

MONOCHROMATIC HOMOTHETIC COPIES
OF $\{1, 1 + s, 1 + s + t\}$

TOM C. BROWN, BRUCE M. LANDMAN AND MARNI MISHNA

ABSTRACT. For positive integers s and t , let $f(s, t)$ denote the smallest positive integer N such that every 2-colouring of $[1, N] = \{1, 2, \dots, N\}$ has a monochromatic homothetic copy of $\{1, 1 + s, 1 + s + t\}$.

We show that $f(s, t) = 4(s + t) + 1$ whenever s/g and t/g are not congruent to 0 (modulo 4), where $g = \gcd(s, t)$. This can be viewed as a generalization of part of van der Waerden's theorem on arithmetic progressions, since the 3-term arithmetic progressions are the homothetic copies of $\{1, 1 + 1, 1 + 1 + 1\}$. We also show that $f(s, t) = 4(s + t) + 1$ in many other cases (for example, whenever $s > 2t > 2$ and t does not divide s), and that $f(s, t) \leq 4(s + t) + 1$ for all s, t .

Thus the set of homothetic copies of $\{1, 1 + s, 1 + s + t\}$ is a set of triples with a particularly simple Ramsey function (at least for the case of two colours), and one wonders what other "natural" sets of triples, quadruples, etc., have simple (or easily estimated) Ramsey functions.

1. Introduction. Van der Waerden's Theorem on Arithmetic Progressions [5] states that for every positive integer k there exists a smallest positive integer $w(k)$ such that for every 2-colouring of $[1, w(k)] = \{1, 2, \dots, w(k)\}$, there is a monochromatic k -term arithmetic progression. (In other words, if $[1, w(k)]$ is partitioned in any way into two parts A and B , then either A or B must contain a k -term arithmetic progression.) The only known non-trivial values of $w(k)$ are $w(3) = 9$, $w(4) = 35$, $w(5) = 178$. Furthermore the estimation of the function $w(k)$ for large k is one of the most outstanding (and presumably one of the most difficult) problems in Ramsey theory. For a discussion of this, see [2].

The function $w(k)$ is often called the *Ramsey function* for the set of k -term arithmetic progressions. Landman and Greenwell ([3], [4]) considered the Ramsey function $g(n)$ of the set of all n -term sequences that are homothetic copies (see the definition below) of $\{1, 2, 2 + t, 2 + t + t^2, \dots, 2 + t + t^2 + \dots + t^{n-2}\}$ for some positive integer t . They obtained a lower bound for $g(n)$ and an upper bound for $g^{(r)}(3)$, where the (r) indicates that r colours are used. Other "substitutes" for the set of k -term arithmetic progressions were introduced in [1].

In contrast, in this paper we consider the Ramsey function associated with a much smaller set of sequences, namely the set of homothetic copies of $\{1, 1 + s, 1 + s + t\}$ for given positive integers s and t .

A *homothetic copy* of $\{1, 1 + s, 1 + s + t\}$ is any set of the form $\{x, x + ys, x + ys + yt\}$, where x and y are positive integers. From now on, let us agree to use the term " (s, t) -progression"

The first and third authors were partially supported by NSERC.

Received by the editors January 4, 1996.

AMS subject classification: 05D10.

©Canadian Mathematical Society 1997.

to refer to a homothetic copy of $\{1, 1 + s, 1 + s + t\}$.

Instead of considering 3-term arithmetic progressions, as in the case $k = 3$ of van der Waerden's theorem, we consider the set of (s, t) -progressions for given positive integers s and t . (Note that the $(1, 1)$ -progressions are the 3-term arithmetic progressions.)

For positive integers s and t we define $f(s, t)$ to be the smallest positive integer N such that every 2-colouring of $[1, N]$ has a monochromatic (s, t) -progression. Note that $f(s, t) = f(t, s)$. We will use this fact several times.

We show that for all positive integers s and t , if $s/g \not\equiv 0$ and $t/g \not\equiv 0 \pmod{4}$, where $g = \gcd(s, t)$, then $f(s, t) = 4(s + t) + 1$. A special case of this is $w(3) = f(1, 1) = 9$. Thus this result can be viewed as a generalization of the case $k = 3$ of van der Waerden's theorem.

