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CONGRUENCE NETWORKS FOR COMPLETELY SIMPLE SEMIGROUPS

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Abstract

The operators K, k, T and t are defined on the lattice $\mathscr{C}(S)$ of congruences on a Rees matrix semigroup S as follows. For $\rho \in \mathscr{C}(S)$, ρK and ρk (ρT and ρt) are the greatest and the least congruences with the same kernel (trace) as ρ, respectively. We determine the semigroup generated by the operators K, k, T and t as they act on all completely simple semigroups. We also determine the network of congruences associated with a congruence $\rho \in \mathscr{C}(S)$ and the lattice generated by it. The latter is then represented by generators and relations.

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1. Introduction and summary

Completely simple semigroups, that is semigroups without proper ideals and with a primitive idempotent, have their best representation as Rees matrix semigroups, which we write as $\mathscr{M}(I, G, \Lambda; P)$. Congruences on a Rees matrix semigroup have their best representation as triples (r, N, π) where r is a partition of I, N is a normal subgroup of G and π is a partition of Λ satisfying a single condition. The sandwich matrix P can always be normalized. This makes it possible to replace that single condition by two simpler and more transparent ones. If we wish to regard a congruence as being given by its kernel and its trace, this means that in the triple representation (r, N, π) we must fuse the partitions r and π into a single entity which can be thought of as a special type of partition of the Cartesian product $I \times \Lambda$. This we call normal. In this way, we represent a congruence on a Rees matrix semigroup by a pair (N, θ) where N is a normal subgroup of G and θ is a normal partition of $I \times \Lambda$, connected by two very simple conditions.

For any regular semigroup S, we have two relations of fundamental importance on the congruence lattice $\mathscr{C}(S)$ of S. For any $\rho \in \mathscr{C}(S)$, the kernel of ρ is the set of elements of S ρ-related to idempotents, the trace of ρ is the restriction of ρ to the set of idempotents of S. We now denote by ρK and ρk (respectively ρT and ρt) the greatest and the least congruences on S with the same kernel (respectively trace) as ρ. This creates the set $\Gamma = \{K, k, T, t\}$ of operators on $\mathscr{C}(S)$. By iteration, we get the semigroup Γ^+, or monoid Γ^*, of operators on $\mathscr{C}(S)$. In particular, we may fix a congruence ρ on S and consider Γ^* acting on ρ alone, which amounts to producing a network of congruences ρ, ρK, $\rho k, \ldots$ ordered by inclusion. The set Γ of operators may be enlarged by adding to it T_l, T_r, t_l and t_r pertaining to left and right traces of congruences.

Various aspects of the problem area indicated above can be found in the literature. The sudy of networks of congruences was inaugurated (for inverse semigroups) by Petrich and Reilly [7]. Networks on the lattice of varieties of completely regular semigroups were considered by Pastijn and Trotter [3] and Petrich and Reilly [8]. Different aspects of various networks were examined for arbitrary regular semigroups by the author in [5] and [4].

For a Rees matrix semigroup S and for a fixed congruence ρ on S, we study the network of congruences $\rho\Gamma^*$ and construct the lattice L_ρ generated by it. On the way, we determine the defining relations for the generators Γ as Γ^+ acts on all completely simple semigroups. Our success is due primarily to the most felicitous description of congruences on a Rees matrix semigroup in terms of admissible pairs as indicated above. It would be definitely intriguing to see whether our results can be extended to more general classes of semigroups.

In Section 2 we list a few preliminaries; for the remainder we refer to the relevant literature. We treat the representation of congruences on a Rees matrix semigroup in Section 3 in considerable detail as this is fundamental to the remainder of the paper. In Section 4, we introduce the operators K, k, T and t, as well as certain functions they induce on normal subgroups of G and normal partitions of $I \times \Lambda$. This section contains all the crucial lemmas needed for the main results of the paper. We consider in Section 5 an example of a Rees matrix semigroup which shows that all the important congruences we will have encountered earlier are distinct. In Section 6, we determine the semigroup generated by our operators on all completely simple semigroups. We determine in Section 7 the network of congruences associated with a congruence and in Section 8 the lattice generated by it and represent it in terms of generators and relations.

