INTEGRAL $\vee$-IDEALS

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1. Introduction. Let R be an integral domain with quotient field K. A fractional ideal I of R is a $\vee$-ideal if I is the intersection of all the principal fractional ideals of R which contain I. If I is an integral $\vee$-ideal, at first one is tempted to think that I is actually just the intersection of the principal integral ideals which contain I. However, this is not true. For example, if R is a Dedekind domain, then all integral ideals are $\vee$-ideals. Thus a maximal ideal of R is an intersection of principal integral ideals if and only if it is actually principal. Hence, if R is a Dedekind domain, each integral $\vee$-ideal is an intersection of principal integral ideals precisely when R is a PID.

In this paper we study domains in which each integral $\vee$-ideal is an intersection of principal integral ideals. We also study the weaker property that each integral $\vee$-ideal be contained in a principal integral ideal. The relationship between these properties and the PSP-property introduced by Arnold and Sheldon [**1**] is then investigated.

R will always be an integral domain with quotient field K. For a fractional ideal I of R, let $I^{-1}=R:I=\{x\in K \mid xI\subseteq R\}$. We will usually denote $(I^{-1})^{-1}$ by $I_\vee$. Here $I_\vee$ is just the intersection of all the principal fractional ideals of R which contain I. If $I=I_\vee$, I is called a *divisorial* or $\vee$*-ideal*. A $\vee$-ideal I is of *finite type* if $I=J_\vee$ for some finitely generated fractional ideal J of R. Our general reference is Gilmer [**6**]. Also, $\subset$ will denote proper inclusion.

2. Integral $\vee$-Ideals. For an integral ideal I of R, let I_p be the intersection of all the principal integral ideals of R which contain I. Clearly $I_\vee\subseteq I_p$. We will say that R satisfies the *IP-property* if $I_\vee=I_p$ for all integral ideals I of R. Thus R satisfies the IP-property if and only if each integral $\vee$-ideal of R is an intersection of principal integral ideals of R. One is tempted to try to define a $*$-operation [**6**, p. 392] on the set $F(R)$ of fractional ideals of R by first defining the $*$-operation on the integral ideals of R by $I^*=I_p$, and then extending this to $F(R)$ [**6**, p. 393]. However, for any $*$-operation, $I^*\subseteq I_\vee$ [**6**, p. 417]. Thus the p-operation defines a $*$-operation if and only if it corresponds with the $\vee$-operation; that is, if and only if R satisfies the IP-property.

We will say that R satisfies the *CP-property* if each proper integral $\vee$-ideal of R is contained in a proper principal integral ideal of R. Clearly the IP-property implies the CP-property. We recall that R is a GCD-domain if each pair of nonzero elements of R has a greatest common divisor. It is well known that R is a GCD-domain if and only if each pair of nonzero elements of R has a least common multiple, or equivalently, the intersection of two principal fractional ideals of R is principal. First we give easily proved characterizations of these properties in terms of the intersection of two fractional ideals.

LEMMA 2.1. (1) *R is a* GCD*-domain if and only if $R\cap xR$ is principal for each $x\in K$.*

(2) *R satisfies the IP-property if and only if $R\cap xR$ is an intersection of principal integral ideals for each $x\in K$.*

(3) *R satisfies the CP-property if and only if whenever* $I = R \cap xR \subset R$ *for some* $x \in K$ *then I is contained in a proper principal integral ideal of R.*

Lemma 2.1 shows that a GCD-domain satisfies the IP-property, and that the IP-property implies the CP-property. However, we will see that none of these implications is reversible. We next show that the GCD-, IP-, and CP-properties are all equivalent if R satisfies the ascending chain condition on principal ideals. In fact, in this case the properties are all equivalent for the trivial reason that all $\vee$-ideals of R are actually principal. In particular, these properties are equivalent if R is either Noetherian or a Krull domain.

