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ENUMERATING DISSECTIBLE POLYHEDRA BY THEIR AUTOMORPHISM GROUPS

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1. Introduction. A dissectible polyhedron is a natural extension of a concept whose history dates back to at least 1758 and Euler [**7**]—the concept of a dissection of a polygon. An interesting historical survey of dissections of a polygon is given by Brown [**4**]. Some approaches to the classical problem have been given by Moon and Moser [**9**] and by Guy [**8**]; the latter provides an approach which is the basis of the work in this paper. A summary of enumeration results on dissections of polygons and polyhedra by automorphism groups has been given by the authors [**2**]. Recent extensions of the problem have been investigated in a series of papers by Brown and Tutte [**3**; **5**; **14**; **15**] and by Takeo [**10**; **11**; **12**; **13**].

The mathematical object which is the basis of this study can be looked at in several ways in that it can be viewed as a particular type of 3-dimensional simplicial complex or 3-tree, as a type of polyhedron, and as a type of planar triangulation.

A *dissectible polyhedron* can be defined inductively in this way: A triangle and a tetrahedron are both dissectible polyhedra, and a dissectible polyhedron with $n + 1$ tetrahedra is obtainable from a dissectible polyhedron P with n tetrahedra by adding a new tetrahedron having precisely an exterior triangle in common with P. Figure 1 shows all dissectible polyhedra with up to five tetrahedra.

This concept is a special case of that of a 3-tree, with the essential difference being that in a 3-tree a triangle can be shared by any number of tetrahedra, while in a dissectible polyhedron it can be shared by at most two. Thus, a dissectible polyhedron is equivalent to a 3-tree embeddable in 3-space. For more formal definitions and a further discussion of 3-trees, see Beineke and Pippert [**1**], where labeled dissectible polyhedra are enumerated as a special case of labeled k-ball dissections.

Dissectible polyhedra are also related to triangulations of a polygon. From one point of view, they are the 3-dimensional analogue of the 2-dimensional concept of triangulating a polygon using nonintersecting diagonals. From another point of view, they are a special type of a triangulation of a disk with internal vertices. In the references cited earlier, Tutte and Takeo enumerated some variations of the labeled case in the form of decomposable rooted triangulated maps.

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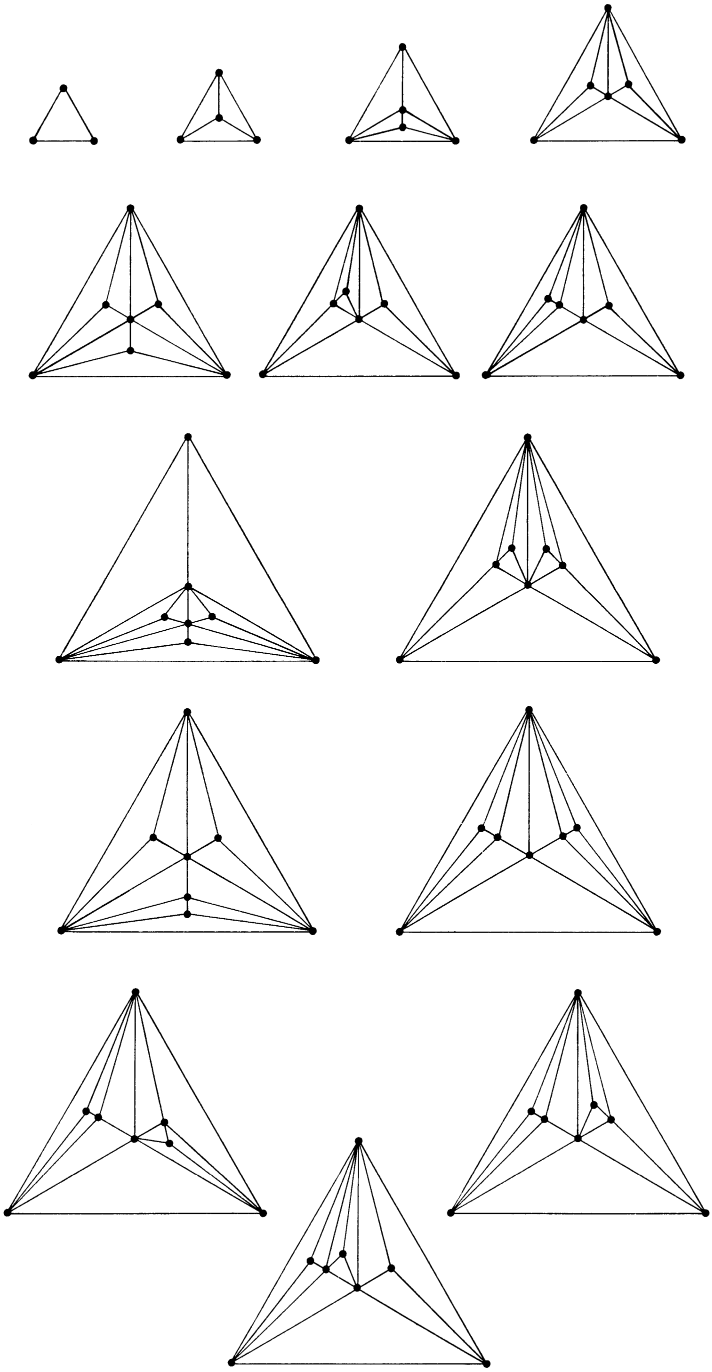

FIGURE 1

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The procedure for enumerating unlabeled dissectible polyhedra will be as follows. In Section 2, the types of automorphisms of dissectible polyhedra will be analyzed and the automorphism groups computed. Some relationships between special types of dissectible polyhedra will be explored in the lemmas and preliminary results obtained in Section 3. Section 4 comprises the heart of the theorem—the determination of formulas for the number of dissectible polyhedra having each of the automorphism groups computed in the second part. The main result is given in Section 5, along with a table of values for the number of dissectible polyhedra of each automorphism type when there are up to 14 tetrahedra.