2. Preliminaries

Throughout the paper, let

$$S = \mathscr{M}(I, G, \Lambda; P)$$

be a *Rees matrix semigroup* over a group G and normalized sandwich matrix P. Recall that S is defined on the set $I \times G \times \Lambda$ and $P : \Lambda \times I \to G$ is a mapping, in notation $P = (p_{\lambda i})$, with multiplication

$$(i, g, \lambda)(j, h, \mu) = (i,\ g p_{\lambda j} h,\ \mu).$$

We suppose, as we may, that P is *normalized* at $1 \in I \cap \Lambda$, that is $P_{\lambda 1} = P_{1i} = e$, the identity element of G, for all $i \in I$ and $\lambda \in \Lambda$. Note that

$$E = \left\{(i, p_{\lambda i}^{-1}, \lambda) \mid\ i \in I,\ \lambda \in \Lambda\right\}$$

is the set of idempotents of S. We will identify e with the singleton set $\{e\}$.

For any set X, we denote by X^* and X^+ the *free monoid* and the *free semigroup* on X, respectively. The elements of X^+ are nonempty words over X under concatenation and $X^* = X^+ \cup \{1\}$ where 1 denotes the empty word and acts on X^* as an identity element. If Σ is a set of relations among generators X, then $\langle X / \Sigma \rangle$ denotes the corresponding quotient semigroup.

For any set X, we denote by ϵ and ω the *equality* and the *universal relations* on X, respectively.

3. Congruences on a Rees matrix semigroup

Congruences on S are described by the following device. Let r be a partition of I, N be a normal subgroup of G and π be a partition of Λ satisfying

(1) if $\quad i\ r\ j$ or $\lambda\ \pi\ \mu$, then $p_{\mu i} p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1} \in N$.

In such a case, (r, N, π) is an *admissible triple* for S and the relation ρ defined on S by

$$(i, g, \lambda)\ \rho\ (j, h, \mu) \Leftrightarrow i\ r\ j, \qquad gh^{-1} \in N, \quad \lambda\ \pi\ \mu$$

is a congruence on S. Conversely, every congruence on S can be so represented for a unique admissible triple. The inclusion and the meet and join operations are naturally transferred to the set of admissible triples thereby producing the following formulae:

$$\begin{aligned}
(r, N, \pi) \leq (r', N', \pi') &\Leftrightarrow r \subseteq r',\ N \subseteq N',\ \pi \subseteq \pi', \\
(r, N, \pi) \wedge (r', N', \pi') &= (r \wedge r',\ N \wedge N',\ \pi \wedge \pi'), \\
(r, N, \pi) \vee (r', N', \pi') &= (r \vee r',\ N \vee N',\ \pi \vee \pi')
\end{aligned}$$

where the joins $r \vee r'$ and $\pi \vee \pi'$ are of equivalence relations.

These results and the accompanying considerations can be found in ([1, III.4]). We now modify the representation of congruences on S by admissible triples by essentially fusing the partitions r and π together into a single entity θ which can be thought of as a partition of the set $I \times \Lambda$.

For an admissible triple (r, N, π) for S, since the matrix P is normalized, we may replace condition (1) by the following two simpler conditions

$$(2) \qquad \begin{cases} irj \ \Rightarrow \ p_{\lambda i} p_{\lambda j}^{-1} \in N & \text{for all } \lambda \in \Lambda, \\ \lambda \pi \mu \ \Rightarrow \ p_{\lambda i} p_{\mu i}^{-1} \in N & \text{for all } i \in I. \end{cases}$$

Next we write $\theta = (r, \pi)$ and call θ a *normal partition* of $I \times \Lambda$, and in (2) write θ instead of both r and π. We are thus led to the following concept.

