabelian subgroup of finite index. Then G is hyperabelian.*

Proof. Clearly, G contains a normal hyperabelian subgroup H of finite index. Suppose G is not hyperabelian. Then there exists a quotient G/R which does not possess an abelian normal subgroup. Let the images of G and H in G/R be denoted again by G and H respectively. Let A be a nontrivial normal abelian subgroup of H. Then $\langle A^G\rangle$ is the product of finitely many abelian normal subgroups of H and so is nilpotent. Thus $Z\langle A^G\rangle$ is a nontrivial normal abelian subgroup of G.

Lemma 2. *Let π be a set of primes, G a locally finite π'-group acted on by a finite π-group V.*

(i) If N is a normal V-invariant subgroup of G then $C_{G/N}(V)=C_G(V)N/N$.

(ii) $[G,V]=[G,V,V]$.

(iii) $G=[G,V]C_G(V)$.

Proof. Each of these facts follows immediately from the corresponding finite case [7].

Lemma 3. *Let G be a locally finite group acted on by an involutory automorphism v in such a manner that $C_G(v)$ is Chernikov. Let $0(G)$ be the largest normal $2'$-subgroup of G, and let F be a divisible abelian 2-subgroup of G such that $x^v=x^{-1}$ for every $x\in F$. Then* $[0(G),F]=1$.

Proof. Since $F\subseteq[G,v]$, we have $[0(G),F]\subseteq[G,v]$ so by [2] and [9, Theorem B] $[0(G),F,F]$ is Chernikov whence, by [4], $[0(G),F,F,F]=1$ and, by the preceding lemma, $[0(G),F]=1$, as required.

Throughout the remaider of this section let G stand for a locally finite $2'$-group, V an

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elementary group of automorphisms of order 2^n. Let $V_1, V_2, \ldots, V_{2^n-1}$ be the set of maximal subgroups of V, $G_i = C_G(V_i)$, $J_i = \{x \in G_i; x^v = x^{-1}$ for $v \in V - V_i\}$; $1 \leqq i \leqq 2^n - 1$.

Lemma 4. *Let $n=2$ and v_i be the involution from V_i; $1 \leqq i \leqq 3$. Suppose that $x, y \in G$ and v_3 sends x into x^{-1}. Then*

(i) *there exists a unique pair of elements $a \in J_1$, $b \in J_2$ such that $x = bab$;*

(ii) *there exist elements $y_1 \in G_1$, $y_2 \in G_2$, $y_3 \in G_3$ such that $y = y_1 y_2 y_3$.*

Proof. See [15, Lemmas 1.4 and 1.6].

Remark. Suppose that the x above is conjugate in G to some element of G_1. Then $b \in G'$.

Indeed, let $\bar{z}$ denote the image of z in $\bar{G} = G/G'$. Then $\bar{x} \in C_{\bar{G}}(v_1)$, whence $\bar{x} = \bar{a}$. So $\bar{b} = 1$ as required.

Lemma 5. *If $G = [G, V]$ then*

(i) $G = \langle J_i;\ 1 \leqq i \leqq 2^n - 1\rangle$;

(ii) *$C_G(V)$ is generated by its subgroups $C_G(V) \cap \langle J_i \rangle$, $1 \leqq i \leqq 2^n - 1$.*

Proof. It is not hard to see that any element of G is contained in some finite V-invariant subgroup which satisfies the conditions of the lemma. Thus it is sufficient to consider the case in which G is finite. Let us prove (i). By [6, Lemma 2.1] we can assume that G is a p-group for some prime p. By [15, Lemma 1.6] $G = G_1 G_2 \ldots G_{2^n-1}$ and by [7, Lemma 10.4.1] $G_i = J_i C_G(V)$. It follows from $G = [G, V]$ that $C_G(V) \subseteq G' \subseteq \Phi(G)$. So $G = \langle J_i; 1 \leqq i \leqq 2^n - 1\rangle$.

