L-SEMIRINGS

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By a *topological semiring* I mean a Hausdorff space together with two continuous associative operations, addition and multiplication, such that the multiplication distributes across the addition from both sides.

In the study of semirings it seems natural to ask what can be concluded about one operation in terms of information about the other and about the topology. In this note I shall point out a class of semirings whose multiplicative and topological structures form a commonly occurring product-like semigroup. I then prove a theorem about the additive structure, the possibilities for which I suspect will eventually turn out to be very limited.

DEFINITION. Let J be an I-semigroup, i.e., a semigroup homeomorphic with an arc (which will be identified with the interval $[0, 1]$) in which 0 is a zero and 1 is an identity element. The structure of such semigroups (here written multiplicatively) is completely determined in [1]. Let G be a compact group and H a closed normal subgroup of it. On the product semigroup $G \times J$ define the closed congruence α by $(g, x)\alpha(h, y)$ if (1) $x = y$ and $g = h$ or (2) $x = y = 0$ and $gh^{-1} \in H$. A semiring whose topological and multiplicative structures are given by the semigroup $G \times J/\alpha$ will be called an *L-semiring*. According to theorem C of [1], a semigroup with identity on a compact manifold with connected regular boundary B such that B is a subsemigroup is such a semigroup. Perhaps the simplest example of this type of semigroup is the unit complex disk.

EXAMPLE. Let $(G, \cdot)$ be a finite group and let 0 not be in G. Extend the multiplication on G to $G \cup \{0\}$ so that it becomes a group with zero. Define $+$ on $G \cup \{0\}$ by

$$x + y = \begin{cases} x & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \text{ for all } x \text{ and } y \text{ in } G \cup \{0\}.$$

Let $(J, \cdot)$ be the ordinary unit interval and define $+$ on J by $x + y = \min\{x, y\}$ for all x and y in J. Now $(G \cup \{0\}) \times J$ is a semiring under coordinatewise addition and multiplication. Also $A = (G \times \{0\}) \cup (\{0\} \times H)$ is both an additive

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and multiplicative ideal of $(G \cup \{0\}) \times J$. Now

$$\frac{(G \cup \{0\}) \times J}{(G \times \{0\}) \cup (\{0\} \times J)}$$

is an L-semiring in which each element is an additive idempotent. If we replace J in the above example by the one point compactification of $[1, \infty)$ under ordinary real addition and multiplication then the same procedure yields an L-semiring in which there is exactly one additive idempotent, a zero.

THEOREM. *In an L-semiring each additive subgroup is trivial.*

PROOF. Let S be an L-semiring with multiplicative and topological structures given by G, H and J as in the definition. We shall identify elements of S with the corresponding equivalence classes in $(G \times J)/\alpha$. Clearly the set of additive idempotents, $E[+]$, is the multiplicative ideal and thus of the form $(G \times [0, a])/\alpha$, where $a \in J = [0, 1]$.

It will be convenient to know that a is a multiplicative idempotent. Suppose this were not so. Then $0 < a$. Let

$$b = \inf \{x | x \in J, x^2 = x, \text{ and } a < x\}.$$

Clearly $\{x | x \in J, x^2 = x, \text{ and } a < x\}$ is closed and thus contains b. Therefore $b^2 = b$ and, since $a^2 \neq a$, we have $a < b$. Let g_1 denote the unit of G and $\alpha(n_g, n_j)$, which will be identified with the positive integer n, represent the n-fold sum of $\alpha(g_1, 1)$. Now

$$n\alpha(g_1, b) = \alpha(n_g, n_j)\alpha(g_1, b) = \alpha(n_g g_1, n_j b)$$

and $n_j b \leqq b$ for all positive integers n. If, for each n, $n_j b = b$ then

$$\{n\alpha(g_1, b) | n \geqq 1\} \subset (G \times [b, 1])/\alpha$$

which is closed. Consequently $\Gamma[+](\alpha(g_1, b)) \subset (G \times [b, 1])/\alpha$ which misses $(G \times [0, a])/\alpha$. (By $\Gamma[+](s)$ we mean $\{s, 2s, 3s, \cdots\}^*$ where s is any element of a semiring S and * denotes the closure in the topology of S. A discussion of such semigroups can be found in [2].) Thus we have a compact additive semigroup not intersecting $E[+]$. This leads us to the existence of an additive idempotent not in $E[+]$ which is a contradiction. Thus there exists a positive integer m such that $m_j b < b$. Because $a < b$, there is also an element c of J such that $cb = a$ so $\alpha(g_1, c)\alpha(g_1, b) = \alpha(g_1, a)$. Now, if $m_j b < a$ then