DEFINITION. Let N be a normal subgroup of G and let θ be a normal partition of $I \times \Lambda$ satisfying

$$\begin{aligned}
i\theta j \ &\Rightarrow \ p_{\lambda i} p_{\lambda j}^{-1} \in N \qquad \text{for all } \lambda \in \Lambda, \\
\lambda\theta\mu \ &\Rightarrow \ p_{\lambda i} p_{\mu i}^{-1} \in N \qquad \text{for all } i \in I.
\end{aligned}$$

Then (N, θ) is an *admissible pair* for S. In such a case, define a relation ρ on S by

$$(i, g, \lambda)\ \rho\ (j, h, \mu) \Leftrightarrow i\theta j, \qquad gh^{-1} \in N, \quad \lambda\ \theta\ \mu,$$

and write $\rho \sim (N, \theta)$. Denote by $\mathscr{A}\mathscr{P}$ the set of all admissible pairs for S and by $\mathscr{C}$ the congruence lattice of S.

In view of the above discussion of admissible triples and their conversion into admissible pairs, we conclude that ρ is a congruence and, conversely, that every congruence on S admits a unique representation as an admissible pair. In addition, for two such pairs, we get

$$\begin{aligned}(N,\theta) \leq (N',\theta') &\Leftrightarrow N \subseteq N',\ \theta \subseteq \theta',\\ (N,\theta) \wedge (N',\theta') &= (N \wedge N',\ \theta \wedge \theta'),\\ (N,\theta) \vee (N',\theta') &= (N \vee N',\ \theta \vee \theta').\end{aligned}$$

This makes $\mathscr{A}\mathscr{P}$ into a lattice isomorphic to $\mathscr{C}$. We continue with the lattice notation: $N \wedge N' = N \cap N',\ N \vee N' = NN',\ \theta \wedge \theta' = \theta \cap \theta'$, etcetera for the sake of symmetry.

We will need the following special congruences on S:

$\mu = \mathscr{H}$ - the greatest idempotent separating congruence,

τ - the greatest idempotent pure congruence,

σ - the least group congruence.

4. Operators on the congruence lattice

This section contains the key lemmas to be used later for proving the main results. These lemmas pertain mainly to the four operators defined below.

We specialize some concepts from general regular semigroups to completely simple semigroups. For any $\rho \in \mathscr{C}$, we define the *kernel* and the *trace* of ρ by

$$\begin{aligned}\ker\rho &= \{a \in S \mid a\ \rho\ e \text{ for some } e \in E\},\\ \operatorname{tr}\rho &= \rho\,|_E,\end{aligned}$$

respectively. On the lattice $\mathscr{C}$ we define the relations $\mathscr{K}$ and $\mathscr{T}$ by

$$\lambda \mathscr{K} \rho \Leftrightarrow \ker\lambda = \ker\rho, \qquad \lambda T \rho \Leftrightarrow \operatorname{tr}\lambda = \operatorname{tr}\rho.$$

Then $\mathscr{K}$ and $\mathscr{T}$ are the *kernel* and *trace relations* on $\mathscr{C}$, respectively.

For any $\rho \in \mathscr{C}$, we will use the following notation:

ρK - the greatest congruence on S with the same kernel as ρ,

ρk - the least congruence on S with the same kernel as ρ,

ρT - the greatest congruence on S with the same trace as ρ,

ρt - the least congruence on S with the same trace as ρ.

The existence of these congruences follows from Lemma 2 below in the present case. For general regular semigroups, it was established in [2]. This produces the set

(3) $$\Gamma = \{K, k, T, t\}$$

of operators on $\mathscr{C}$.