We also show that $f(s, t) \leq 4(s + t) + 1$ for all s and t , and we show that even if $s/g \equiv 0$ or $t/g \equiv 0 \pmod{4}$, the equality $f(s, t) = 4(s + t) + 1$ still holds, except for a small number of possible exceptions. For example, we are unable to find the exact value of $f(4m, 1)$, although we show in Theorem 4 that $4(4m + 1) \leq f(4m, 1) \leq 4(4m + 1) + 1$. The remaining cases where $f(s, t)$ is unknown are described in Section 4.

2. Upper bounds. First we give a simple proof of the weak bound $f(s, t) \leq 9s + 8t$, which is subsequently refined (in Theorem 2 below) to give the stronger bound $f(s, t) \leq 4(s + t) + 1$. The equality $w(3) = 9$ will be used in our proof of this weak bound, but will not be used again.

We prove $f(s, t) \leq 9s + 8t$ by contradiction. Assume that $f(s, t) > 9s + 8t$, and let $[1, 9s + 8t]$ be 2-coloured, using the colours Red and Blue, in such a way that there is no monochromatic (s, t) -progression. Since $w(3) = 9$, the set $\{s, 2s, 3s, \dots, 9s\}$ contains a monochromatic (say in the colour Red) 3-term arithmetic progression. Let us suppose, in order to simplify our notation, that this Red progression is $\{s, 5s, 9s\}$. (In all other cases, the argument is essentially the same.)

Consider the (s, t) -progressions $\{s, 5s, 5s + 4t\}$, $\{5s, 9s, 9s + 4t\}$, $\{s, 9s, 9s + 8t\}$. Since by assumption none of these is monochromatic, and $s, 5s, 9s$ are all Red, it follows that $\{5s + 4t, 9s + 4t, 9s + 8t\}$ is a Blue (s, t) -progression, a contradiction, completing the proof.

The following theorem will be useful in obtaining both upper and lower bounds for $f(s, t)$.

THEOREM 1. *Let s, t, c be positive integers. Then $f(cs, ct) = c(f(s, t) - 1) + 1$.*

PROOF. Let $M = f(s, t)$. Let B be a 2-colouring of $[1, c(M - 1) + 1]$. Since every 2-colouring of $[0, M - 1]$ contains a monochromatic (s, t) -progression, every 2-colouring of $\{0, c, 2c, \dots, (M - 1)c\}$ contains a monochromatic (cs, ct) -progression. Hence, every 2-colouring of $\{1, c + 1, 2c + 1, \dots, (M - 1)c + 1\}$ contains a monochromatic (cs, ct) -progression. Thus, $f(cs, ct) \leq c(M - 1) + 1$.

On the other hand, we know there is a 2-colouring, B , of $[1, M - 1]$ that contains no monochromatic (s, t) -progressions. Define B' on $[1, c(M - 1)]$ by

$B'([c(i - 1) + 1, ci]) = B(i)$, for $i = 1, \dots, M - 1$. We will show that B' avoids monochromatic (cs, ct) -progressions, which will complete the proof.

Assume, by way of contradiction, that x_1, x_2, x_3 is a (cs, ct) -progression, contained in $[1, c(M - 1)]$, that is monochromatic under B' . Then there exists $r > 0$ such that $x_3 - x_2 = rct, x_2 - x_1 = rcs$. Let $y_j = \lceil x_j/c \rceil$ for $j = 1, 2, 3$. Then $y_3 - y_2 = \lceil x_3/c \rceil - \lceil x_2/c \rceil = rt$, and similarly $y_2 - y_1 = rs$.

Hence y_1, y_2, y_3 is an (s, t) -progression. Also, $B(y_j) = B(\lceil x_j/c \rceil) = B'(x_j)$, for each j . This contradicts our assumption that there is no monochromatic (s, t) -progression relative to the colouring B . ■

Note that this proof easily extends to a proof that if $f(a_1, \dots, a_k) = M$, then $f(ca_1, \dots, ca_k) = c(M - 1) + 1$, where $f(a_1, \dots, a_k)$ denotes the least positive integer N such that every 2-colouring of $[1, N]$ will contain a monochromatic homothetic copy of $\{1, 1 + a_1, 1 + a_1 + a_2, \dots, 1 + a_1 + a_2 + \dots + a_k\}$.