2. Automorphism groups. All types of possible automorphisms of dissectible polyhedra will be determined, then all of the automorphism groups computed. The automorphism types are obtained by considering two cases according as to whether or not a tetrahedron remains fixed under the automorphism.

A couple of comments regarding notation and terminology are in order. Lower case Greek letters will represent faces of a tetrahedron, except that ϵ will denote the identity permutation. The names of permutations are an extension of those used by Coxeter [**6**] for the permutations of a tetrahedron.

Case 1. A tetrahedron is fixed: There are five possible types of permutations of the faces of the tetrahedron (including the identity), with names given for future reference:

(i)	ϵ	identity,
(ii)	$(\alpha\beta)\ (\gamma)\ (\delta)$	reflection,
(iii)	$(\alpha\beta)\ (\gamma\delta)$	digonal rotation,
(iv)	$(\alpha\beta\gamma)\ (\delta)$	trigonal rotation,
(v)	$(\alpha\beta\gamma\delta)$	tetragonal rotation.

Case 2. There is no fixed tetrahedron, so that, except for the trivial case, there is a pair of tetrahedra with a common face which are interchanged. The other three faces of one tetrahedron will be denoted by α, β, γ, and the corresponding faces of the other tetrahedron by α', β', γ'.

There are three possible types of permutations of the faces:

(vi)	$(\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma')$	reversal,
(vii)	$(\alpha\beta')\ (\beta\alpha')\ (\gamma\gamma')$	half-turn,
(viii)	$(\alpha\beta'\gamma\alpha'\beta\gamma')$	hexagonal rotation.

Since the permutations listed in Case 1 include all the types of permutations of a tetrahedron, it is clear that none have been omitted. In Case 2, there are two possibilities. If one face γ is interchanged with the corresponding face γ', one obtains permutation types (vi) and (vii). If no face is interchanged with the corresponding face of the other tetrahedron, there is only one possible permutation type due to the restrictions imposed by having to maintain adjacency properties of the faces.

With a list of the automorphism types of dissectible polyhedra, we can now obtain all of their automorphism groups. We observe that every automorphism group of a dissectible polyhedron is isomorphic to a subgroup of the automorphism group of a single tetrahedron or that of two tetrahedra with a common face.

The first part of the list contains those groups which leave a tetrahedron and one of its faces fixed. (These are subgroups of both types of groups.) The capital letters used in the listing will be used for reference later, e.g. B-symmetry, K-symmetric.

Figure 2 might be helpful in an analysis of these groups. Diagrams C and D as well as L through Q represent the six exterior faces of two joined tetrahedra

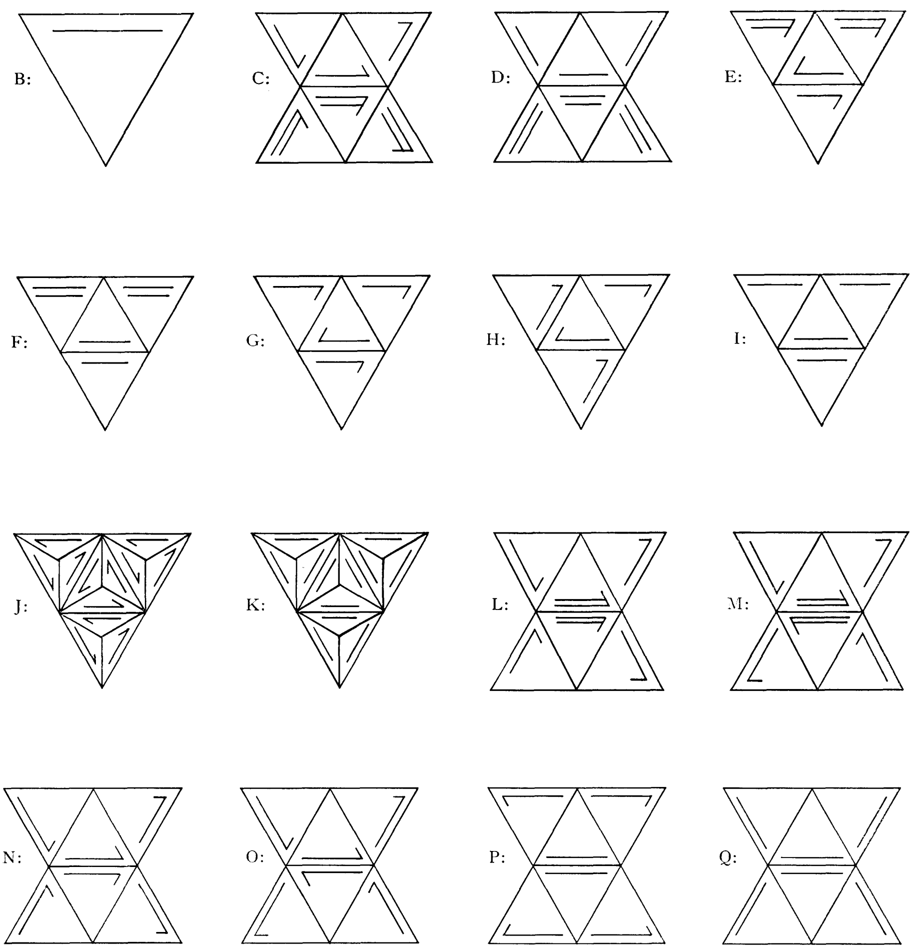

FIGURE 2

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which have been cut along certain edges. Similarly, diagrams E through I represent the four faces of a single tetrahedron, while B is a single face and J and K give the twelve exterior faces obtained by adding a new tetrahedron to each of the four faces of a tetrahedron. For a particular group, the symbols on the faces of the diagram indicate how the faces are permutable, with arrows implying an orientation.