PROPOSITION 2.2. *If R satisfies the ascending chain condition on principal ideals the following are equivalent.*

(1) *R is a UFD.*

(2) *R is a GCD-domain.*

(3) *All* $\vee$*-ideals of R are principal.*

(4) *R satisfies the IP-property.*

(5) *R satisfies the CP-property.*

Proof. It is well known that (1) and (2) are equivalent if R satisfies the ascending chain condition on principal ideals. Clearly (2)$\Rightarrow$(4), (3)$\Rightarrow$(2), and (3)$\Rightarrow$(4)$\Rightarrow$(5) hold without any chain conditions. For (5)$\Rightarrow$(3), let I be a proper integral $\vee$-ideal of R. Let $\mathscr{C} = \{xR \mid I \subseteq xR \subset R\}$. By (5), $\mathscr{C} \neq \emptyset$. Since $\mathscr{C}$ is bounded below, $\mathscr{C}^{-1} = \{x^{-1}R \mid xR \in \mathscr{C}\}$ is bounded above. But R satisfies the ascending chain condition on principal ideals, so $\mathscr{C}^{-1}$ has a maximal element $y^{-1}R$. Hence $\mathscr{C}$ has a minimal element yR. If $I \subset yR$, then $y^{-1}I$ is a proper integral $\vee$-ideal, and thus $y^{-1}I \subseteq zR \subset R$ for some $z \in R$. But then $I \subseteq yzR \subset yR$, a contradiction.

In order to have a ready supply of non-trivial examples, we characterize when the $D+M$ construction yields rings which satisfy the IP- or CP-property. For more details on the $D+M$ construction see [**2**] or [**6**].

PROPOSITION 2.3. *Let V be a nontrivial valuation ring of the form* $K+M$, *where K is a field and M is the maximal ideal of V. Let R be the subring* $D+M$, *where D is a proper subring of the field K.*

(1) *R satisfies the IP-property if and only if D satisfies the IP-property and D is not a field.*

(2) *R satisfies the CP-property if and only if D satisfies the CP-property and D is not a field.*

Proof. We will prove (1); the proof of (2) is similar. Suppose that R satisfies the IP-property. Since M is always a $\vee$-ideal of R [**2,** Theorem 4.1], if D is a field, then M is a principal ideal of R. But then $D = K$, a contradiction. Let I be an integral $\vee$-ideal of D. Then $I+M$ is an integral $\vee$-ideal of R [**2,** Theorem 4.1]. Since R satisfies the IP-property, $I+M = \bigcap y_\alpha R$ for some $y_\alpha \in R$. Each $y_\alpha = x_\alpha + m_\alpha$ for some $m_\alpha \in M$ and $0 \neq x_\alpha \in D$. But then $I+M = \bigcap y_\alpha R = \bigcap x_\alpha R = \bigcap x_\alpha D + M$ [**2,** Lemma 3.12], so $I = \bigcap x_\alpha D$, and thus D satisfies the IP-property.

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Conversely, suppose that D satisfies the IP-property and is not a field. Let J be an integral $\vee$-ideal of R. If $J \supset M$, then $J = I + M$ for some integral $\vee$-ideal I of D [**2**, Theorems 2.1 and 4.1]. Since I is an intersection of principal integral ideals of D, just as above, $J = I + M$ is an intersection of principal integral ideals of R. If $J = M$, then $M = \bigcap \{xR \mid 0 \neq x \in D\}$ [**2**, Theorem 4.1]. Thus we may assume that $J \subset M$ [**2**, Theorem 2.1]. Let $J = \bigcap z_\alpha R$ for some z_α in the quotient field of R. Since each $z_\alpha R$ compares with V under inclusion [**2**, Theorem 3.1], we may assume that each $z_\alpha = x_\alpha + m_\alpha$ with $x_\alpha \in K$ and $m_\alpha \in M$. If $x_\alpha \neq 0$, then $z_\alpha R = x_\alpha R = x_\alpha D + M \supset M$ [**2**, Lemma 3.12]. Thus $J = \bigcap \{z_\alpha R \mid x_\alpha = 0\}$.

EXAMPLE 2.4. Let $V = \mathbf{R}[[X]] = \mathbf{R} + M$, where $M = XV$. Then $R = \mathbf{Z}_{(2)} + M$ satisfies the IP-property by the previous proposition. But R is not a GCD-domain [**2**, Theorem 3.13].

EXAMPLE 2.5. (I would like to thank J. Matijevic for suggesting this example.) Let $D = \mathbf{Q}[[X^2, X^3]]$ be the subring of $\mathbf{Q}[[X]]$ which consists of those power series with zero linear term. Write $D = \mathbf{Q} + M$, where M is the unique maximal ideal of D. Then let $R = \mathbf{Z}_{(2)} + M$ be the subring of D whose constant terms lie in $\mathbf{Z}_{(2)}$. Clearly R satisfies the CP-property since R is quasi-local with principal maximal ideal $2\mathbf{Z}_{(2)} + M = 2R$. We will show that R does not satisfy the IP-property by showing that the $\vee$-ideal $I = R \cap XR = X^3\mathbf{Q} + X^4\mathbf{Q} + \ldots$ is not an intersection of principal integral ideals. In fact,

Claim. Let J be the intersection of all the principal integral ideals of R that contain I. Then $J = M$.

