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PRINCIPAL TRAIN ALGEBRAS OF RANK 3 AND DIMENSION≦5

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A commutative algebra A over the field F, endowed with a non-zero homorphism $\omega: A\to F$ is principal train if it satisfies the identity $x^r+\gamma_1\omega(x)x^{r-1}+\cdots+\gamma_{r-1}\omega(x)^{r-1}x=0$ where $\gamma_1,\ldots,\gamma_{r-1}$ are fixed elements in F. We present in this paper, after the introduction of the concept of "type" of A, some results concerning the classification in the case $r=3$. In particular we describe all these algebras of dimension $\leqq 5$.

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1. Introduction

Let F be an infinite field of characteristic not 2 and A a finite-dimensional, non-associative algebra over F. The principal powers of $x\in A$ are defined by $x^1=x$ and $x^i=x^{i-1}x$ for $i\geqq 2$. If $\omega: A\to F$ is a non-zero homomorphism, the ordered pair (A,ω) is called a baric algebra and ω its weight function. (A,ω) is a principal train algebra (train algebra, for short) if we have identically in A:

$$x^r+\gamma_1\omega(x)x^{r-1}+\cdots+\gamma_{r-1}\omega(x)^{r-1}x=0 \tag{1}$$

where $\gamma_1,\ldots,\gamma_{r-1}$ are fixed elements in F. The equation like (1) with minimum degree is the rank equation of A, r is the rank of A and the roots of the algebraic equation $x^r+\gamma_1x^{r-1}+\cdots+\gamma_{r-1}x=0$, in some extension field of F, are the train roots of A. Most algebras appearing in the algebraic formalism of Genetics are in this class (see [**8**, chapter 3, **3, 4**]).

The following properties of a train algebra are immediate:

(a) $1+\gamma_1+\cdots+\gamma_{r-1}=0$;

(b) All $x\in A$ such that $\omega(x)=1$ satisfy the same equation;

(c) The kernel B of ω is an ideal of codimension 1 satisfying the identity $x^r=0$. Also it is true that ω is the only non-zero homomorphism from A to F. We say that B is the kernel of A.

In this paper we deal only with train algebras of rank 3.

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Every baric algebra with an idempotent of weight 1 can be obtained in the following way. Suppose B is an arbitrary commutative finite-dimensional algebra over F. Take the direct sum B' of F and B and define a multiplification in B' by

$$(\alpha, a)(\beta, b) = (\alpha\beta, ab + \tau(\alpha b + \beta a)); \quad \alpha, \beta \in F; \quad a, b \in B \tag{2}$$

where $\tau: B \to B$ is an arbitrary F-linear mapping. Then $\omega: B \to F$ given by $\omega(\alpha, a) = \alpha$ is a non-zero homomorphism, (B, ω) is a baric algebra and $(1,0)$ is an idempotent of weight 1. Two different τ's may give rise to isomorphic algebras. Note that $(1,0)(0,a) = (0, \tau(a))$. If B satisfies the identity $a^3 = 0$, it is easy to see that B' satisfies the rank equation

$$x^3 - (1+\gamma)\omega(x)x^2 + \gamma\omega(x)^2 x = 0 \tag{1'}$$

if and only if the following identities hold:

$$2\tau(a)a + \tau(a^2) = (1+\gamma)a^2; \quad a \in B \tag{3}$$

$$2\tau^2 - (1+2\gamma)\tau + \gamma I = 0 \quad (I = \text{identity operator}), \tag{4}$$

Note that any τ satisfying (3) is determined by its values on a generating system of the algebra B. It also follows from (3) by induction that the powers B^i defined by $B^1 = B$ and $B^i = B^{i-1}B$ $(i \geqq 2)$ are invariant under τ. The same holds for the ideal $An(B)$ of absolute divisors of zero in B.

