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On algebraic rings

M. Chacron

A ring R is π-regular (periodic) if for each element x of R there is $n = n(x)$ so that $x^n = x^n.a.x^n$ ($x^n = x^n.1.x^n$) (a depending on x). Let R be an algebraic algebra over a commutative ring F with identity. In this paper we prove that if every π-regular image of the ring F is periodic, then R is periodic. This result applies in particular to the algebraic rings R (over the integers) considered by Drazin and to the algebraic algebras R over algebraically prime fields. It extends a result of Drazin on torsin-free algebraic rings and a generalization by this author of Drazin's result.

In [5, Cor. 3.1] Drazin proved that if R is a torsion-free ring which is algebraic over the integers, then R is a nil ring. In [3, Prop. 2] we proved that every ring R which is algebraic over the integers, must in fact be a periodic ring (i.e., $\forall\, x \in R\ \exists\, n > m \geq 1$, $x^n = x^m$ [1, Introduction]). Since a periodic ring can be characterized as a ring such that given x, we can find $n = n(x)$ such that x^n generates a finite subring [1, p. 6 and 3, Prop. 1], it is clear that Drazin's result is a particular case of [3, Prop. 2]. In [5, Th. 5.6] (and in other well-known papers) rings R algebraic over finite fields or, more generally, over periodic fields were considered. It is therefore natural to seek a generalization of [3, Prop. 2] to algebraic algebras R over arbitrary commutative rings F with 1 (in the sense of Drazin[1]) with F extending

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[1] However, in this paper we let the elements of F be pure 'scalars' of the algebra A. Thus A might be annihilated by non-zero scalars of F.

suitably the case of the integers and that of the periodic rings. In this paper we prove that if the ring F satisfies the condition that

(c) every π-regular homomorphic image of F is periodic, then every algebra algebraic over F must in fact be periodic.

Since every π-regular image of the integers is finite, hence periodic, we see that our result contains as particular case [3, Prop. 2].

If R is an algebraic algebra with the property (c) , say a (c)-algebra, then each idempotent element e of R will generate a locally finite subring $F.e$ (i.e., every finitely generated subring of $F.e$ is finite). In fact, $F.e$ is a subalgebra of R ring homomorphic to F. Since $F.e$ is algebraic, $F.e$ is π-regular. By (c) , $F.e$ is periodic. Since F has a unit and is commutative, $F.e$ is a unitary commutative periodic ring, hence $F.e$ is locally finite. (This result was shown in [4, Th. 2] in the more general case of a periodic ring with 1 satisfying a polynomial identity.) We have proved the following

LEMMA. *If R is a (c)-algebra, then each of its cyclic subalgebras generated by an idempotent is locally finite (as a ring).*

We are now in a position to prove the

THEOREM. *Every (c)-algebra R is periodic.*

Notations. For $a \in R$, $[a]$ $(\langle a \rangle)$ denotes the multiplicative subsemigroup (subring) of R generated by a .

Proof. We have to prove that $[a]$ is finite for all $a \in R$, which will be certainly the case for any nilpotent element a of R. Assume that a is non-nilpotent. We can find a polynomial $p(t)$ in one indeterminate having as coefficients elements of F such that $a^m = p(a).a^{m+1}$. We may assume (without loss of generality) that $p(t)$ is a non-constant polynomial without constant term, whence $p(a) \in R$. By standard computation

$$a^m = p(a).a.a^m = p^2(a)a^2.a^m = \dots = a^m.p^m(a).a^m ,$$

so, for $e = p^m(a).a^m$, we have

$$0 \neq e = e^2 = p^m(a).a^m , \quad a^m.e = e.a^m = a^m .$$

Set: $R_e = e.R.e$; $b = ae = ea$. We have $p(ae) = p(a).e$ so that

$$e = b^m.p^m(b) \ , \quad b^m = a^m \quad \text{and} \quad b^m = p(b).b^{m+1} \ .$$

From $b \in R_e$ and $e = b^m.p^m(b)$, we derive that b is an invertible element of the ring R_e . Then $b^m = p(b).b^{m+1}$ yields $e = p(b).b$, which is to say that the inverse c of b in R_e is precisely $p(b)$; in symbols:

$$c = b^{-1} = p(b) = \alpha_1 b + \ldots + \alpha_k b^k$$

for some $\alpha_1,\ldots,\alpha_k \in F$. By multiplying both sides of the equality by b^{-k-1} we obtain an equality of the form

$$c^{k+2} = \alpha_k c + \ldots + \alpha_1 c^k \ .$$

By standard computation

$$[c] \subseteq \sum_{j=1}^{k} A.c^j$$

where A stands for the subring of F generated by the set $\{\alpha_1,\ldots,\alpha_k\}$. By the Lemma, $F.e$ is locally finite. It follows that the subring $\langle \alpha_j.e \rangle_j$ generated by the set $\{\alpha_1.e,\ldots,\alpha_k.e\}$ is finite. Clearly, $\langle \alpha_j.e \rangle_j = \langle \alpha_j \rangle.e = A.e$. Therefore $A.e$ is finite, *a fortiori* $A.c^j = (A.e)c^j$ is finite, whence $\sum_j A.c^j$ is finite. All in all we have proved that $[c]$ is finite, which, combined with the property for c to possess the inverse b in R_e, tells us that $b^l = c^l = e$ for some $l \geq 1$.

From $b^m = a^m$, follows $e = e^2 = e^m = b^{lm} = b^{ml} = a^{ml} \in [a]$, proving thereby that $[a]$ is finite; this for every non-nilpotent element a of R , and R is periodic.

COROLLARY 1. *Let R be a ring. The following conditions are equivalent.*

(i) R *is algebraic over the integers.*

(ii) R *is periodic.*

(iii) $\forall\, x \in R\ \exists\, n,\ m \geq 1\ ,\ (x^{n+1}-x)^m = 0$.

(iv) For some two-sided ideal A *of* R *,* R/A *and* A *are periodic.*

This Corollary is an immediate consequence of the Theorem. It tells us by standard argumentation that if R is an arbitrary ring, one can define a maximal periodic ideal L such that R/L is periodic-simple (i.e., it has no non-zero periodic ideals). Also, by *(iii)*, every periodic ring R is a quasi-radical extension of the subring generated by the nilpotent elements of R . From [7] follows immediately

COROLLARY 2. *Every* (c)*-algebra having all its nilpotent elements central is commutative.*

This Corollary extends [5, Th. 5.5]. We note that in [5, Th. 5.5] or [6, Th.], the authors used [8, Th.], which is more general than [7]. The following extends [5, Th. 5.6].

COROLLARY 3. *Let* F *be a commutative ring with* 1 . *Let* R *be an algebraic algebra over* F . *Assume that* F *is algebraic over its prime subring* $\langle 1 \rangle$. *If, further, all nilpotent elements of* R *are central, then* R *is commutative.*

Proof. Clearly F is a (c)-algebra with respect to its subring $\langle 1 \rangle$. Therefore, F is periodic (Theorem). Consequently, R is a (c)-algebra, whence periodic. By Corollary 2, R is commutative.

REMARK. By a general property of periodic rings [2, Th. 9], if R is an algebra as in Corollary 3, its subdirect irreducible components A are local rings. Also, R , modulo its prime radical, is a ring in which every element x satisfies $x = x^{n+1}$, for some $n \geq 1$ depending on x .

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University of Windsor,
Windsor, Ontario,
and
University of British Columbia,
Vancouver, British Columbia.