We can now transfer, in an obvious way, these four operators from $\mathscr{C}$ to $\mathscr{A}\mathscr{P}$. This means that

$$(N,\theta)\gamma = (N',\theta') \text{ if } \rho\gamma = \rho' \qquad \text{where} \quad \rho \sim (N,\theta) \text{ and } \rho' \sim (N',\theta'),$$

for $\gamma \in \Gamma$.

In order to describe the value of $(N,\theta)\gamma$, we will need the following constructions. For a normal subgroup N of G, let $\overline{N}$ be the normal partition of $I \times \Lambda$ defined by

$$\begin{array}{lcll} i\overline{N}j & \Leftrightarrow & p_{\lambda i}p_{\lambda j}^{-1} \in N & \text{for all } \lambda \in \Lambda, \\ \lambda\overline{N}\mu & \Leftrightarrow & p_{\lambda i}p_{\mu i}^{-1} \in N & \text{for all } i \in I. \end{array}$$

For a normal partition θ of $I \times \Lambda$, let $\overline{\theta}$ be the normal subgroup of G generated by the set

$$\left\{p_{\lambda i}p_{\lambda j}^{-1} \mid i\ \theta\ \mu,\ \lambda \in \Lambda\right\} \cup \left\{p_{\lambda i}p_{\mu i}^{-1} \mid i \in I, \lambda\ \theta\ \mu\right\}.$$

In terms of the above notation, we have the following simple characterization of an admissible pair.

LEMMA 1. *Let N be a normal subgroup of G and θ be a normal partition of $I \times \Lambda$. Then $(N,\theta) \in \mathscr{A}\mathscr{P}$ if and only if $\theta \subseteq \overline{N}$.*

PROOF. Indeed, $(N,\theta) \in \mathscr{A}\mathscr{P}$ if and only if

$$\begin{array}{lcll} i\ \theta\ j & \Rightarrow & p_{\lambda i}p_{\lambda j}^{-1} \in N & \text{for all } \lambda \in \Lambda, \\ \lambda\ \theta\ \mu & \Rightarrow & p_{\lambda i}p_{\mu i}^{-1} \in N & \text{for all } i \in I \end{array}$$

which is equivalent to

$$i\ \theta\ j \Rightarrow i\overline{N}j, \qquad \lambda\ \theta\ \mu \Rightarrow \lambda\overline{N}\mu$$

that is, $\theta \subseteq \overline{N}$.

The next lemma, of fundamental importance to our condsiderations, will be used throughout the paper without express reference.

LEMMA 2. *For* $(N, \theta) \in \mathscr{A}\mathscr{P}$, *we have*

$$\begin{aligned} (N,\theta)K &= (N,\overline{N}), & (N,\theta)T &= (G,\theta), \\ (N,\theta)k &= (N,\epsilon), & (N,\theta)t &= (\overline{\theta},\theta). \end{aligned}$$

PROOF. Straightforward.

Basic properties of the bar functions are the subject of the next result.

LEMMA 3. *The following statements hold.*

(i) $\overline{G} = \omega$.

(ii) $\overline{\epsilon} = e$.

(iii) *For* N, N' *normal subgroups of* G, $N \subseteq N'$ *implies* $\overline{N} \subseteq \overline{N'}$.

(iv) *For* θ, θ' *normal partitions of* $I \times \Lambda$, $\theta \subseteq \theta'$ *implies* $\overline{\theta} \subseteq \overline{\theta'}$.

(v) *For any normal subgroup* N *of* G, *we have* $\overline{\overline{N}} \subseteq N$ *and* $\overline{N} = \overline{\overline{\overline{N}}}$.

(vi) *For any normal partition* θ *of* $I \times \Lambda$, *we have* $\overline{\overline{\theta}} \supseteq \theta$ *and* $\overline{\theta} = \overline{\overline{\overline{\theta}}}$.

PROOF. The proofs of items (i) - (iv) are straightforward and are omitted.