2. Invariants

In this paragraph we will assume that $\gamma \neq 1/2$ in equation (1'). By (4), if we have a train algebra B' of rank 3, constructed as explained above, the proper values of τ will be 1/2 and/or γ. Decompose $B = B_1 \oplus B_2$ where $B_1 = \ker(\tau - (1/2)I)$, $B_2 = \ker(\tau - \lambda I)$. The linearized form of (3) gives the following relations:

$$B_1 B_1 \subset B_2 \tag{5}$$

$$B_1 B_2 \subset B_1 \tag{6}$$

$$B_2 B_2 = 0. \tag{7}$$

To show this, note that if $x_1, x_2 \in B_1$, $\tau(x_1x_2) = (1+\gamma)x_1x_2 - \tau(x_1)x_2 - x_1\tau(x_2) = \gamma x_1 x_2$ so (5). (6) is similar. For (7), if $x_1, x_2 \in B_2$, $\tau(x_1x_2) = (1-\gamma)x_1x_2$ so $x_1x_2 = 0$ because $1-\gamma$ is not a proper value. Take now $x = \alpha x_1 + \beta x_2$, $\alpha, \beta \in F$, $x_1 \in B_1$, $x_2 \in B_2$. Then $x^3 = 0$ implies $0 = \alpha\beta^2(x_1x_2)x_2 + \alpha^2\beta x_1(x_1x_2)$. Each component must be zero so:

$$(x_1x_2)x_2 = 0 \tag{8}$$

$$x_1(x_1x_2) = 0 \tag{9}$$

and the linearized forms

$$(x_1x_2)y_2+(x_1y_2)x_2=0 \tag{8'}$$

$$x_1(y_1x_2)+y_1(x_1x_2)=0. \tag{9'}$$

Suppose conversely that the algebra B is given. If a linear mapping $\tau:B\to B$ satisfies (4) and $B_1=\ker(\tau-(1/2)I)$ and $B_2=\ker(\tau-\gamma I)$ satisfy the above relations (5),...,(9), then B', defined by (2), satisfies (1').

Idempotents in B' must have weight 1 because B is nil. $(1,a)\in B'$ is idempotent if and only if $2\tau(a)=a-a^2$. Decomposing $a=a_1+a_2$, $a_i\in B_i$, we are led to the equations

$$\begin{cases} a_1a_2=0 \\ a_1^2+2\gamma a_2=a_2 \end{cases}$$

so idempotents have the form $(1,a_1+(1-2\gamma)^{-1}a_1^2)$, $a_1\in B_1$.

Proposition 1. *The function $a_1\in B_1\to(1,a_1+(1-2\gamma)^{-1}a_1^2)$ is a bijection between the subspace B_1 and the set of idempotents of B'. In particular, the dimension of B_1 is independent of the operator τ used to construct B'. The same holds for the dimension of B_2.*

Definition. The *type* of B' is the ordered pair of non-negative integers $(1+\dim B_1,\dim B_2)$.

Algebras having the extreme types are very simple, in any dimension. If the type is $(1,n)$, we take any basis $\{c_1,\ldots,c_n\}$ of B and call $c_0=(1,0)$. The table is: $c_0^2=c_0$, $c_0c_i=\gamma c_i$ $(i=1,\ldots,n)$ and $c_ic_j=0$ $(i,j=1,\ldots,n)$. If the type is $(n+1,0)$, we have a similar table with γ replaced by 1/2. This algebra satisfies in fact the equation $x^2=\omega(x)x$.

Proposition 2. *If B' has type $(2,n-1)$, there is a basis $\{c_0,x_1,\ldots,x_n\}$ of B' such that its multiplication table is:*

$$c_0^2=c_0,\quad c_0x_1=\tfrac{1}{2}x_1,\quad c_0x_i=\gamma x_i\ (i=2,\ldots,n),\quad x_1^2=\varepsilon x_2,$$

where $\varepsilon=0$ or 1, other products are zero.

Proof. Start with $B_1=\langle c_1\rangle$ and $B_2=\langle c_2,\ldots,c_n\rangle$. By (5), (6) and (7), $c_1c_j=\lambda_jc_1$ $(j=2,\ldots,n)$, $c_jc_k=0$ $(j,k=2,\ldots,n)$, $c_1^2\in B_2$. But by (8), $0=(c_1c_j)c_j=\lambda_j^2c_1$ so $\lambda_j=0$. If $c_1^2=0$, we are done. Otherwise replace some $c_j(2\leqq j\leqq n)$ by c_1^2 where possible, permute so that c_1^2 becomes the first vector. This is the case $\varepsilon=1$.