Now consider (ii). By the Feit-Thompson Theorem [5] G is soluble. We shall prove the lemma by induction on the derived length of G. Suppose that for $[G', V]$ the assertion is true. Put $A = \bigcup_i J_i$, $B = A - (G' \cap A)$, $B_i = B \cap J_i$. Let S denote the set of finite formal sequences of elements of A, i.e.

$$S = \{(a_1, a_2, \ldots, a_r); a_i \in A\}.$$

For each $s \in S$ define numbers $\alpha(s)$ and $\beta(s)$ as follows. Let $s = (a_1, a_2, \ldots, a_r)$. Then $\alpha(s) = |\{i;\ a_i \in B\}|$. We put $\beta(s) = 0$ if and only if for any $m \in \{1, 2, \ldots, 2^n - 1\}$ s contains at most one element of B_m. Otherwise, $\beta(s) = \min_m \{|i-j|; i \neq j; a_i, a_j \in B_m\}$. We denote by $\bar{s}$ the element $a_1 a_2 \ldots a_r$ of G. By (i) G is generated by A therefore the mapping $S \to G$ defined by $s \to \bar{s}$ is surjective. Let h be an arbitrary element of $H = C_G(V)$. Then

$$\alpha(h) = \min_s \{\alpha(s);\ \bar{s} = h\}$$

$$\beta(h) = \min_s \{\beta(s);\ \bar{s} = h, \alpha(s) = \alpha(h)\}.$$

We note that if $\alpha(h) = 0$ then $h \in [G', V]$ and consequently h belongs to $H_0 =$

$\langle H \cap \langle J_i \rangle; 1 \leqq i \leqq 2^n - 1 \rangle$. Let us show that $\beta(h) = 0$ is possible only when $\alpha(h) = 0$. Suppose that $\beta(h) = 0$. Then there exists $s \in S$ such that $\bar{s} = h$, $\beta(s) = 0$. Let $s = (a_1, a_2, \ldots, a_r)$. Note that $h \in G'$ as $G = [G, V]$. So in the factor group G/G' we have $b_1 b_2, \ldots, b_{2^n - 1} = 1$ where $b_m = a_k G'$ for $a_k \in B_m$. This immediately gives us $b_1 = b_2 = \cdots = b_{2^n - 1} = 1$ which implies $\alpha(h) = 0$.

Suppose that α is the least number such that $\alpha(h) = \alpha$ does not imply $h \in H_0$ and β is the least number such that $\alpha(h) = \alpha$ and $\beta(h) = \beta$ does not imply $h \in H_0$. As we have shown above both α and β are positive. Choose an element $h \in H$ such that $\alpha(h) = \alpha$, $\beta(h) = \beta$ and $h \notin H_0$. There exists a sequence $s = (a_1, a_2, \ldots, a_r) \in S$ such that $\bar{s} = h$, $\alpha(s) = \alpha$, $\beta(s) = \beta$. We have $a_i, a_j \in B_m$ where $j - i = \beta$. Denote by s_1 the initial segment of s consisting of $j - 2$ elements, by s_2 denote the final one consisting of $r - j$ elements. First suppose that $a_{j-1} \in J_m$. By [7, Lemma 10.4.1] $a_{j-1} a_j = b h_1$ where $b \in B_m$, $h_1 \in H \cap \langle J_m \rangle$. Then $h h_1^{-1} = \bar{s}_1 b h_1 \bar{s}_2 h_1^{-1}$. Clearly, $H \subseteq N_G(J_k)$ for $k = 1, 2, \ldots, 2^n - 1$ so if $i = j - 1$ then $\alpha(h h_1^{-1}) \leqq \alpha - 1$. If $i \neq j - 1$ then $\beta(h h_1^{-1}) \leqq \beta - 1$. In any case under our assumptions $h h_1^{-1} \in H_0$, whence $h \in H_0$.

Let now $a_{j-1} \notin J_m$. By Lemma 4(i) there exist $m_1, m_2 \in \{1, 2, \ldots, 2^n - 1\}$ such that $a_j^{-1} a_{j-1} a_j = cdc$ for suitable elements $c \in J_{m_1} \cap G'$ and $d \in J_{m_2}$. Therefore $h = \bar{s}_1 a_j c d c \bar{s}_2$, whence we get that $\beta(h) \leqq \beta - 1$. Lemma 5 is now established.