Proof. Clearly $J \subseteq M$, since $M = \bigcap 2^n\mathbf{Z}_{(2)} + M$. Suppose that $I \subseteq yR \subset R$, say $y = a_0 + a_2X^2 + \ldots$. Since all $(1/n)X^3 \in I$, $a_0 \neq 0$. But then $M \subseteq yR$ because y is a unit in D.

Example 2.4 shows that neither the IP nor the CP-property is preserved by localization. For if we let $S = \mathbf{Z}\backslash\{0\}$, then $R_S = \mathbf{Q} + M$ does not satisfy the IP- or CP-property by Proposition 2.3.

Example 2.4 may be easily modified to give an example of an integrally closed domain which satisfies the IP-property, but it is not a GCD-domain. For example, we could let $V = K[[X]] = K + M$ where $K = \mathbf{Q}(S, T)$, and then $R = \mathbf{Q}[S] + M$ is integrally closed [**2**, Theorem 2.1], but not a GCD-domain [**2**, Theorem 3.13]. However, the above examples are not completely integrally closed (recall that R is *completely integrally closed* if for $x \in K$, $0 \neq a \in R$, $ax^n \in R$ for all $n \geq 1$, then $x \in R$). We do not know if a completely integrally closed domain that satisfies the IP-property is necessarily a GCD-domain. Probably it is not. The difficulty is that the $D + M$ and similar constructions never yield completely integrally closed domains. Also, one can use the Kaplansky–Krull–Jaffard–Ohm Theorem [**6**, p. 215] to construct completely integrally closed domains, but these are necessarily Bezout domains. However, if R is completely integrally closed, the IP-property and CP-property are equivalent.

PROPOSITION 2.6. *If R is completely integrally closed, then R satisfies the IP-property if and only if R satisfies the CP-property.*

Proof. Let I be a proper integral $\vee$-ideal of R. Then J, the intersection of the principal integral ideals of R which contain I, is a proper integral $\vee$-ideal with $I \subseteq J$. Since R is completely integrally closed, the set of $\vee$-ideals of R forms a group. If $I \subset J$, then $(IJ^{-1})_\vee \subset R$. So by hypothesis, $(IJ^{-1})_\vee \subseteq xR \subset R$ for some $x \in R$. But then $I \subseteq xJ \subset J$. But xJ is also an intersection of principal integral ideals, so $J \subseteq xJ$, a contradiction.

If R satisfies the CP-property, then any integral ideal I which is maximal with respect to being a $\vee$-ideal is principal. If, in addition, R is completely integrally closed, I is necessarily a principal prime ideal [**5**, p. 13].

If each maximal ideal of R is principal, clearly R satisfies the CP-property. In the special case that R is also quasi-local with principal maximal ideal we can ask when R is a GCD-domain, or equivalently, (by Proposition 2.7) when R is a valuation domain. Example 2.4 shows that in general this is not true. Recall that R is a *finite conductor domain* if the intersection of two principal ideals of R *is* finitely generated.

Proposition 2.7. *Let R be a quasi-local domain with principal maximal ideal $M = xR$.*

(1) *If R is integrally closed, then R is a valuation ring if and only if R is a finite conductor domain.*

(2) *R is a valuation ring if any of the following conditions hold:*

(a) *R is completely integrally closed.*

(b) *R satisfies the ascending chain condition on principal ideals.*

(c) *R has Krull dimension one.*

(d) *R is a GCD-domain.*

(e) *R is coherent.*

Proof. (1) follows from a result of Zafrullah [**12,** Lemma 5] about t-ideals. (2) (a) and (c) follow because then $\bigcap x^nR = 0$ [**6,** p. 74].

If R satisfies the ascending chain condition on principal ideals, then (5)$\Rightarrow$(1) of Proposition 2.2, shows that R is a UFD, and hence a DVR. Thus (b) holds. Finally, (d) is a special case of (1), while (e) is a special case of [**11,** p. 60 Lemma 3.9].

Note that in (a), (b), and (c) above, R is actually a DVR.