Other numerical invariants of train algebras B' of rank 3 are the dimensions of the ideals B^i of B'. In fact, B^i is invariant under τ by (3) so it is an ideal in B'. We have $B^2=B_1B_2\oplus B_1^2$, $B^3=((B_1B_2)B_2+B_1^3)\oplus(B_1B_2)B_1$ and so on. For some k, $B^k=0$ ([1, Theorem 1]). Etherington introduced in [3] the concepts of "nil products" and "nil squares" and also the ideal generated by all the nil squares. In our context, this ideal will be $J=B_1B_2\oplus B_2$. In fact, the sum $B_1B_2+B_2$ is direct by (6) and the relations

$$x_1(u_1u_2+v_2)=x_1(u_1u_2)+x_1v_2\in B_2\oplus B_1B_2$$

$$x_2(u_1u_2+v_2)=x_2(u_1u_2)\in B_2B_1$$

if $x_1,u_1\in B_1$ and $x_2,u_2,v_2\in B_2$, show that J is an ideal of B. As J is obviously invariant under τ, it is an ideal of B'. If J' is the ideal generated by all nil squares $x^2-\omega(x)x$, for $x\in B'$, then $J'\subset J$. In fact, if $x=(\alpha,a)$, where $a=u_1+u_2$, $u_i\in B_i$, then $x^2-\alpha x=2u_1u_2+(u_1^2+(2\gamma-1)\alpha u_2)\in J$. Let us show that B_1B_2 and B_2 are contained in J'. For the second inclusion, if $u_2\in B_2$, take $x=(1,u_2)\in B'$, then $x^2-x=(2\gamma-1)u_2$, $u_2\in J'$. If u_1u_2 is a generator of B_1B_2 take $x=(1,u_1+u_2)$. Then $x^2-x-(u_1^2+(2\gamma-1)u_2)=2u_1u_2$ and so $B_1B_2\subset J'$. Hence $J=J'$ and the dimension of J is a numerical invariant of B'.

Remark. $B'\supset B\supsetneqq J\supset B^2\supset B^3\supset\cdots$ (see [4, p. 140]). The only relation to be proved is $J\neq B$. In fact, if $J=B$ then it would follow that $B^2=B^3$, contrary to Abraham's Theorem 1 of [1].

Remark. Train algebras of rank 3, with $\gamma=0$, are Bernstein algebras satisfying two additional conditions (see [8, Theorem 9.12] or [4, Theorem XII]). The ideal J coincides, in this case, with the ideal appearing in [8, equation 9.56].

We describe now train algebras of a given type having the smallest possible ideal J, that is $J=B_2$. Consider the set of all triples (B_1,B_2,ψ) where B_1 and B_2 are arbitrary finite dimensional vector spaces over the field F and $\psi:B_1\times B_1\to B_2$ is an arbitrary symmetric bilinear function.

Two triples (B_1,B_2,ψ) and (C_1,C_2,ϕ) are equivalent if and only if there exist bijective linear mappings $v:B_1\to C_1$ and $\mu:B_2\to C_2$ such that the diagram commutes:

$$\begin{array}{ccc} B_1\times B_1 & \xrightarrow{\psi} & B_2 \\ \downarrow{\scriptstyle v\times v} & & \downarrow{\scriptstyle \mu} \\ C_1\times C_1 & \xrightarrow[\phi]{} & C_2 \end{array}$$

This is clearly an equivalence relation. Given now (B_1,B_2,ψ) we construct a train algebra in the following way. Take in the vector space $B=B_1\oplus B_2$ the multiplication

$$(u_1,v_1)(u_2,v_2)=(0,\psi(u_1,u_2));\quad u_1,u_2\in B_1;\quad v_1,v_2\in B_2$$