THEOREM 2. For all positive integers s and t , $f(s, t) \leq 4(s + t) + 1$.

PROOF. Let s, t be given. We may assume without loss of generality that $s \leq t$. We may also assume that $\gcd(s, t) = 1$, for if we knew the result in this case then, with $g = \gcd(s, t)$, Theorem 1 would give $f(s, t) = g[f(s/g, t/g) - 1] + 1 \leq g[4(s/g + t/g) + 1 - 1] + 1 = 4(s + t) + 1$.

Consider the following set of 20 triples contained in $[1, 4(s + t) + 1]$, which are all (s, t) -progressions:

$$\begin{aligned} & \{1, s + 1, s + t + 1\}, \{s + 1, 2s + 1, 2s + t + 1\}, \\ & \{2s + 1, 3s + 1, 3s + t + 1\}, \{3s + 1, 4s + 1, 4s + t + 1\}, \\ & \{1, 2s + 1, 2s + 2t + 1\}, \{s + 1, 3s + 1, 3s + 2t + 1\}, \\ & \{2s + 1, 4s + 1, 4s + 2t + 1\}, \{1, 3s + 1, 3s + 3t + 1\}, \\ & \{s + 1, 4s + 1, 4s + 3t + 1\}, \{1, 4s + 1, 4s + 4t + 1\}, \\ & \{s + t + 1, 2s + t + 1, 2s + 2t + 1\}, \{2s + t + 1, 3s + t + 1, 3s + 2t + 1\}, \\ & \{3s + t + 1, 4s + t + 1, 4s + 2t + 1\}, \{s + t + 1, 3s + t + 1, 3s + 3t + 1\}, \\ & \{2s + t + 1, 4s + t + 1, 4s + 3t + 1\}, \{s + t + 1, 4s + t + 1, 4s + 4t + 1\}, \\ & \{2s + 2t + 1, 3s + 2t + 1, 3s + 3t + 1\}, \{3s + 2t + 1, 4s + 2t + 1, 4s + 3t + 1\}, \\ & \{2s + 2t + 1, 4s + 2t + 1, 4s + 4t + 1\}, \{3s + 3t + 1, 4s + 3t + 1, 4s + 4t + 1\}. \end{aligned}$$

It is straightforward to check (under the assumptions that $s \leq t$ and $\gcd(s, t) = 1$) that except in the cases $s = 1, 1 \leq t \leq 3$, the 15 integers which appear in these 20 triples are distinct. It is then a simple matter to check all 2-colourings of these 15 integers and verify that each 2-colouring has a monochromatic triple from the above list of 20 triples. (If one identifies these 15 integers with the numbers $1, 2, \dots, 15$ via the correspondence

$$1 \leftrightarrow 1, s + 1 \leftrightarrow 2, 2s + 1 \leftrightarrow 3, 3s + 1 \leftrightarrow 4, 4s + 1 \leftrightarrow 5,$$

$$\begin{aligned} s+t+1 &\leftrightarrow 6, 2s+t+1 \leftrightarrow 7, 3s+t+1 \leftrightarrow 8, 4s+t+1 \leftrightarrow 9, \\ 2s+2t+1 &\leftrightarrow 10, 3s+2t+1 \leftrightarrow 11, 4s+2t+1 \leftrightarrow 12, \\ 3s+3t+1 &\leftrightarrow 13, 4s+3t+1 \leftrightarrow 14, 4s+4t+1 \leftrightarrow 15, \end{aligned}$$

the resulting set of 20 triples contained in $[1, 15]$ has a particularly pleasing form.) The cases $s = 1, 1 \leq t \leq 3$ can be checked separately. In all cases we obtain $f(s, t) \leq 4(s+t)+1$.