Lemma 6. *Suppose the hypotheses of Lemma 4 hold. Then for any element x of $[G, V]$ there exist elements x_1, x_2, x_3 such that $x_i \in \langle J_i \rangle$ and $x = x_1 x_2 x_3$.*

Proof. By [8, Lemma 4] there exist elements y_1, y_2, y_3, h such that $x = y_1 y_2 y_3 h$; $y_i \in J_i$; $h \in C_G(V)$. By the preceding lemma $h = h_1 h_2 h_3$ for $h_i \in C_G(V) \cap \langle J_i \rangle$. Put $x_1 = y_1 h_1$, $x_2 = h_1^{-1} y_2 h_1 h_2$, $x_3 = h_2^{-1} h_1^{-1} y_3 h_1 h_2 h_3$. Then $x_i \in \langle J_i \rangle$ and $x = x_1 x_2 x_3$.

Lemma 7. *With the hypotheses of Lemma 4 assume that $G = [G, V]$ and R is a normal V-invariant subgroup of G such that $R \cap C_G(V) = 1$. Then R possesses a GV-invariant series all of whose quotients are abelian.*

Proof. Suppose that R has no non-trivial abelian subgroup which is normal in GV. By [14, Lemma 2.2] there exists an element $a \in R \cap G_i$ such that $\langle a^G \rangle \cap C_G(a) \neq \langle a^G \rangle \cap G_i$ for some $i \in \{1, 2, 3\}$. We assume that $a \in G_1$. Put $R_i = R \cap G_i$, $T = \langle a^G \rangle$, $T_i = T \cap G_i$, $D = C_T(a)$, $D_i = D \cap G_i$, $1 \leqq i \leqq 3$.

Evidently,

$$\langle J_i \rangle \subseteq C_G(R_i). \tag{*}$$

So $D_1 = T_1$. By the choice of a we can assume that $D_2 \neq 1$. Put $K = C_{T_1}(D_2)$, $L = C_{T_3}(K)$.

Let x be an arbitrary element of G. By Lemma 6 $x = x_1 x_2 x_3$ where $x_i \in \langle J_i \rangle$. We have $a^x = a^{x_2 x_3}$. By Lemma 4(i) $x_2^{-1} a x_2 = c a_1 c$ where $c \in J_3$, $a_1 \in J_1$. By (*) $a^{x_2} \in C_T(D_2)$, whence $a_1 \in K$, $c \in C_T(D_2)$. Again by (*) $x_3 \in C_G(c)$, therefore $(c a_1 c)^{x_3} = c a_1^{x_3} c$. By Lemma 4(i) $a_1^{x_3} = b a_2 b$ where $b \in J_2$, $a_2 \in J_1$. We note that $a_1^{x_3} \in C_T(L)$, whence $b, a_3 \in C_T(L)$. This argument shows that $L \subseteq Z(T)$. Suppose that $L = 1$. Then v_3 acts fixed-point-freely on $C_T(K)$ which gives us that $T_1 \subseteq C_G(D_2)$, that is $K = T_1$.

Consequently, $D_2 \subseteq C_G(\langle T_1, T_2 \rangle)$. By [14, Lemma 1.5, Corollary 1] $\langle T_1, T_2 \rangle = [T, v_3]$. So

$$D_2 \subseteq C_G([T, v_3]). \qquad (**)$$

Evidently,

$$G_3 \subseteq N_G([T, v_3]). \qquad (***)$$

Now let y be an arbitrary element of G. Again by Lemma 6 $y = y_1 y_3 y_2$ where $y_i \subseteq \langle J_i \rangle$. We have $a^{y_1} = a$. Using (**) and (***), we get $D_2 \in C_G(a^{y_3})$. By (*) $[D_2, y_2] = 1$, whence $D_2 = D_2^{y_2} \subseteq C_G(a^{y_3 y_2}) = C_G(a^y)$. Therefore $1 \neq D_2 \subseteq Z(T)$. This contradicts the assumption that R has no non-trivial abelian subgroup which is normal in GV. The lemma is established.