Then $(u,v)^3=0$ for all $(u,v)\in B$. Now $\tau:B\to B$ given by $\tau(u,v)=((1/2)u,\gamma v)$ satisfies (3) and (4) so B' is a train algebra of rank 3, of type $(1+\dim B_1,\dim B_2)$ and also $J=B_2$. Denote this algebra by $[B_1,B_2,\psi]$. If (B_1,B_2,ψ) and (C_1,C_2,ϕ) are equivalent the corresponding algebras will be isomorphic, $1_F\oplus v\oplus\mu$ being an isomorphism. On the other hand every train algebra of rank 3 with $J=B_2$ is obtained in this way, by taking $\psi: B_1 \times B_1 \to B_2$ as the product already existing. Moreover two isomorphic train algebras B' and C' must come from equivalent triples. To see this, consider their kernels $B_1 \oplus B_2$ and $C_1 \oplus C_2$. Then B_2 (resp. C_2) is formed by absolute divisors of zero in B (resp. C). An isomorphism from B to C must therefore take B_2 to C_2. The result follows by passage to quotients. We summarize these results in:

Proposition 3. (a) *The train algebras* $[B_1, B_1, \psi]$ *and* $[C_1, C_2, \phi]$ *are isomorphic if and only if the triples* (B_1, B_2, ψ) *and* (C_1, C_2, ϕ) *are equivalent.*

(b) *Every train algebra of rank* 3 *with minimum* J *is isomorphic to some* $[B_1, B_2, \psi]$.

As a particular case, when the type of B' is $(r+1, r)$, the classification of train algebras with minimum J is equivalent to the classification of commutative algebras of dimension r. In another particular case, when the type is $(n, 1)$ and $\dim J = 1$, the problem reduces to the classification of bilinear forms in spaces of dimension $n-1$.

Proposition 4. *If* B' *has type* $(n, 1)$ *then* $\dim J \leqq \frac{1}{2}(n+1)$.

Proof. Start with $B_1 = \langle c_1, \ldots, c_{n-1} \rangle$ and $B_2 = \langle c_n \rangle$. Then J is generated by $\{c_n, c_1 c_n, \ldots, c_{n-1} c_n\}$. If $\dim J = 1 + k$ $(k \geqq 0)$ there are exactly k linearly independent vectors in the set $\{c_1 c_n, \ldots, c_{n-1} c_n\}$. We may suppose they are $c_1 c_n, \ldots, c_k c_n$. The set $\{c_1 c_n, \ldots, c_k c_n, c_1, c_2, \ldots, c_n\}$ which generates B, must contain a basis of the form $\{c_1 c_n, \ldots, c_k c_n, c_{i_1}, \ldots, c_{i_{n-k-1}}, c_n\}$. These vectors give a new generating system of J, namely $\{c_{i_1} c_n, \ldots, c_{i_{n-k-1}} c_n, c_n\}$ because $(c_i c_n) c_n = 0$ $(i = 1, \ldots, k)$. Then $k + 1 = \dim J \leqq n - k$ and so $\dim j \leqq \frac{1}{2}(n+1)$.

Proposition 5. *For every train algebra* B' *of rank* 3, J^2 *is an ideal. If the type of* B' *is* $(n, 1)$ *then* $J^2 = 0$.

Proof. Clearly J^2 is invariant under τ. As $J^2 = (B_1 B_2) B_2 \oplus (B_1 B_2)^2$, the following relations show that J^2 is an ideal in B:

$$x_1((u_1 u_2) v_2) = -(u_1 u_2)(x_1 v_2) \in (B_1 B_2)^2$$

$$x_1((u_1 u_2)(v_1 v_2)) = -(u_1 u_2)(x_1(v_1 v_2)) - (v_1 v_2)(x_1(u_1 u_2)) \in (B_1 B_2) B_2$$

$$x_2((u_1 u_2) v_2) = ((u_1 u_2) v_2) x_2 \in (B_1 B_2) B_2$$

$$x_2((u_1 u_2)(v_1 v_2)) = 0$$

for $x_i, u_i, v_i \in B_i$ $(i = 1, 2)$.