■

3. Lower bounds and exact values for $f(s, t)$.

THEOREM 3. *Let s, t be positive integers, and let $g = \gcd(s, t)$. If $s/g \not\equiv 0$ and $t/g \not\equiv 0 \pmod{4}$ then $f(s, t) = 4(s+t) + 1$.*

PROOF. The proof splits naturally into two cases.

CASE 1. Assume that s/g and t/g are both odd. In view of Theorem 2, we only need to show that $f(s, t) \geq 4(s+t) + 1$.

First, assume $g = 1$. Now colour $[1, 4(s+t)]$ as

$$101010 \cdots 1010101 \cdots 01,$$

where each of the two long blocks has length $2(s+t)$. Assume x, y, z is a monochromatic (s, t) -progression. Then $y = x + ds$ and $z = y + dt$, for some positive integer d . Let B_1 and B_2 represent $[1, 2(s+t)]$ and $[2(s+t) + 1, 4(s+t)]$, respectively.

In case d is odd, then x and y have opposite parity, and y and z have opposite parity. Since x and y have the same colour and opposite parity, x is in B_1 , while y is in B_2 . Hence z is in B_2 , so that y and z cannot have the same colour, a contradiction.

If d is even, then x, y and z all have the same parity, so they all must be in the same B_i . But then $d(s+t) = z - x \leq 2(s+t)$, and hence $d = 1$, a contradiction.

If g is unequal to 1, then by Theorem 1 and the case in which $g = 1$, $f(s, t) = g[f(s/g, t/g) - 1] + 1 \geq g[4(s/g + t/g) + 1 - 1] + 1 = 4(s+t) + 1$. This finishes the proof of Case 1.

CASE 2. Assume without loss of generality that $s/g \equiv 2 \pmod{4}$. First we assume that $g = 1$. Then $s \equiv 2 \pmod{4}$ and t is odd.

By Theorem 2, we only need to provide a 2-colouring of $[1, 4(s+t)]$ that contains no monochromatic (s, t) -progression. Let C be the colouring $11001100 \cdots 1100$ (i.e., $s+t$ consecutive blocks each having the form 1100).

We proceed by contradiction. Assume that x, y, z is a monochromatic (s, t) -progression. So there exists a $d > 0$ such that $y - x = ds$ and $z - y = dt$. By the way C is defined, if $C(i) = C(j)$ and $j - i$ is even, then 4 divides $j - i$. Now since $z - x = d(s+t) \leq 4(s+t) - 1$, we must have that $d < 4$. The case $d = 2$ is impossible, for if $d = 2$, then $C(z) = C(x)$, $z - x = d(s+t)$ is even, but 4 does not divide $z - x$, a contradiction. Hence d is odd. But then, since $s \equiv 2 \pmod{4}$, $y - x$ is even yet 4 doesn't divide $y - x$, again a contradiction.

This shows that $f(s, t) \geq 4(s+t) + 1$ in the case $g = 1$.

If g is unequal to 1, we proceed just as at the end of Case 1. ■

Suppose that $s/g \equiv 0 \pmod{4}$, where $g = \gcd(s, t)$. Then t/g is odd, and in the case $t/g = 1$, that is, t divides s , we have the following result.

THEOREM 4. *Let m, t be positive integers. Then either*

$$f(4mt, t) = 4(4mt + t) - t + 1 \quad \text{or} \quad f(4mt, t) = 4(4mt + t) + 1.$$

PROOF. By Theorem 1, it is sufficient to show that $4(4m + 1) \leq f(4m, 1) \leq 4(4m + 1) + 1$. By Theorem 2, we only need to show that $4(4m + 1) \leq f(4m, 1)$. Thus it suffices to find a 2-colouring of $[1, 16m + 3]$ that avoids monochromatic $(4m, 1)$ -progressions.

Let χ be the colouring 1A0B0C1D0, where

$A = 00110011 \cdots 0011$ has length $4m$

$B = 11001100 \cdots 11$ has length $4m - 2$

$C = 11001100 \cdots 1100$ has length $4m$

$D = 00110011 \cdots 0011$ has length $4m$.