(v) Let N be a normal subgroup of G. Then $(N, \epsilon) \in \mathscr{A}\mathscr{P}$ so let $\rho \sim (N, \epsilon)$. Hence

$$\rho K \sim (N, \overline{N}), \qquad \rho K t \sim (\overline{\overline{N}}, \overline{N}), \qquad \rho K t K \sim (\overline{\overline{N}}, \overline{\overline{\overline{N}}}). \tag{4}$$

Since $\rho K t \subseteq \rho K$, it follows that $\overline{\overline{N}} \subseteq N$ which by part (iii) gives $\overline{\overline{\overline{N}}} \subseteq \overline{N}$. On the other hand, $\rho K t \subseteq \rho K t K$ so that $\overline{N} \subseteq \overline{\overline{\overline{N}}}$ and equality holds.

(vi) Let θ be a normal partition of $I \times \Lambda$. Then $(G, \theta) \in \mathscr{A}\mathscr{P}$ so let $\rho \sim (G, \theta)$. Hence

$$\rho t \sim (\overline{\theta}, \theta), \qquad \rho t K \sim (\overline{\theta}, \overline{\overline{\theta}}), \qquad \rho t K t \sim (\overline{\overline{\overline{\theta}}}, \overline{\overline{\theta}}), \qquad . \tag{5}$$

Since $\rho t \subseteq \rho t K$, it follows that $\theta \subseteq \overline{\overline{\theta}}$ which by part (iv) gives $\overline{\theta} \subseteq \overline{\overline{\overline{\theta}}}$. On the other hand, $\rho t K t \subseteq \rho t K$ so that $\overline{\overline{\overline{\theta}}} \subseteq \overline{\theta}$ and equality holds.

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Note that by Lemma 2 and Lemma 3(iii) and (iv), all the operators K, k, T and t are order preserving. We are interested in the form of $(N,\theta)w$ where $w \in \Gamma^*$. To this end, we start with the following special cases for w. Note that $\mu = \epsilon T$, $\tau = \epsilon K$ and $\sigma = \omega t$.

LEMMA 4. *For any $\rho \in \mathscr{C}$, we have*

(i) $\rho TK = \omega \sim (G, \omega)$,

(ii) $\rho TKt = \sigma \sim (\overline{\omega}, \omega)$,

(iii) $\rho kT = \rho Tk = \mu \sim (G, \epsilon)$,

(iv) $\rho TKtk = \sigma \wedge \mu \sim (\overline{\omega}, \epsilon)$.

(v) $\rho kt = \epsilon \sim (e, \epsilon)$.

(vi) $\rho ktK = \tau \sim (e, \overline{e})$,

(vii) $\rho ktKT = \tau \vee \mu \sim (G, \overline{e})$.

PROOF.

(i) $\rho TK \sim (N,\theta)TK = (G,\theta)K = (G,\overline{G}) = (G,\omega) \sim \omega$ by Lemma 3(i),

(ii) $\rho TKT = \omega t = \sigma \sim (\overline{\omega}, \omega)$,

(iii) $\rho kT \sim (N,\theta)kT = (N,\epsilon)T = (G,\epsilon) \sim \mu$,

$\rho Tk \sim (N,\theta)Tk = (G,\theta)k = (G,\epsilon) \sim \mu$,

(iv) $\rho TKtk \sim (\overline{\omega},\omega)k = (\overline{\omega},\epsilon) \sim \sigma \wedge \mu$,

(v) $\rho kt \sim (N,\theta)kt = (N,\epsilon)t = (\overline{\epsilon},\epsilon) = (e,\epsilon) \sim \epsilon$ by Lemma 3(ii),

(vi) $\rho ktK = \epsilon K = \tau \sim (e, \overline{e})$,

(vii) $\rho ktKT \sim (e,\overline{e})T = (G,\overline{e}) \sim \tau \vee \mu$.