The second assertion: if $B_1 = \langle c_1, \ldots, c_{n-1} \rangle$ and $B_2 = \langle c_n \rangle$, then J is linearly generated by $c_1 c_n, \ldots, c_{n-1} c_n$ and c_n. The product of any two of these elements is 0 by (8).

3. Train algebras of dimension $\leqq 5$

The invariants type, $\dim J$ and $\dim B^2$ classify train algebras of rank 3 (always $\gamma \neq \frac{1}{2}$) up to dimension 5 or reduce this problem to the classification of other algebraic objects. Etherington proved that every train algebra of rank 3 is special triangular, a gap in his proof was filled by Abraham [1]. Algebras in this paragraph are expressed by means of a canonical basis. Almost all computational details are omitted, to save space.

(I) $\dim B' = 2$. The possible types are $(2,0)$ and $(1,1)$, already discussed, see Proposition 2 and the discussion preceding it.

(II) $\dim B' = 3$. The non-extreme type $(2,1)$ is covered by Proposition 2, yielding two non-isomorphic algebras.

(III) $\dim B' = 4$. The only type to be considered is $(3,1)$. The ideal J may have dimension 1 or 2, because $J \neq B$.

(*a*) $\dim J = 1$. The algebras have already been described by Proposition 3. Every essentially distinct bilinear form in a F-vector space of dimension 2 gives an algebra here and conversely.

(b) $\dim J = 2$. The answer is given by the following:

Proposition 6. *There is only one, up to isomorphisms, train algebra of type* $(3,1)$ *such that* $\dim J = 2$.

Proof. If $B_1 = \langle c_1, c_2 \rangle$ and $B_2 = \langle c_3 \rangle$ then J is generated by $\{c_3, c_1c_3, c_2c_3\}$. One of c_1c_3 and c_2c_3 is non-zero, the other is a scalar multiple of it. By symmetry we may suppose $c_1c_3 \neq 0$ and $c_2c_3 = kc_1c_3$, $k \in F$. The set $\{c_1c_3, c_1, c_2, c_3\}$ generates B so it must contain a basis of the form $\{c_1c_3, ?, c_3\}$. There are two possibilities:

(a) $\{c_1c_3, c_1, c_3\}$ is a basis of B. The multiplication table of B, according to (8) is (on the left):

	c_1c_3	c_1	c_3
c_1c_3	0	0	0
c_1		0	c_1c_3
c_3			0

	c_0	x_1	x_2	x_3
c_0	c	$\frac{1}{2}x_1$	γx_2	$\frac{1}{2}x_3$
x_1		0	x_3	0
x_2			0	0
x_3				0

Calling $x_1 = c_1$, $x_2 = c_3$ and $x_3 = c_1c_3$, B' is given by the above (on the right) table.

(b) $\{c_1c_3, c_2, c_3\}$ is a basis of B. The multiplication table is:

	c_1c_3	c_2	c_3
c_1c_3	0	0	0
c_2		λc_3	kc_1c_3
c_3			0

But here J is generated by c_3 and kc_1c_3 so necessarily $k \neq 0$ because $\dim J = 2$. From $0 = c_2^3 = \lambda k c_1 c_3$ we get $\lambda = 0$. Taking now the basis $\{k^{-1}c_1c_3, k^{-1}c_2, k^{-1}c_3\}$ we get the same table but with 1 in place of k. Introducing now x_i as in case (a) we get B' exactly as in case (a).

(IV) $\dim B' = 5$. There are two non-trivial types to consider : (4, 1) and (3, 2).

(A) Algebras of type (4, 1)

By Proposition 4, $1 \leqq \dim J \leqq 2$.

A.1. $\dim J = 1$. The algebras have already been described in Proposition 3. They correspond to essentially distinct bilinear forms in spaces of dimension 3 over the field F.

A.2. $\dim J = 2$. Take $B_1 = \langle c_1, c_2, c_3 \rangle$ and $B_2 = \langle c_4 \rangle$ so J will be generated by $\{c_4, c_1c_4, c_2c_4, c_3c_4\}$. One of the last 3 vectors must be non-zero and the other 2 must be scalar multiples of it. By symmetry, we may suppose that $c_1c_4 \neq 0$ and $c_2c_4 = k_2c_1c_4$, $c_3c_4 = k_3c_1c_4$, with $k_2, k_3 \in F$. The set $\{c_1c_4, c_1, c_2, c_3, c_4\}$ must contain a basis of the form $\{c_1c_4, ?, ?, c_4\}$. Let us examine the three possibilities.