$$m_j a = m_j(bc) = (m_j b)c \leqq m_j b < a,$$

i.e., $m_j a < a$. On the other hand,

$$\alpha(m_g g_1, m_j a) = \alpha(m_g, m_j)\alpha(g_1, \alpha) = m\alpha(g_1, a) = \alpha(g_1, a)$$

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since $\alpha(g_1, a) \in E[+]$. Thus $m_j a = a$. This is a contradiction so that $m_j b \nless a$, i.e., $a \leqq m_j b$. But $b^2 = b$ and $a < b$ gives us that $ba = a$. Now

$$\alpha(m_g g_1, m_j ba) = \alpha(m_g, m_j)\alpha(g_1, ba) = m\alpha(g_1, ba) = m\alpha(g_1, a) = \alpha(g_1, a)$$

since $\alpha(g_1 a) \in E[+]$. Thus $m_j ba = a$. This implies that there is an idempotent of J in the interval $[a, m_j b]$. But recall that $m_j b < b$ and b is the minimal idempotent of $[a, 1]$. This gives us a contradiction so $a^2 = a$.

Let $\alpha(p, q)$ be in $E[+]$ such that $q \neq 0$ and suppose $\alpha(r, s) \in H[+](\alpha(p, q))$, the maximal additive subgroup containing $\alpha(p, q)$, such that $\alpha(r, s) \neq \alpha(p, q)$. Note that since a group contains only one idempotent, $s > a \geqq q$ guaranteeing that $qs = q$. Thus $\alpha(g_1, q)\alpha(r, s) = \alpha(r, q)$ so $\alpha(r, q) \in \alpha(g_1, q)H[+](\alpha(p, q))$. Furthermore $\alpha(r, q) \in E[+]$ because $q \leqq a$. However because multiplicative translations are additive homomorphisms, $\alpha(g_1, q)H[+](\alpha(p, q))$ is a group whose identity element is $\alpha(g_1, q)\alpha(p, q)$ which equals $\alpha(p, q^2)$. Therefore $\alpha(r, q) = \alpha(p, q^2)$ which gives us $q = q^2$ and, since $q \neq 0$, $r = p$. Now for any $\alpha(p, x)$ in $H[+](\alpha(p, q))$ and for any x^* such that $q \leqq x^* \leqq x$, there exists $x^{**} \geqq x^*$ such that $x^{**}x = x^*$. Thus $\alpha(g_1, x^{**})\alpha(p, x) = \alpha(p, x^*)$. Clearly

$$\alpha(p, x^*) \in \alpha(g_1, x^{**})H[+](\alpha(p, q))$$

which is a group having identity $\alpha(g_1, x^{**})\alpha(p, q)$. But $q^2 = q$ and $q \leqq x^{**}$ so $x^{**}q = q$ giving us $\alpha(g_1, x^{**})\alpha(p, q) = \alpha(p, q)$. Thus

$$\alpha(g_1, x^{**})H[+](\alpha(p, q)) \subset H[+](\alpha(p, q))$$

and so $\alpha(p, x^*) \in H[+](\alpha(p, q))$. Since $H[+](\alpha(p, q))$ is closed, we have shown that, for some element t of J, $H[+](\alpha(p, q)) = (\{p\} \times [q, t])/\alpha$, i.e., $H[+](\alpha(p, q)$ is homeomorphic to a closed interval. This cannot happen if the interval is non-trivial (since a non-trivial closed interval is not homogeneous) so we must conclude that $H[+](\alpha(p, q)) = \{\alpha(p, q)\}$.