A. The identity group.

B. Generated by a reflection,

$$(\alpha\beta),\ \epsilon$$

(isomorphic to the symmetric group on two elements).

C. Generated by a trigonal rotation

$$(\alpha\beta\gamma),\ (\alpha\gamma\beta),\ \epsilon$$

(isomorphic to the cyclic group on three elements).

D. Generated by a trigonal rotation and a reflection (the symmetries of a triangle),

$$(\alpha\beta\gamma),\ (\alpha\gamma\beta),\ (\alpha\beta),\ (\alpha\gamma),\ (\beta\gamma),\ \epsilon$$

(isomorphic to the symmetric group on three elements).

We next list those groups which leave a tetrahedron but none of its faces fixed.

E. Generated by a digonal rotation,

$$(\alpha\beta)\ (\gamma\delta),\ \epsilon$$

(isomorphic to the symmetric group on two elements).

F. Generated by two reflections,

$$(\alpha\beta),\ (\gamma\delta),\ (\alpha\beta)\ (\gamma\delta),\ \epsilon$$

(isomorphic to the Klein 4-group).

G. Generated by two digonal rotations,

$$(\alpha\beta)\ (\gamma\delta),\ (\alpha\gamma)\ (\beta\delta),\ (\alpha\delta)\ (\beta\gamma),\ \epsilon$$

(also isomorphic to the Klein 4-group).

H. Generated by a tetragonal rotation,

$$(\alpha\beta\gamma\delta),\ (\alpha\gamma)\ (\beta\delta),\ (\alpha\gamma\beta\delta),\ \epsilon$$

(isomorphic to the cyclic group on four elements).

I. Generated by a tetragonal rotation and a reflection,

$$(\alpha\beta\gamma\delta),\ (\alpha\gamma)\ (\beta\delta),\ (\alpha\delta\gamma\beta),\ (\alpha\gamma),$$
$$(\alpha\delta)\ (\beta\gamma),\ (\beta\gamma),\ (\alpha\beta)\ (\gamma\delta),\ \epsilon$$

(isomorphic to the dihedral group on four elements).

J. Generated by a trigonal rotation and a digonal rotation,

$$(\alpha\beta\gamma),\ (\alpha\gamma\beta),\ (\alpha\beta\delta),\ (\alpha\delta\beta),\ (\alpha\gamma\delta),$$
$$(\alpha\delta\gamma),\ (\beta\gamma\delta),\ (\beta\delta\gamma),\ (\alpha\beta)\ (\gamma\delta),$$
$$(\alpha\gamma)\ (\beta\delta),\ (\alpha\delta)\ (\beta\gamma),\ \epsilon$$

(isomorphic to the alternating group on four elements).

K. Generated by a tetragonal rotation and a trigonal rotation,

$$(\alpha\beta\gamma\delta),\ (\alpha\gamma)\ (\beta\delta),\ (\alpha\delta\gamma\beta),\ (\alpha\beta\gamma),$$
$$(\alpha\gamma\beta),\ (\alpha\gamma\beta\delta),\ (\alpha\beta)\ (\gamma\delta),\ (\alpha\delta\beta\gamma),$$
$$(\alpha\beta\delta),\ (\alpha\delta\beta),\ (\alpha\beta\delta\gamma),\ (\alpha\delta)\ (\beta\gamma),$$
$$(\alpha\gamma\delta\beta),\ (\alpha\gamma\delta),\ (\alpha\delta\gamma),\ (\alpha\beta),\ (\alpha\gamma),$$
$$(\alpha\delta),\ (\beta\gamma),\ (\beta\delta),\ (\gamma\delta),\ (\beta\gamma\delta),$$
$$(\beta\delta\gamma),\ \epsilon$$

(isomorphic to the symmetric group on four elements).

Now come those groups which have a triangle but no tetrahedron fixed.

L. Generated by a reversal,

$$(\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma'),\ \epsilon$$

(isomorphic to the symmetric group on two elements).

M. Generated by a half-turn,

$$(\alpha\beta')\ (\beta\alpha')\ (\gamma\gamma'),\ \epsilon$$

(isomorphic to the symmetric group on two elements).

N. Generated by a hexagonal rotation (or by a trigonal rotation and a reversal),

$$(\alpha\beta'\gamma\alpha'\beta\gamma'),\ (\alpha\gamma\beta)\ (\alpha'\gamma'\beta'),\ (\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma'),$$
$$(\alpha\beta\gamma)\ (\alpha'\beta'\gamma'),\ (\alpha\gamma'\beta\alpha'\gamma\beta'),\ \epsilon$$

(isomorphic to the cyclic group on six elements).

O. Generated by a trigonal rotation and a half-turn,

$$(\alpha\beta\gamma)\ (\alpha'\beta'\gamma'),\ (\alpha\gamma\beta)\ (\alpha'\gamma'\beta'),$$
$$(\alpha\alpha')\ (\beta\gamma')\ (\gamma\beta'),\ (\beta\beta')\ (\alpha\gamma')\ (\gamma\alpha'),$$
$$(\gamma\gamma')\ (\alpha\beta')\ (\beta\alpha'),\ \epsilon$$

(isomorphic to the symmetric group on three elements).