A.2.1. $\{c_1c_4, c_1, c_2, c_4\}$ is a basis of B. The multiplication table:

	c_1c_4	c_1	c_2	c_4
c_1c_4	0	0	0	0
c_1		0	0	c_1c_4
c_2			0	$k_2c_1c_4$
c_4				0

(In fact, $c_1^2 = \mu c_4$ but $0 = c_1^3 = \mu c_1 c_4$ so $\mu = 0$; $c_1c_2 = \lambda c_4$ and $c_2^2 = \nu c_4$ but for all $m, n \in F$, $(mc_1 + nc_2)^3 = 0$ implies $\nu = \lambda = 0$, an easy calculation.)

Calling now $k_2 = k$, $x_1 = c_1$, $x_2 = c_2$, $x_3 = c_4$ and $x_4 = c_1c_4$ the table of B' is:

	c_0	x_1	x_2	x_3	x_4
c_0	c_0	$\frac{1}{2}x_1$	$\frac{1}{2}x_2$	γx_3	$\frac{1}{2}x_4$
x_1		0	0	x_4	0
x_2			0	kx_4	0
x_3				0	0
x_4					0

A.2.2. $\{c_1c_4, c_2, c_3, c_4\}$ is a basis of B, whose table is:

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	c_1c_4	c_2	c_3	c_4
c_1c_4	0	0	0	0
c_2		λc_4	μc_4	$k_2c_1c_4$
c_3			νc_4	$k_3c_1c_4$
c_4				0

where the following relations hold: $\lambda k_2 = \nu k_3 = \lambda k_3 + 2\mu k_2 = \nu k_2 + 2\mu k_3 = 0$, a consequence of the identity $x^3 = 0$. We see that J is generated by the set $\{c_4, k_2c_1c_4, k_3c_1c_4\}$ so necessarily $k_2 \neq 0$ or $k_3 \neq 0$. By symmetry we may study only one case, say $k_2 \neq 0$. This implies that in the above table $\lambda = \mu = \nu = 0$. Introducing now the vectors $x_1 = k_2^{-1}c_2$, $x_2 = c_3$, $x_3 = k_2^{-1}c_4$ and $x_4 = k_2^{-1}\ c_1c_4$ we get the same table for B' obtained in case A.2.1.

A.2.3. $\{c_1c_4, c_1, c_3, c_4\}$ is a basis of B. This is similar to A.2.1, because the roles of c_2 and c_3 can be interchanged. We summarize the facts:

Proposition 7. *Train algebras of type* $(4,1)$ *such that* $\dim J = 2$ *form a one-parameter family, given by the above table of A.2.1.*

(*B*) *Algebras of type* $(3,2)$

We have $2 \leqq \dim J \leqq 3$ because $J \neq B$.

B.1. $\dim J = 2$. These algebras have already been described in Proposition 3. Every essentially distinct commutative algebra of dimension 2 gives an algebra here. The classification of such bidimensional algebras is a problem of its own interest, see for example [**2**] and [**6**].

B.2. $\dim J = 3$. Take $B_1 = \langle c_1, c_2 \rangle$ and $B_2 = \langle c_3, c_4 \rangle$. Then as J is generated by $\{c_3, c_4, c_1c_3, c_1c_4, c_2c_3, c_2c_4\}$ one of the last 4 vectors is non-zero and the remaining 3 are scalar multiples of it. Again by symmetry we may suppose that $c_1c_3 \neq 0$ and $c_1c_4 = k_1c_1c_3$, $c_2c_3 = k_2c_1c_3$, $c_2c_4 = k_3c_1c_3$, $k_i \in F$. The set $\{c_1c_3, c_1, c_2, c_3, c_4\}$ generates B so it must contain a basis of the form $\{c_1c_3, ?, c_3, c_4\}$. There are two possibilities:

B.2.1. $\{c_1c_3, c_1, c_3, c_4\}$ is a basis of B. The multiplication table is:

	c_1c_3	c_1	c_3	c_4
c_1c_3	0	0	0	0
c_1		$\mu(c_4 - k_1c_3)$	c_1c_3	$k_1c_1c_3$
c_3			0	0
c_4				0

(In general. $c_1^2 = \lambda c_3 + \mu c_4$ but $0 = c_1^3 = (\lambda + \mu k_1)c_1c_3$ implies $\lambda = -\mu k_1$.)