In view of Lemma 4 we will use the notation

$$\begin{array}{llllllllllll} \epsilon & = & kt, & \tau & = & ktK, & \tau \vee \mu & = & ktKT, & \mu & = & kT, \\ \omega & = & TK, & \sigma & = & TKt, & \sigma \wedge \mu & = & TKtk, & \mu & = & TK. \end{array}$$

Observe that $\overline{\omega}$ is the normal subgroup of G generated by entries of P and that $\overline{e}$ identifies identical rows and identical columns.

It follows easily from Lemma 4 that

$$(\tau \vee \mu) \wedge \sigma = (\sigma \wedge \mu) \vee \tau \sim (\overline{\omega}, \overline{e}).$$

Let

$$\Delta = \{\epsilon, \tau, \sigma \wedge \mu, \sigma, \mu, \tau \vee \mu, (\tau \vee \mu) \wedge \sigma, \omega\}.$$

Letting $\rho \in \mathscr{C}$ and taking into account Lemma 4, we may represent Δ as the lattice with vertices labelled in relation to ρ; see the diagram in [6].

The next lemma shows the invariance of Δ under Γ.

LEMMA 5. *We have* $\Delta\Gamma \subseteq \Delta$ *up to the relations in* Σ.

PROOF. Using Lemmas 3 and 4, we obtain the desired results which we present as the following table:

	K	k	T	t
ϵ	τ	ϵ	μ	ϵ
τ	τ	ϵ	$\tau \vee \mu$	τ
$\sigma \wedge \mu$	σ	$\sigma \wedge \mu$	μ	ϵ
μ	ω	μ	μ	ϵ
$\tau \vee \mu$	ω	μ	$\tau \vee \mu$	τ
σ	σ	$\sigma \wedge \mu$	ω	σ
ω	ω	μ	ω	σ
$(\tau \vee \mu) \wedge \sigma$	σ	$\sigma \wedge \mu$	$\tau \vee \mu$	τ

We now consider some relations valid for our operators.

LEMMA 6. *Operators* Γ *satisfy the following relations*

$$\Sigma = \{ \quad \text{(i)}\ \ K^2 = kK = K, \quad k^2 = Kk = k \quad \text{(ii)}\ \ KTK = TKT = TK,$$
$$t^2 = Tt = t, \quad T^2 = tT = T, \qquad tkt = ktk = kt,$$
$$\text{(iii)}\quad KtK = Kt,$$
$$tKt = tK, \qquad \text{(iv)} \qquad kT = Tk \qquad \}.$$

PROOF. (i) This follows directly from the definition of the operators K, k, T and t.

(ii) By Lemma 4(i), we have $\rho TK = \omega$ for any $\rho \in \mathscr{C}$. In particular $(\rho K)TK = \omega$ and thus $KTK = TK$. Since $\omega T = \omega$, it follows that $\rho TKT = \omega$ for all $\rho \in \mathscr{C}$ and thus $TKT = TK$.

By Lemma 4(v), we have $\rho kt = \epsilon$ for any $\rho \in \mathscr{C}$. In particular $(\rho t)kt = \epsilon$ and thus $tkt = kt$. Since $\epsilon k = \epsilon$, it follows that $\rho ktk = \epsilon$ for all $\rho \in \mathscr{C}$ and thus $ktk = kt$.

(iii) The first relation follows from Lemma 3(v) and (4); the second from Lemma 3(vi) and (5).

(iv) This was proved in Lemma 4(iii).

Observe that the second line in each of the items (i), (ii) and (iii) can be obtained from the first line by the transformation

$$K \leftrightarrow t, \qquad T \leftrightarrow k.$$

Item (iv) is invariant under this transformation.

In view of Lemma 6, we can now give a system of representatives of the corresponding congruence.

LEMMA 7. *The set*

$$\Omega = \{\, K, T, KT, Kt, kt, Ktk, ktK, ktKT, \\ kT, t, k, tk, tK, TK, tKT, TKt, TKtk \,\} \tag{6}$$

is a system of representatives of the congruence on Γ^+ generated by the relations Σ.