3. Proof of theorem

Let $C = C_G(V)$ be Chernikov, and let S be some maximal 2-subgroup of G containing V. By [13, Lemma 3.1] S is Chernikov. Let F be the minimal subgroup of finite index in S. We denote by Q the maximal normal $2'$-subgroup of G. First, let us prove that $[Q, V]$ is hyperabelian. By Lemma 2(i) it suffices to show that $[Q, V]$ has a nontrivial normal V-invariant abelian subgroup. Suppose that any such subgroup of $[Q, V]$ is trivial. Then by Lemma 7 each normal V-invariant subgroup of $[Q, V]$ has non-trivial intersection with C. In this case $[Q, V]$ contains a minimal non-trivial normal V-invariant subgroup M. By a theorem of McLain [13, p. 11] M is abelian and we obtain a contradiction. Thus $[Q, V]$ is hyperabelian. Therefore in order to prove (i) it suffices to show that $[Q, V]$ contains a subgroup R such that R is normal in G and G/R is Chernikov.

Let v_1, v_2, v_3 be the involutions of V, $G_k = C_G(v_k)$, $F_k = F \cap G_k$, $Q_k = Q \cap G_k$, $J_k = \{x \in G_k;\ x^{v_i} = x^{-1} \text{ for } i \neq k\}$; $1 \leqq k \leqq 3$. Then $F = F_1 F_2 F_3$ [3, Lemma 6] and $F_k = C_F(V)(F \cap J_k)$. We note that $(F \cap J_i)(F \cap J_j) \cap (F \cap J_k) = 1$ implies that $F \cap J_k$ is divisible. By Lemma 5(i) $[Q, V] = \langle J_k \cap Q;\ 1 \leqq k \leqq 3 \rangle$. Let x be an element of $F \cap J_k$ for some $k \in \{1, 2, 3\}$ and y an element of $J_j \cap Q$ for some $j \in \{1, 2, 3\}$. If $j \neq k$, then v_k inverts y^x whence $y^x \in [Q, v_k] \subseteq [Q, V]$. Suppose $j = k$. Then by Lemma 3, $y^x = y$. Thus $F \cap J_k$ normalizes $[Q, V]$. Evidently so does $QC_F(V)$ and we have $QF \subseteq N = N_G([Q, V])$. This gives us [13, Theorem 3.17] that N is of finite index in G. Since by Lemma 2(iii) $Q/[Q, V]$ is Chernikov we have that $N/[Q, V]$ is Chernikov. So $R = \bigcap_x x^{-1}[Q, V]x$ is normal in G and G/R is Chernikov.

Let us now assume that $C_G(V)$ is finite. Then by Lemma 2(iii) $[Q, V]$ has finite index in Q. Since $[Q, V]$ is hyperabelian, by Lemma 1 QV is hyperabelian. Let $r = \text{rank}(F)$. We prove by induction on r that QFV is hyperabelian. If $r = 0$ then $F = 1$ and $QFV = QV$ is hyperabelian. Let $r \geqq 1$. Since $F = F_1 F_2 F_3$ [3, Lemma 6] without loss of generality we can assume that $\text{rank}(F_2 F_3) \leqq r - 1$ and consequently by induction $QF_2 F_3 V$ is hyperabelian. If QFV is not hyperabelian then there exists a quotient T of QFV such that T does not possess a non-trivial normal abelian subgroup. For the sake of simplicity we assume that $T = QFV$. Let B be a non-trivial abelian subgroup of Q, which is normal in QF_2F_3V. Put $B_1 = B \cap G_1$. If $B_1 = 1$ then $B \subseteq Z([Q, v_1])$. By Asar's result [1] $Q/[Q, v_1]$ is soluble so B centralizes some term of the derived series of Q. This shows that Q possesses a non-trivial characteristic abelian subgroup and we obtain a contradiction. Let $B_1 \neq 1$. By Lemma 3 $F_1 \subseteq C_G(B_1)$. whence $\bigcap_{x \in F_1} x^{-1}Bx$ is a non-trivial abelian subgroup which is normal in QFV. This contradicts our assumptions and proves that QF is hyperabelian. As QF is of finite index in G [13, Theorem 3.17], by Lemma 1 G is hyperabelian. The proof is now completed.

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