The ideal B^2 is generated, as a vector space, by c_1c_3, $k_1c_1c_3$ and $\mu(c_4 - k_1c_3)$ so $1 \leqq \dim B^2 \leqq 2$.

B.2.1.1. $\dim B^2=1$. This means that $\mu=0$. Calling $k_1=k$, $x_1=c_1$, $x_2=c_3$, $x_3=c_4$ and $x_4=c_1c_3$ we get the table of B':

	c_0	x_1	x_2	x_3	x_4
c_0	c_0	$\frac{1}{2}x_1$	γx_2	γx_3	$\frac{1}{2}x_4$
x_1		0	x_4	kx_4	0
x_2			0	0	0
x_3				0	0
x_4					0

B.2.1.2. $\dim B^2=2$. In this case $\mu\neq 0$. We look now for the new basis $x_1=\mu^{-1}c_1$, $x_2=\mu^{-1}c_3$, $x_3=\mu^{-1}(c_4-k_1c_3)$, $x_4=\mu^{-1}c_1c_3$. The table of B' will be, for some $k\neq 0$:

	c_0	x_1	x_2	x_3	x_4
c_0	c_0	$\frac{1}{2}x_1$	γx_2	γx_3	$\frac{1}{2}x_4$
x_1		x_3	kx_4	0	0
x_2			0	0	0
x_3				0	0
x_4					0

B.2.2. $\{c_1c_3, c_2, c_3, c_4\}$ is a basis of B. The table is:

	c_1c_3	c_2	c_3	c_4
c_1c_3	0	0	0	0
c_2		$\lambda c_3+\mu c_4$	$k_2c_1c_3$	$k_3c_1c_3$
c_3			0	0
c_4				0

with $\lambda k_2+\mu k_3=0$, coming from $c_2^3=0$. From this table we have J generated by c_3, c_4, $k_2c_1c_3$ and $k_3c_1c_3$ so necessarily $k_2\neq 0$ or $k_3\neq 0$. By symmetry we may study only the case $k_2\neq 0$. This means that $c_2^2=\mu(c_4-(k_3/k_2)c_3)$. The ideal B^2 is generated as a vector space by the vectors $k_2c_1c_3$, $k_3c_1c_3$ and $\mu(c_4-(k_3/k_2)c_3)$ so $1\leqq \dim B^2\leqq 2$.

B.2.2.1. $\dim B^2=1$. This means $\mu=0$. Calling $k_3=k$, $x_1=k_2^{-1}c_2$, $x_2=k_2^{-1}c_3$, $x_3=c_4$, $x_4=k_2^{-1}c_1c_3$, we get for B' the same table already obtained in case B.2.1.1.

B.2.2.2. $\dim B^2=2$. This means $\mu\neq 0$. Using now the basis of B $x_1=\mu^{-1}c_2$, $x_2=\mu^{-1}c_3$, $x_3=\mu^{-1}(c_4-(k_3/k_2)c_3)$ and $x_4=\mu^{-2}c_1c_3$, we get the same table already obtained in case B.2.1.2.

This ends the classification for the case where the type is $(3,2)$. We have obtained one one-parameter family of algebras when $\dim J=3$ and $\dim B^2=1$ and another one-parameter family when $\dim J=3$ and $\dim B^2=2$. The invariant $\dim \mathrm{An}(B)$ can be used to give a little bit more information about isomorphisms between algebras in the same family in the case $\dim B^2=1$.

Added in proof: some improvement of this classification will appear in a forthcoming paper by the author in *Linear Algebra and its Applications.*

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