P. Generated by a reflection and a reversal,

$$(\alpha\beta)\ (\alpha'\beta'),\ (\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma'),$$
$$(\alpha\beta')\ (\beta\alpha')\ (\gamma\gamma'),\ \epsilon$$

(isomorphic to the Klein 4-group).

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Q. Generated by a hexagonal rotation and a reflection,

$$(\alpha\beta'\gamma\alpha'\beta\gamma'),\ (\alpha\gamma\beta)\ (\alpha'\gamma'\beta'),\ (\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma'),$$
$$(\alpha\beta\gamma)\ (\alpha'\beta'\gamma'),\ (\alpha\gamma'\beta\alpha'\gamma\beta'),\ (\alpha\beta)\ (\alpha'\beta'),$$
$$(\alpha\gamma)\ (\alpha'\gamma'),\ (\beta\gamma)\ (\beta'\gamma'),\ (\alpha\alpha')\ (\beta\gamma')\ (\gamma\beta'),$$
$$(\beta\beta')\ (\alpha\gamma')\ (\alpha'\gamma),\ (\gamma\gamma')\ (\alpha\beta')\ (\alpha'\beta),\ \epsilon$$

(isomorphic to the dihedral group on six elements).

That completes the list of possible automorphism groups, as indicated in the following discussion.

The groups A through K are simply a listing of all possible subgroups of the symmetric group S_4 on four elements, with A through D being those with at least one fixed element. It is not difficult to show, using arguments involving order and the types of permutation products, that these include all possibilities up to isomorphism.

Because groups which interchange two tetrahedra with a shared face are less common, we present here a brief argument indicating that groups L through Q exhaust all possibilities for this type of symmetry group.

The entire automorphism group of two joined tetrahedra is the product of the symmetric groups on 2 and 3 elements, and hence has 12 elements, six of which leave the tetrahedra fixed:

$$\epsilon,\ (\alpha\beta\gamma)\ (\alpha'\beta'\gamma'),\ (\alpha\gamma\beta)\ (\alpha'\gamma'\beta'),$$
$$(\alpha\beta)\ (\alpha'\beta'),\ (\alpha\gamma)\ (\alpha'\gamma'),\ (\beta\gamma)\ (\beta'\gamma')$$

and six of which interchange them:

$$(\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma'),\ (\alpha\beta'\gamma\alpha'\beta\gamma'),\ (\alpha\gamma'\beta\alpha'\gamma\beta'),$$
$$(\alpha\alpha')\ (\beta\gamma')\ (\beta'\gamma),\ (\alpha\gamma')\ (\alpha'\gamma)\ (\beta\beta'),\ (\alpha\beta')\ (\alpha'\beta)\ (\gamma\gamma').$$

Each group we are seeking must contain one of these. Each of the groups L, M, and N is generated by a single permutation, and Q consists of all 12 permutations. Hence, no subgroup other than N or Q can contain an element of order 6. The only other way to obtain a group of order 6 is to have elements of orders 2 and 3, namely one of two cases. Now

$$(\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma')\cdot(\alpha\beta\gamma)\ (\alpha'\beta'\gamma') = (\alpha\beta'\gamma\alpha'\beta\gamma'),$$

and

$$(\alpha\alpha')\ (\beta\gamma')\ (\beta'\gamma)\cdot(\alpha\beta\gamma)\ (\alpha'\beta'\gamma') = (\alpha\beta')\ (\alpha'\beta)\ (\gamma\gamma').$$

The first yields N, which we had, and the second yields O. We can have no subgroup here of order 3, unless there is one of order 6, so 4 is the only other

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order possible. In this case, we look at four possibilities for combining two elements of order 2:

$$(\alpha\beta)\ (\alpha'\beta') \cdot (\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma') = (\alpha\beta')\ (\alpha'\beta)\ (\gamma\gamma')$$
$$(\alpha\beta)\ (\alpha'\beta') \cdot (\alpha\beta')\ (\alpha'\beta)\ (\gamma\gamma') = (\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma')$$
$$(\alpha\alpha')\ (\beta\beta')\ (\gamma\gamma') \cdot (\alpha\beta')\ (\alpha'\beta)\ (\gamma\gamma') = (\alpha\beta)\ (\alpha'\beta')$$
$$(\alpha\beta)\ (\alpha'\beta') \cdot (\alpha\alpha')\ (\beta\gamma')\ (\beta'\gamma) = (\alpha\gamma'\beta\alpha'\gamma\beta').$$

The first three all give the same group P, while the fourth gives an element of order 6.

Therefore, there are precisely these seventeen kinds of automorphism groups for dissectible polyhedra.

3. Computational lemmas. The lemmas of this section will serve to simplify succeeding calculations, and seem to be representative of the types of relationships often encountered in enumeration problems.

A recurrence formula for the number $T(n)$ of dissectible polyhedra rooted at an exterior face has been established [**1**]. Because of its brevity, the proof is included here for completeness.

LEMMA 1.

$$T(n) = \sum_{i+j+k=n-1} T(i)T(j)T(k).$$

Proof. Begin with the root triangle and add a fourth vertex to form a tetrahedron. Allocate the remaining $n - 4$ vertices to the three new faces of the tetrahedron. With each face as root triangle and the original edge oriented, say counterclockwise, form new rooted polyhedra on these faces with the appropriate number of vertices (see Figure 3). Each rooted dissection is constructible just once in this manner, so the result follows.