Assume x, y, z is a monochromatic $(4m, 1)$ -progression. We shall reach a contradiction. We know there exists a positive integer d such that $y - x = 4md$ and $z - y = d$. Hence, $d(4m + 1) \leq 16m + 2$, so that $d \leq 3$. Let

$S_1 = [2, 4m + 1]$ (corresponds to A above)

$S_2 = [4m + 3, 8m]$ (corresponds to B above)

$S_3 = [8m + 2, 12m + 1]$ (corresponds to C above)

$S_4 = [12m + 3, 16m + 2]$ (corresponds to D above).

CASE 1. $d = 1$. Then y, z belong to the same S_i , for some $1 \leq i \leq 4$. Denote by $S(i, j)$ the j -th element of S_i . We see that $y = S(i, j)$ for some odd j . Note that for each even p , if $i = 2$ or 4 , then $\chi(S(i - 1, p))$ is unequal to $\chi(S(i, p - 1))$. Now if $i = 2$ or $i = 4$, then $x = y - 4m = S(i - 1, j + 1)$, so that (by the preceding remark), $\chi(x)$ is different from $\chi(y)$, a contradiction. Now if $i = 3$ and $j > 1$, then $y - 4m = S(2, j - 1)$, and $\chi(x) = \chi(y - 4m)$ is unequal to $\chi(y)$, a contradiction. If $i = 3$ and $j = 1$, then $x = 4m + 2$ and $y = 8m + 2$, and these again have different colours.

CASE 2. $d = 2$. Then $y - x = 8m$ and $z - y = 2$. If $\chi(y) = \chi(z)$ then y must be one of the following: $4m + 1, 8m, 12m + 1$; and since $y - x = 8m$, this reduces the possibilities for y to only $12m + 1$. However we see that $\chi(4m + 1)$ is unequal to $\chi(12m + 1)$, a contradiction.

CASE 3. $d = 3$. Then $y - x = 12m$ and $z - y = 3$. Clearly x belongs to $[1, 4m]$, so that y belongs to $[12m + 1, 16m]$. Now $[1, 4m]$ has colouring 1 0011 \cdots 001100 1 while $[12m + 1, 16m]$ has colouring 0100110011 \cdots 001100. Hence, since $\chi(x) = \chi(y)$, y belongs to the set $\{12m + 3, 12m + 5, 12m + 7, \dots, 16m - 1\}$. Now z belongs to $[12m + 4, 16m + 3]$, so let's compare the colouring of $[12m + 1, 16m]$ to that of $[12m + 4, 16m + 3]$: $[12m + 1, 16m]$ has colouring as noted above, while $[12m + 4, 16m + 3]$ has colouring 0 11001100 \cdots 11 0. Hence, in order for $\chi(y) = \chi(z)$, y must belong to the set $\{12m + 1, 12m + 2, 12m + 4, 12m + 6, \dots, 16m\}$, a contradiction. ■

THEOREM 5. *Let s, t be positive integers such that $s > t > 1$ and t does not divide s . If $\lfloor s/t \rfloor$ is even or $\lfloor 2s/t \rfloor$ is even, where $\lfloor \cdot \rfloor$ is the floor function, then $f(s, t) = 4(s+t) + 1$. If $\lfloor s/t \rfloor$ and $\lfloor 2s/t \rfloor$ are both odd, then $f(s, t) = 4(s+t) + 1$ provided s, t satisfy the additional condition $s/t \notin (1.5, 2)$.*

PROOF. Let s, t satisfy the hypotheses of the theorem. By Theorems 1 and 2, it suffices to show that $f(s, t) \geq 4(s+t) + 1$ under the additional assumption that $\gcd(s, t) = 1$, hence throughout the proof we assume $\gcd(s, t) = 1$.

Let $a = \lfloor s/t \rfloor$ and $b = \lfloor 2s/t \rfloor$. Then $s = at + r$, where $0 < r < t$. Also, $2s = 2at + 2r$, so if $2r = t$ we would have $t = 2$. However, since $\gcd(s, t) = 1$, the case $t = 2$ is already covered by Theorem 3. Therefore we assume throughout the proof that $2r \neq t$.

CASE 1. We assume that a is even and b is odd. Then $b = 2a + 1, 2r > t$, and $2(s+t) = 2(at+r) + 2t = (b-1)t + 2r + 2t = (b+2)t + (2r-t)$.