Since $E[+]$ is a multiplicative ideal and $(G \times \{0\})/\alpha$ is the minimal multiplicative ideal, we see that $\alpha(d, 0) \in E[+]$ for any d in G. Suppose $\alpha(h^*, k^*) \in H[+](\alpha(d, 0))$ where $\alpha(h^*, k^*) \neq \alpha(d, 0)$. Then

$$\alpha(d^{-1}h^*, k^*) = \alpha(d^{-1}, 1)\alpha(h^*, k^*)$$

which is in $\alpha(d^{-1}, 1)H[+](\alpha(d, 0))$. But this set is an additive group with identity $\alpha(d^{-1}, 1)\alpha(d, 0)$ which equals $\alpha(g_1, 0)$. Therefore it is contained in $H[+](\alpha(g_1, 0))$. Thus $\alpha(d^{-1}h^*, k^*) \in H[+](\alpha(g_1, 0))$. Furthermore, because $\alpha(h^*, k^*) \neq \alpha(d, 0)$ we have $\alpha(d^{-1}h^*, k^*) \neq \alpha(g_1, 0)$, i.e., $H[+](\alpha(g_1, 0))$ is non trivial. If $\alpha(h, k) \in H[+](\alpha(g_1, 0))$ then

$$\alpha(h, 0) = \alpha(g_1, 0)\alpha(h, k) \in \alpha(g_1, 0)H[+](\alpha(g_1, 0))$$

which is an additive group having unit $\alpha(g_1, 0)$. Thus $\alpha(h, 0) \in H[+](\alpha(g_1, 0))$.

On the other hand $\alpha(h, 0) \in E[+]$ so $\alpha(h, 0) = \alpha(g_1, 0)$. Therefore $h \in H$ and we have shown that $H[+](\alpha(g_1, 0) \subset (H \times [0, 1])/\alpha$. Let

$$A = \{x | \text{ there is a } y \text{ in } G \text{ so that } \alpha(y, x) \in H[+](\alpha(g_1, 0))\}$$

and $\bar{k} = \sup A$. Since A is closed, $\bar{k} \in A$, i.e., there is an $\bar{h}$ such that $\alpha(\bar{h}, \bar{k}) \in H[+](\alpha(g_1, 0))$. Clearly $H[+](\alpha(g_1, 0)) \subset (H \times [0, \bar{k}])/\alpha$. If $\alpha(u, v) \in (H \times [0, \bar{k}])/\alpha$ then $v \leqq \bar{k}$ so there is a w in J such that $v = w\bar{k}$. Thus

$$\alpha(u, v) = \alpha(u\bar{h}^{-1}\bar{h}, w\bar{k}) = \alpha(u\bar{h}^{-1}, w)\alpha(\bar{h}, \bar{k})$$

which is an element of the additive group $\alpha(u\bar{h}^{-1}, w)H[+](\alpha(g_1, 0))$. Now

$$\alpha(u\bar{h}^{-1}, w)\alpha(g_1, 0) = \alpha(u\bar{h}^{-1}g_1, w0) = \alpha(g_1, 0))$$

because $u\bar{h}^{-1}g_1$ is in H. Therefore the additive idempotent $\alpha(g_1, 0) \in \alpha(u\bar{h}^{-1}, w)H[+](\alpha(g_1, 0))$ so this additive group is contained in $H[+](\alpha(g_1, 0))$. This implies $\alpha(u, v) \in H[+](\alpha(g_1, 0))$. We have now shown that $H[+](\alpha(g_1, 0)) = (H \times [0, \bar{k}])/\alpha$. But, according to [3], $H[+](\alpha(g_1, 0)$ is totally disconnected. Therefore $\bar{k} = 0$ and $H[+](\alpha(g_1, 0)) = \{\alpha(g_1, 0)\}$ which contradicts our assumption and concludes the proof.

An interesting special case of L-semirings is that in which the group G is trivial. In this situation, that is in a semiring which is multiplicatively an I-semigroup, we have much more information about the addition. If the addition is commutative then the interval is the union of three subintervals (disjoint except at the endpoints) each of which is a subsemiring. Of these $x+y = \min \{x, y\}$ in one interval and $x+y = \max \{x, y\}$ in another. The third interval has only one additive idempotent which is at one end and which is an additive zero. This last type of semiring has not yet been fully studied.

A description of this special case can be found in [4] for commutative additions. A more complicated description including the additively non-commutative case appears in [5] and [6].

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