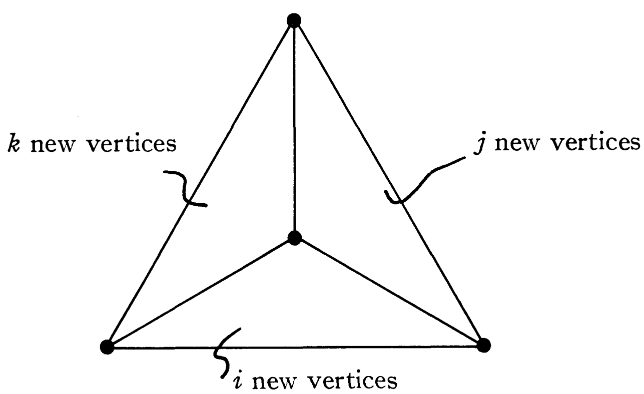

FIGURE 3

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The next result is similar in nature.

LEMMA 2. $(3n + 1)T(n) = (n + 1) \sum_{i+j=n} T(i)T(j)$.

Proof. Since a dissectible polyhedron with n tetrahedra has $3n + 1$ triangles, the left-hand side gives the number of rooted dissectible polyhedra with a distinguished triangle (in addition to the root face). To see that the right-hand side gives the same number, form two rooted dissectible polyhedra with i and j tetrahedra, where $i + j = n$. Join them on their root triangles, which now become the distinguished triangle, and choose any of the $2n + 2$ exterior triangles as the new root. Summing over all possible choices of i and j gives each of these dissectible polyhedra twice, and from this the equality follows.

In addition to the total number $T(n)$ of rooted dissectible polyhedra with n tetrahedra, there are some other combinations which will be found useful. Let $U(n)$ denote the number of dissectible polyhedra which are symmetric in the vertical axis, that is, when the direction of the root edge is reversed, the map is unchanged.

LEMMA 3. $U(n) = \sum_{2r+s=n-1} T(r)U(s)$.

Proof. As in the proof of Lemma 1, we begin with a tetrahedron (see Figure 3), and add three rooted polyhedra. Two of these, having r vertices each, must be alike (actually, mirror images), while the third, with s vertices, must have reflectional symmetry. By choosing these in all possible ways, we obtain the result.

LEMMA 4.

$$U(n) = \begin{cases} T(n/2), & \textit{if } n \textit{ is even} \\ [(3n-1)/(n+1)]T((n-1)/2), & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Case 1. n even: We proceed by induction. The result clearly holds for two tetrahedra; assume it holds when there are fewer than n tetrahedra. In the formula of Lemma 3, when n is even, s must be odd, and if $2j + k = s - 1$, k is even, say $k = 2i$. Then

$$U(s) = \sum_{2j+k=s-1} T(j)U(k) = \sum_{2i+2j=s-1} T(i)T(j)$$

so that

$$U(n) = \sum_{2(r+i+j)=n-2} T(r)T(i)T(j)$$

$$= T(n/2) \text{ (by Lemma 1).}$$

Case 2. n odd: In this case, s is even, say $s = 2q$, and by Case 1, $U(s) = T(s/2)$. Hence

$$\begin{aligned} U(n) &= \sum_{2r+s=n-1} T(r)U(s) \\ &= \sum_{2r+s=n-1} T(r)T(s/2) \\ &= \sum_{2r+2q=n-1} T(r)T(q) \\ &= \frac{3((n-1)/2)+1}{((n-1)/2)+1} T((n-1)/2) \text{ (by Lemma 2)} \\ &= ((3n-1)/(n+1))T((n-1)/2)). \end{aligned}$$

In the course of proving Lemma 4, a useful formula arose in Case 1, which we note here for future reference.

LEMMA 5. $U(2r + 1) = \sum_{i+j=r} T(i)T(j)$.

We have seen several sums involving combinations of $T(n)$'s and $U(n)$'s. Another is the following, which we denote $V(n)$:

$$V(n) = \sum_{r+s=n} U(r)U(s).$$

LEMMA 6.

$$V(n) = \begin{cases} 2U(n+1), & \textit{if } n \textit{ is odd}, \\ 2U(n+1) - U(n), & \textit{if } n \textit{ is even}. \end{cases}$$

Proof. We first consider the case of n odd, when one of r and s is always even and each summand appears twice. Then

$$\begin{aligned} V(n) &= \sum_{r+s=n} U(r)U(s) \\ &= 2\sum_{i+2j=n} U(i)U(2j) \\ &= 2\sum_{i+2j=n} U(i)T(j) \\ &= 2U(n+1) \quad \text{(by Lemma 3)}. \end{aligned}$$

The case of n even is a bit more difficult. We first split the sum into two:

$$\begin{aligned} V(n) &= \sum_{r+s=n} U(r)U(s) \\ &= \sum_{i+j=n/2} U(2i)U(2j) + \sum_{i+j=n/2-1} U(2i+1)U(2j+1). \end{aligned}$$

The first of these is easily simplified:

$$\begin{aligned} \sum_{2i+2j=n} U(2i)U(2j) &= \sum_{2i+2j=n} T(i)T(j) \\ &= U(n+1). \end{aligned}$$

The second sum we simplify in the following way:

$$\begin{aligned}\sum_{2(i+j)=n-2} U(2i+1)U(2j+1) &= \sum_{2(i+j)=n-2} \left(\sum_{p+q=i} T(p)T(q)\right)\left(\sum_{r+s=j} T(r)T(s)\right) \\ &= \sum_{2(p+q+r+s)=n-2} T(p)T(q)T(r)T(s) \\ &= \sum_{2p+2(q+r+s)=n-2} T(p)(T(q)T(r)T(s)) \\ &= \sum_{p=0}^{n/2-1} T(p)T(n/2-p) \\ &= \sum_{p=0}^{n/2} T(p)T(n/2-p) - T(n/2) \\ &= U(n+1) - U(n).\end{aligned}$$

Therefore

$$V(n) = 2U(n+1) - U(n).$$

For completeness, we conclude this section with a theorem tabulating the values of $T(n)$, $U(n)$, and $V(n)$ in factorial form.