Hence we can colour $[1, 4(s+t)]$ as follows. Let

$$C = QRQR \cdots QRQJ RQRQ \cdots RQRJ',$$

where $Q = 11 \cdots 1$ and $R = 00 \cdots 0$ each have length t , $J = 00 \cdots 0$ and $J' = 11 \cdots 1$ each have length $2r - t$, and where each of Q and R appears $b + 2$ times.

Suppose x, y, z is any (s, t) -progression in $[1, 4(s+t)]$ with $y - x = ds, z - y = dt$. We will show that $\{x, y, z\}$ is not monochromatic. Clearly $d \leq 3$, since $d(s+t) = z - x \leq 4(s+t) - 1$.

If $d = 2$, then $z - x = 2(s+t)$, so $C(z) \neq C(x)$. (This is because the colouring on the second half of $[1, 4(s+t)]$ is the reversal of the colouring on the first half.)

If $d = 3$, then, since $z = y + 3t$ and $C(i) \neq C(i+t)$ for all $i > 2(s+t)$, if $C(y) = C(z)$ we must have $y \leq 2(s+t)$; but then $x = y - 3s \leq 2t - s$. However, the conditions $s > t, s = at + r, 0 < r < t, a$ even, imply that $s > 2t$, hence $x < 0$, a contradiction.

Now assume that $d = 1$ and $C(y) = C(z)$. Since $z = y + t$, y must occur in the block J , so $C(y) = 0$. Since J has length $2r - t < r$, we see that $y - r$ must occur in the block Q just to the left of block J , so that $y - at - r = x$ also occurs in a block Q , and $C(x) = 1$.

Hence there is no monochromatic (s, t) -progression with respect to the colouring C , therefore $f(s, t) \geq 4(s+t) + 1$. This finishes Case 1.

CASE 2. We assume that a is odd and b is even. Again we have $s = at + r, 0 < r < t$, but now $b = 2a, 2r < t$, and $2(s+t) = (b+2)t + 2r$.

Now colour $[1, 4(s+t)]$ with the colouring

$$D = QRQR \cdots QRK RQRQ \cdots RQK',$$

where Q, R are defined as in Case 1, and $K = 11 \cdots 1, K' = 00 \cdots 0$ each have length $2r$.

Assume x, y, z is an (s, t) -progression contained in $[1, 4(s+t)]$, with $y - x = ds, z - y = dt$; then $d \leq 3$.

If $d = 2$, then as in Case 1, $D(x) \neq D(z)$.

If $d = 3$ and $D(y) = D(z)$, then as in Case 1, $y \leq 2(s+t)$. In fact, since K and R have opposite colours, $y \leq 2(s+t) - 2r$. On the other hand, $y \geq 1 + 3s \geq 2s + t + r + 1$,

so y is an element of the last occurrence of R in $[1, 2(s + t)]$, hence $D(y) = 0$. Then $x = y - 3s \leq 2(s + t) - 2r - 3s < t$, so $D(x) = 1$ and $D(x) \neq D(y)$.

Now assume $d = 1$ and $D(y) = D(z)$. Then y belongs to the last occurrence of R in $[1, 2(s + t)]$, and $y \equiv i \pmod{t}$, where $2r < i \leq t$. Hence, since a is odd, $x = y - (at + r)$ lies in one of the Q 's, and $D(x) = 1, D(y) = 0$.

Thus, no monochromatic (s, t) -progression exists in $[1, 4(s + t)]$, hence $f(s, t) \geq 4(s + t) + 1$.

CASE 3. We assume that both a and b are even. Then $s = at + r, b = 2a, 0 < 2r < t$, and $2(s + t) = (b + 2)t + 2r$. Note that $a \geq 2$, since $s > t$.