3. The PSP-Property and Schreier Rings. Based on earlier work of Tang [**10**], Arnold and Sheldon [**1**] defined a finitely generated integral ideal I of R to be *primitive* if it is contained in no proper principal integral ideal of R and to be *super-primitive* if $I^{-1} = R$. Clearly a super-primitive ideal is primitive. They defined R to satisfy the PSP-property if each primitive ideal is super-primitive. Our next result, implicit in their work, shows that the CP-property implies the PSP-property.

Proposition 3.1. *R satisfies the PSP-property if and only if each proper integral $\vee$-ideal of finite type is contained in a proper principal integral ideal of R.*

Proof. Let I be a finitely generated ideal of R with $I_\vee \subset R$. If $I_\vee$ is not contained in any proper principal integral ideal, then neither is I. Thus I is primitive, so $I^{-1} = R$ because R satisfies the PSP-property. But then $I_\vee = (I^{-1})^{-1} = R$, a contradiction. The converse may be proved similarly.

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Thus the difference between the CP- and PSP-properties is whether we consider all the integral $\vee$-ideals or just those of finite type. In [**1,** p. 49], it is shown that if $V = K + M$ is a rank-one non-discrete valuation ring, then for any proper subfield F of K, $R = F + M$ satisfies the PSP-property; but R does not satisfy the CP-property by Proposition 2.3.

However, if R is a finite conductor domain, then the PSP-property and CP-property are equivalent. This follows easily from Lemma 2.1.

If $R[X]$ satisfies either the IP- or CP-property, then so does R. But polynomial extensions, like localizations, need not preserve either property [**1,** Theorem 3.3].

An element x of R is *primal* if $x \mid ab$ implies $x = cd$ with $c \mid a$ and $d \mid b$. An integrally closed domain in which each element is primal is called a *Schreier ring* [**3**], [**4**]. A GCD-domain is a Schreier ring, but the converse need not be true [**3,** p. 256], or Example 3.2. If R satisfies the ascending chain condition on principal ideals, then the PSP-property or Schreier property imply that R is a UFD [**1,** p. 42], [**3,** Theorem 2.3]. This gives another proof of (5)$\Rightarrow$(1) of Proposition 2.2.

In general, there is no relationship between Schreier rings and the IP- or CP-property. For the $F + M$ examples of Arnold and Sheldon mentioned earlier are Schreier rings as long as F is algebraically closed in K [**8,** p. 80]. McAdam and Rush show that if all elements of R are primal, then R satisfies the PSP-property [**8**, p. 80]. They also ask if the PSP-property implies that all elements of R are primal. Example 2.5 shows that it does not. For that R satisfies the CP-property, and $X^3 \mid X^2X^4$, but there do not exist $a, b \in R$ with $X^3 = ab$ so that $a \mid X^2$ and $b \mid X^4$. Another example, due to G. M. Bergman, is in [**3,** p. 262]. However, neither of these examples is integrally closed.

Finally, we give an example of a completely integrally closed domain D which satisfies the PSP, but not the CP-property. This is the example of Heinzer and Ohm [**7**] of an essential domain that is not a Prüfer $\vee$-multiplication domain. We follow their notation.

EXAMPLE 3.2. Let k be a field and $R = k(x_1, x_2, \ldots)[y, z]_{(y,z)}$. For each i, let V_i be the DVR containing $k(\{x_j\}_{j \neq i})$ obtained by giving x_i, y, and z the value 1 and then taking infimums. Then let $D = R \cap \{V_i \mid i = 1, 2, \ldots\}$. D is completely integrally closed since R and each V_i is completely integrally closed. G, the group of divisibility of D, is $\vee$-embedded in $H \oplus (\prod \mathbf{Z}_i)$, where H is the group of divisibility of R. A proper integral $\vee$-ideal I of D of finite type corresponds to an element $w = (h, t_1, t_2, \ldots .)$ with $h, t_i \geq 0$ and some $t_n > 0$. But the principal integral ideal x_nD corresponds to $(0, e_n) \leq w$. Thus $x_nD \supseteq I$; so D satisfies the PSP-property. However, the proper integral $\vee$-ideal $J = (\{y/x_i \mid i = 1, 2, \ldots\})_\vee = D \cap y/zD$ is contained in no proper principal integral ideal of D. For J corresponds to $(h, 0, 0, \ldots)$ with $h > 0$, and a positive element of G of the form $(h, t_1, t_2, \ldots)$ with $h > 0$ necessarily has $t_n > 0$ for all large n. Thus D does not satisfy the CP-property. It may be shown that D is actually a Schreier ring because it is an ascending union of UFD's [**9**].

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