THEOREM 1.

$$T(n) = \frac{(3n)!}{n!(2n+1)!}$$

$$U(n) = \begin{cases} \dfrac{(3m)!}{m!(2m+1)!}, & \textit{for } n = 2m \\ \dfrac{(3m+1)!}{(m+1)!(2m+1)!}, & \textit{for } n = 2m+1 \end{cases}$$

$$V(n) = \begin{cases} \dfrac{(3m)!(5m+1)}{(m+1)!(2m+1)!}, & \textit{for } n = 2m \\ \dfrac{6(3m+2)!}{m!(2m+3)!}, & \textit{for } n = 2m+1. \end{cases}$$

4. The enumeration process. We now determine formulas for the number of dissectible polyhedra for each of the symmetry groups. If the group is listed above as X, the number of dissectible polyhedra with n tetrahedra is denoted $X(n)$. Some of the quantities are expressed in terms of others as well as the numbers $T(n)$, $U(n)$, and $V(n)$, which are zero if the argument is not an integer. The diagrams of Figure 2 are again helpful in deriving the formulas. The symbols on the faces now indicate the presence of rooted dissectible polyhedra. Similar symbols represent isomorphic structures placed accordingly, and the absence of an arrow indicates a U-type symmetry in the polyhedron. We work from the highly symmetric cases to those which are less so.

K. A dissectible polyhedron with this symmetry is obtained when a tetrahedron is added to each face of the root tetrahedron, and then a U-type dissectible polyhedron is added to each of the 12 faces in the appropriate way. Therefore, for $n > 1$, $n - 5$ must be a multiple of 12, and

$$K(n) = U((n-5)/12), \text{ with } K(1) = 1.$$

J. This type of dissectible polyhedron arises when four tetrahedra are added to the base tetrahedron as before, but a dissectible polyhedron which is not of U-type is added to each of the twelve faces. Thus, from the T-type we must take away the U-type. Furthermore, however, each is counted twice, from the two possible orientations. Hence,

$$J(n) = \tfrac{1}{2}[T((n-5)/12) - K(n)].$$

I. When each face of a tetrahedron has the same U-type dissectible polyhedron attached, this type of symmetry arises. This of course includes the K-symmetric case, so

$$I(n) = U((n-1)/4) - K(n).$$

H. In this case, the same T-type dissectible polyhedron must be added to each of the four faces of the base tetrahedron (as shown). Each K-symmetry and each I-symmetry arises here, but a J-symmetry does not (without being a K-symmetry); that is, group J does not contain H as a subgroup. Each dissectible polyhedron is counted twice here, because of the orientation. Hence,

$$H(n) = \tfrac{1}{2}[T((n-1)/4) - I(n) - K(n)].$$

G. As in the preceding case, the same T-type dissectible polyhedron must be appropriately added to each face of the base tetrahedron. This includes the first three cases, with each K-symmetry arising once, each J-symmetry twice (once for each orientation) and each I-symmetry three times (once for each of the three choices of the axis of symmetry of the U-type dissectible polyhedron). Furthermore, each resulting dissectible polyhedron is counted six times, for the reflection and rotation possibilities. Therefore,

$$G(n) = \tfrac{1}{6}[T((n-1)/4) - 3I(n) - 2J(n) - K(n)].$$

F. This type of automorphism group is obtained when one U-type dissectible polyhedron is added to two faces of a tetrahedron and another of U-type to the other two faces in an appropriate manner. Each I-symmetry and each K-symmetry arises here, and from interchanging the two kinds of U-type dissectible polyhedra, each F-symmetry occurs twice. Hence,

$$F(n) = \frac{1}{2}\left[\sum_{2i+2j=n-1} U(i)U(j) - I(n) - K(n)\right].$$

By the definition of $V(n)$, this can be written as

$$F(n) = \tfrac{1}{2}[V((n-1)/2) - I(n) - K(n)].$$

E. This is similar to case F, except that E includes all the preceding cases. Two kinds of T-type dissectible polyhedra are appropriately added to the faces of a tetrahedron. Doing this in all possible ways, we get each K-symmetry once and each J-symmetry twice (because of orientation). Furthermore, each I-symmetry is obtained three times (depending on where the axis of symmetry occurs), and each H-symmetry is obtained twice (orientation again). Each G-symmetry is obtained six times (three rotational choices for the T-type in G and two possible orientations each), and each F-symmetry twice (because of interchange). Each resulting object appears four times, twice for orientation and twice for interchange, so

$$E(n) = \frac{1}{4}\left[\sum_{2i+2j=n-1} T(i)T(j) - 2F(n) - 6G(n) - 2H(n) - 3I(n) - 2J(n) - K(n)\right].$$

By Lemma 5, the first term is equal to $U(n)$ when n is odd. When n is even, all the other terms including $E(n)$ vanish; and since $T(n/2)$ is equal to $U(n)$ for n even and zero for n odd, a single expression can be obtained:

$$E(n) = \tfrac{1}{4}[U(n) - T(n/2) - 2F(n) - 6G(n) - 2H(n) - 3I(n) - 2J(n) - K(n)].$$

We now go back to the groups in which two adjacent tetrahedra are interchanged, beginning with the most symmetric of these. In these cases, we usually view the construction as placing two tetrahedra together and putting dissectible polyhedra on each of the six faces.