We define the colouring E on $[1, 4(s + t)]$ as follows. Let us use the notation $\sim 0 = 1$ and $\sim 1 = 0$. Then we define, in turn,

- (1) $E(i) = 1, 1 \leq i \leq r,$
- (2) $E(i) = \sim E(i - r), r < i \leq t,$
- (3) $E(i) = \sim E(i - t), t < i \leq 2(s + t),$
- (4) $E(i) = \sim E(i - 2(s + t)), 2(s + t) < i \leq 4(s + t).$

That is,

$$E = XYXY \cdots XYLYXYX \cdots YXL',$$

where X has length t and consists of $\lfloor t/r \rfloor$ blocks, each block of length r , followed by a single block of length $t - \lfloor t/r \rfloor r$, the blocks alternating in colour; Y is the same as X , except the colours are reversed; L is X restricted to $[1, 2r]$; and L' is the same as L , except the colours are reversed.

Let x, y, z be an (s, t) -progression contained in $[1, 4(s + t)]$, with $y - x = ds, z - y = dt$.

If $d = 2$, then by (4), $E(x) = \sim E(z)$.

If $d = 3$ and $E(y) = E(z)$, then $y \leq 2(s + t)$, hence $x = y - 3s \leq 2t - s = 2t - (at + r) \leq -r < 0$, a contradiction.

If $d = 1$ and $E(y) = E(z)$, then $y \leq 2(s + t)$. We consider two subcases.

The first subcase is $y \equiv i \pmod{t}, r + 1 \leq i \leq t$. Then y and $y - r$ are in the same block (X, Y , or L), hence by (2) $E(y) = \sim E(y - r)$. By (3), and the fact that a is even, $E(y) = \sim E(y - r) = \sim E(y - at - r) = \sim E(x)$.

The second subcase is $y \equiv i \pmod{t}, 1 \leq i \leq r$. Since $E(y) = E(z) = E(y + t)$, y must belong to the block L , that is, $y = (b + 2)t + i = (2a + 2)t + i, 1 \leq i \leq r$. Since $x = y - s = (2a + 2)t + i - at - r = (a + 1)t + (i + t - r)$, and $1 \leq i + t - r \leq t$, by (3) $E(x) = \sim E(i + t - r)$. Also, since $y = 2(s + t) - 2r + i$, we have $z = y + t = 2(s + t) + (i + t - 2r)$, so by (4), $E(z) = \sim E(i + t - 2r)$. Since $1 \leq i + t - 2r \leq t$, (2) gives $E(z) = E(i + t - r) = \sim E(x)$.

Thus, under the colouring E , there is no monochromatic (s, t) -progression in $[1, 4(s + t)]$, hence $f(s, t) \geq 4(s + t) + 1$.

CASE 4. Assume that both a and b are odd, and $s/t \notin (1.5, 2)$. It follows that $s = at + r, 0 < r < t, b = 2a + 1, t < 2r$, and $2(s + t) = (b + 2)t + (2r - t)$. Also, $a \geq 3$, as a consequence of the assumption $s/t \notin (1.5, 2)$.

Let $p = t - r$. Then $p < t/2$. Define the colouring F by setting, in turn,

- (5) $F(i) = 1, 1 \leq i \leq p,$

- (6) $F(i) = \sim F(i - p), p < i \leq t,$
 (7) $F(i) = \sim F(i - t), t < i \leq 2(s + t),$
 (8) $F(i) = \sim F(i - 2(s + t)), 2(s + t) < i \leq 4(s + t).$

That is,

$$F = ABAB \cdots ABAM \text{ BABA} \cdots BABM',$$

where A and B are the same as the blocks X and Y in Case 3, except that p replaces r ; M is B restricted to $[1, 2r - t]$; and M' is the same as M with the colours interchanged.

Let x, y, z be an (s, t) -progression contained in $[1, 4(s + t)]$, with $y - x = ds, z - y = dt$.

If $d = 2$, then by (8), $E(x) = \sim E(z)$.

If $d = 3$ and $E(y) = E(z)$, then $y \leq 2(s + t)$, hence (since $a \geq 3$) $x = y - 3s \leq 2t - s = 2t - (at + r) < 0$, a contradiction.

If $d = 1$ and $E(y) = E(z)$, then $y \leq 2(s + t)$, and we again consider two subcases.