Q. In this case we can put any U-type dissectible polyhedron on all six faces, so

$$Q(n) = U((n-2)/6).$$

P. In this instance, we have a U-type dissectible polyhedron on two faces and some two T-types on the other four. All the Q-symmetric ones are counted here, so

$$P(n) = \sum_{2(i+2j)=n-2} U(i)T(j) - Q(n),$$

which is, by Lemma 3,

$$P(n) = U(n/2) - Q(n).$$

O. For this automorphism group, we put the same T-type dissectible polyhedron on all six faces. From the number of these, we must subtract the Q-symmetric ones and then take one-half the remainder because of the two possible orientations. Hence

$$O(n) = \tfrac{1}{2}[T((n-2)/6) - Q(n)].$$

N. As in the case of O-symmetry, we begin with the same T-type dissectible polyhedron on each face (but their orientations differ in the two cases). Each

Q-symmetry again appears, and we must again divide by 2 because of the orientation. So, we get the same expression as for $O(n)$:

$$N(n) = \tfrac{1}{2}[T((n-2)/6) - Q(n)].$$

M. Here, we begin by putting each of three kinds of T-type dissectible polyhedra on two faces each. In this case, each P-symmetry and each Q-symmetry is counted once and each O-symmetry is counted twice because of orientation. Each M-symmetry has been counted twice, so that

$$M(n) = \frac{1}{2}\left[\sum_{2(i+j+k)=n-2} T(i)T(j)T(k) - 2O(n) - P(n) - Q(n)\right].$$

By Lemma 1, this simplifies to

$$M(n) = \tfrac{1}{2}[T(n/2) - 2O(n) - P(n) - Q(n)].$$

L. As in the M-symmetric case, we can put each of three T-type dissectible polyhedra on two faces. Several quantities must be subtracted: each Q-symmetry can arise just once, each N-symmetry twice, and each P-symmetry three times. Furthermore, in the result we have counted each L-symmetry six times. Again using Lemma 1, we have

$$L(n) = \tfrac{1}{6}[T(n/2) - 2N(n) - 3P(n) - Q(n)].$$

We now come to some of the simplest of the symmetry groups, but their solutions are among the most difficult to find.

D. When we have the symmetries of a triangle, we can again put dissectible polyhedra on faces. However there are now two cases to consider. First, we can have two U-types on three faces each. Then the Q-symmetries and twice the K-symmetries must be subtracted, and the result must be halved because of the possibility of interchanging the two U-types. In this case, $n \equiv 2 \pmod 3$ and we have

$$D(n) = \frac{1}{2}\left[\sum_{3(i+j)=n-2} U(i)U(j) - 2K(n) - Q(n)\right].$$

By the definition of $V(n)$, this gives

$$D(n) = \tfrac{1}{2}[V((n-2)/3) - 2K(n) - Q(n)] \quad \text{for } n \equiv 2 \pmod 3.$$

In the second case, we have a U-type dissectible polyhedron on three faces of a tetrahedron, so

$$D(n) = U((n-1)/3) \quad \text{for } n \equiv 1 \pmod 3.$$

We observe that all of the terms in the expression for $D(n)$, $n \equiv 2 \pmod 3$, vanish for other values of n. A similar observation for the case $n \equiv 1 \pmod 3$, together with the fact that $D(n) = 0$ if $n \equiv 0 \pmod 3$, allows us to combine the expressions to write for all n,

$$D(n) = \tfrac{1}{2}[2U((n-1)/3) + V((n-2)/3) - 2K(n) - Q(n)].$$

C. For this type of symmetry, we again have two cases to consider. First, two T-type dissectible polyhedra can be put on three faces each. This gives each D-symmetry twice, each J-symmetry four times, each K-symmetry twice, each N-symmetry twice, each Q-symmetry once, and each O-symmetry twice. Each C-symmetry has then been counted four times, so we get

$$C(n) =$$
$$\frac{1}{4}\left[\sum_{3(i+j)=n-2} T(i)T(j) - 2D(n) - 4J(n) - 2K(n) - 2N(n) - 2O(n) - Q(n)\right],$$

which, by Lemma 5, is

$$C(n) = \tfrac{1}{4}[U((2n-1)/3) - 2D(n) - 4J(n) - 2K(n) - 2N(n) - 2O(n) - Q(n)] \quad \text{for } n \equiv 2 \pmod 3.$$

In the other case, we simply put a T-type on three of the faces of a tetrahedron. We must subtract the D-symmetries and halve the result because of orientation, obtaining

$$C(n) = \tfrac{1}{2}[T((n-1)/3) - D(n)] \quad \text{for } n \equiv 1 \pmod 3.$$

Reasoning as in the case of $D(n)$, we can combine the terms to obtain, for all n,

$$C(n) = \tfrac{1}{4}[2T((n-1)/3) + U((2n-1)/3) - 2D(n) - 4J(n) - 2K(n) - 2N(n) - 2O(n) - Q(n)].$$

B. The analysis of this case requires a rather different procedure than we have used in previous cases. We cannot, as might first appear, simply begin with two U-types of dissectible polyhedra with a common face, because a B-symmetric dissectible polyhedron can be obtained from many pairs. In other words, there may be many interior triangles for which both attached dissectible polyhedra are of U-type. However, there are only two exterior triangles which give U-types. So we simply begin with $U(n)$ and note that we must divide by 2 at the end. By this same reasoning it follows that each D-symmetry is counted twice. Also each F-symmetry occurs twice because of its being "U-type in two ways." But because of their own "top-to-bottom" symmetry, each I-, K-, P-, and Q-symmetry is counted just once. Hence

$$B(n) = \tfrac{1}{2}[U(n) - 2D(n) - 2F(n) - I(n) - K(n) - P(n) - Q(n)].$$

A. The dissectible polyhedra with the identity group are counted by using all the others. The number of ways in which a dissectible polyhedron with X-symmetry (where X can be any one of our 17 types) can be rooted is $6(2n+2)/(\text{order of group})$. This is because (a) given an exterior face there are 6 possible vertex/edge rootings and (b) the number of kinds (or orbits) of faces is $2n+2$ divided by the number of symmetries. Thus

$$T(n) = 12(n+1)\sum_X \frac{X(n)}{\sigma(X)},$$

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where the sum is over all seventeen types of automorphism groups ($X = A, B, \ldots, Q$), $\sigma(X)$ is the order of the group, and $X(n)$ is the number of dissectible polyhedra with that group. This expression gives us $A(n)$.