The first subcase is $y = ut + i, 1 \leq i \leq r$. Then $1 \leq i < i + p = i + t - r \leq t$, so by (6), $F(i + p) = \sim F(i)$. Using (7) and the oddness of a , we get $F(x) = F(y - at - r) = F(ut - (a + 1)t + i + t - r) = F(ut + i + t - r) = F(ut + i + p) = \sim F(ut + i) = \sim F(y)$.

The second subcase is $y = ut + i, r + 1 \leq i \leq t$. Since $F(y) = F(y + t)$ and M has fewer than i elements, y must belong to the last occurrence of the block A in $[1, 2(s + t)]$. Since $2(s + t) = (b + 2)t + (2t - r)$, this means that $y = (b + 1)t + i$, hence by (7), $F(y) = F(i)$. Since $x = y - at - r = (b + 1)t + i - at - r$, we have $F(x) = \sim F(i - r) = F(i + t - r) = F(i + p) = \sim F(i) = \sim F(y)$.

Thus, under the colouring F , there is no monochromatic (s, t) -progression in $[1, 4(s + t)]$, hence $f(s, t) \geq 4(s + t) + 1$. ■

4. Remarks. By Theorems 1 and 3, we would know the value of $f(s, t)$ for all s, t provided we knew the value of $f(4m, t)$ when t is odd, and $\gcd(m, t) = 1$. (Here we are using $f(s, t) = f(t, s)$.) Theorem 4 shows $4(4m + 1) \leq f(4m, 1) \leq 4(4m + 1) + 1$. Theorem 5 takes care of many of the cases where $t > 1$. For example, Theorem 5 shows that $f(4m, 3) = 4(4m + 3) + 1$ whenever 3 does not divide m . By examining the cases not covered by Theorem 5, one sees that these are exactly the cases $f(t + r, t)$ where $0 < r < t < 2r$, and 4 divides t or 4 divides $t + r$.

The computations $f(4, 1) = 20, f(8, 1) = 36, f(12, 1) = 52$ suggest that perhaps $f(4m, 1) = 4(4m + 1)$ for all $m \geq 1$.

For positive integers r, a_1, \dots, a_n , let $f^{(r)}(a_1, \dots, a_n)$ denote the smallest positive integer N such every r -colouring of $[1, N]$ has a monochromatic homothetic copy of $\{1, 1 + a_1, \dots, 1 + a_1 + \dots + a_n\}$. Of course $f^{(r)}(a_1, \dots, a_n)$ always exists (by a statement of van der Waerden's theorem which involves any number of colours), but perhaps one can say something about the rate of growth of $f^{(r)}(a_1, \dots, a_n)$ as a function of $a_1 + \dots + a_n$. The computations $f^{(2)}(1, 1, 1) = 35, f^{(2)}(1, 1, 2) = 38, f^{(2)}(1, 1, 3) = 44, f^{(2)}(1, 1, 4) = 56, f^{(2)}(1, 1, 5) = 59$ suggest that $f^{(2)}(1, 1, n)$ does not grow linearly with n . Perhaps $f^{(2)}(1, 1, n) \sim c2^n$.

We have no idea of the growth rate of $f^{(3)}(s, t)$ as a function of $s + t$.

REFERENCES

1. T. C. Brown and P. Erdős and A. R. Freedman, *Quasi-progressions and descending waves*, J. Combin. Theory Ser. A **53**(1990), 81–95.
2. R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, 2nd ed., John Wiley and Sons, New York, 1990.
3. Bruce M. Landman and Raymond N. Greenwell, *Values and bounds for Ramsey numbers associated with polynomial iteration*, Discrete Math. **68**(1988), 77–83.
4. ———, *Some new bounds and values for van der Waerden-like numbers*, Graphs Combin. **6**(1990), 287–291.
5. B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wisk. **15**(1927), 212–216.

Department of Mathematics and Statistics
Simon Fraser University
Burnaby, BC V5A 1S6
tbrown@sfu.ca

Department of Mathematical Sciences
University of North Carolina
Greensboro, NC 27412
landman@steffi.uncg.edu

Faculty of Mathematics
University of Waterloo
Waterloo, ON N2L 3G1
mjmishna@undergrad.math.uwaterloo.ca