We conclude this section with the theorem giving the number of dissectible polyhedra of each automorphism type. They can all be reduced to expressions in $T(n)$, $U(n)$, and $V(n)$, for which formulas are given in Theorem 1.

THEOREM 2.

$$A(n) = \frac{T(n)}{12(n+1)} - \sum_{X \neq A} \frac{X(n)}{\sigma(X)}.$$

$$B(n) = \tfrac{1}{2}(U(n) - 2D(n) - 2F(n) - I(n) - K(n) - P(n) - Q(n)).$$

$$C(n) = \tfrac{1}{4}(2T((n-1)/3) + U((2n-1)/3) - 2D(n) - 4J(n) - 2K(n) - 2N(n) - 2O(n) - Q(n)).$$

$$D(n) = \tfrac{1}{2}(2U((n-1)/3) + V((n-2)/3) - 2K(n) - Q(n)).$$

$$E(n) = \tfrac{1}{4}(U(n) - T(n/2) - 2F(n) - 6G(n) - 2H(n) - 3I(n) - 2J(n) - K(n)).$$

$$F(n) = \tfrac{1}{2}(V((n-1)/2) - I(n) - K(n)).$$

$$G(n) = \tfrac{1}{6}(T((n-1)/4) - 3I(n) - 2J(n) - K(n)).$$

$$H(n) = \tfrac{1}{2}(T((n-1)/4) - I(n) - K(n)).$$

$$I(n) = U((n-1)/4) - K(n).$$

$$J(n) = \tfrac{1}{2}(T((n-5)/12) - K(n)).$$

$$K(n) = U((n-5)/12) \text{ for } n > 1 \text{ and } K(1) = 1.$$

$$L(n) = \tfrac{1}{6}(T(n/2) - 2N(n) - 3P(n) - Q(n)).$$

$$M(n) = \tfrac{1}{2}(T(n/2) - 2O(n) - P(n) - Q(n)).$$

$$N(n) = \tfrac{1}{2}(T((n-2)/6) - Q(n)).$$

$$O(n) = \tfrac{1}{2}(T((n-2)/6) - Q(n)).$$

$$P(n) = U(n/2) - Q(n).$$

$$Q(n) = U((n-2)/6).$$

5. The main result. By adding together the numbers of dissectible polyhedra computed in the previous section for each of the seventeen permutation groups, we obtain the total number of different (unlabeled and unrooted) dissectible polyhedra:

$$S(n) = \sum X(n).$$

While we shall not carry out the computations here, it turns out that many of the terms can be combined in obtaining the sum. We conclude with a theorem giving the result, which is surprisingly simple, together with a table of values for small n.

THEOREM 3. *The number of dissectible polyhedra with n tetrahedra is*

$$S(n) = \frac{1}{12(n+1)} T(n) + \frac{5}{24} T\left(\frac{n}{2}\right) + \frac{1}{3} T\left(\frac{n-1}{3}\right) + \frac{1}{4} T\left(\frac{n-1}{4}\right) + \frac{1}{6} T\left(\frac{n-2}{6}\right) + \frac{3}{8} U(n) + \frac{1}{6} U\left(\frac{2n-1}{3}\right),$$

where a term $T(m)$ *or* $U(m)$ *is zero if* m *is not an integer, and if* m *is an integer,*

$$T(m) = U(2m) = \frac{(3m)!}{m!(2m+1)!}$$

$$U(2m+1) = \frac{(3m+1)!}{(m+1)!\,(2m+1)!}$$

TABLE

		Number of dissectible polyhedra with n tetrahedra													
Group	Order	$n = 1$	2	3	4	5	6	7	8	9	10	11	12	13	14
A	1	0	0	0	0	2	11	71	370	2005	10,682	58,167	320,116	1,789,210	10,121,965
B	2	0	0	0	0	2	5	11	25	66	131	349	708	1,911	3,856
C	3	0	0	0	0	0	0	1	1	0	5	6	0	26	32
D	6	0	0	0	1	0	0	1	1	0	2	3	0	3	5
E	2	0	0	0	0	1	0	6	0	32	0	176	0	952	0
F	4	0	0	1	0	1	0	3	0	5	0	12	0	23	0
G	4	0	0	0	0	0	0	0	0	0	0	0	0	1	0
H	4	0	0	0	0	0	0	0	0	1	0	0	0	5	0
I	8	0	0	0	0	0	0	0	0	1	0	0	0	2	0
J	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
K	24	1	0	0	0	1	0	0	0	0	0	0	0	0	0
L	2	0	0	0	0	0	1	0	8	0	42	0	232	0	1,277
M	2	0	0	0	1	0	5	0	26	0	133	0	708	0	3,860
N	6	0	0	0	0	0	0	0	0	0	0	0	0	0	1
O	6	0	0	0	0	0	0	0	0	0	0	0	0	0	1
P	4	0	0	0	1	0	2	0	2	0	7	0	12	0	29
Q	12	0	1	0	0	0	0	0	1	0	0	0	0	0	1
Sum		1	1	1	3	7	24	93	434	2110	11,002	58,713	321,776	1,792,133	10,131,027

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