PROOF. We consider the following clusters of words:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & KT & \\
K \arrow[ur, dash] \arrow[dr, dash] & & \\
 & Kt \arrow[dr, dash] & \\
 & & KtK
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & tKT \\
 & tK \arrow[ur, dash] & \\
t \arrow[ur, dash] \arrow[dr, dash] & & \\
 & tk &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & TK = \omega \arrow[dr, dash] & & \\
T \arrow[ur, dash] & & TKt = \sigma \arrow[dr, dash] & \\
 & & & TKtk = \sigma \wedge \mu
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & kT = \mu & & ktKT = \tau \vee \mu \\
k \arrow[ur, dash] \arrow[dr, dash] & & ktK = \tau \arrow[ur, dash] & \\
 & kt = \epsilon \arrow[ur, dash] & &
\end{tikzcd}$$

For each word w we consider four words wK, wk, wT, wt. If the last letter of w is K or k, because of $K^2 = kK = K$ and $k^2 = Kk = k$, wK and wk have representatives of the same length as w, so it suffices to consider the words wT and wt. Symmetrically for the case when the last letter of w is either T or t.

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In the remaining cases we search among the relations Σ whether the resulting word can be replaced by a shorter one and replace Tk by kT. If in this process we reach an element δ of Δ, we may refer to Lemma 5 for values of δK, δKT, etcetera. In this way we obtain the above clusters of words as the only possible ones. That no two distinct elements of Ω are related by the congruence generated by Σ will follow from the example in Section 5.

In the arrangement of elements of Ω in Lemma 7, each word in the second line can be obtained from the word above it by the transformation

$$K \leftrightarrow t, \qquad T \leftrightarrow k.$$

Moreover, this transformation maps kT onto Tk (where $kT = Tk$ is one of relations in Σ).

Using the notation introduced afer Lemma 4, we can write the elements of Ω as:

$$\begin{gathered} K, T, KT, Kt, \epsilon, Ktk, \tau, \tau \vee \mu, \\ \mu, t, k, tk, tK, \omega, tKT, \sigma, \sigma \wedge \mu. \end{gathered}$$

For a given $(N, \theta) \in \mathscr{A}P$, the elements of $\Omega \cup \{1\}$ applied to (N, θ), in the above arrangement give

$$(7) \quad \begin{cases} (N,\theta) \ (N,\overline{N}) \ (G,\theta) \ (G,\overline{N}) \ (\overline{\overline{N}},\overline{N}) \ (e,\epsilon) \ (\overline{\overline{N}},\epsilon) \ (e,\overline{e}) \ (G,\overline{e}) \\ (G,\epsilon) \ (\overline{\theta},\theta) \ (N,\epsilon) \ (\overline{\theta},\epsilon) \ (\overline{\theta},\overline{\overline{\theta}}) \ (G,\omega) \ (G,\overline{\overline{\theta}}) \ (\overline{\omega},\omega) \ (\overline{\omega},\epsilon) \end{cases}$$

LEMMA 8. *The lattices generated by the projections of the set in* (7) *into the lattice of normal subgroups of* G *and the lattice of normal partitions of* $I \times \Lambda$ *form the lattices* L_k *and* L_t *in Diagram* 1.

PROOF. The projection into the lattice of normal subgroups of G equals $\{e, \overline{\theta}, \overline{\overline{N}}, N, \overline{\omega}, G\}$. By Lemma 1, we have $\theta \subseteq \overline{N}$ and hence $\overline{\theta} \subseteq \overline{\overline{N}}$ by Lemma 3(iv). Also $\overline{\overline{N}} \subseteq N$ by Lemma 3(v). Finally $\overline{N} \subseteq \omega$ which by Lemma 3(iii) implies that $\overline{\overline{N}} \subseteq \overline{\omega}$. This establishes the order in the set $\{e, \overline{\theta}, \overline{\overline{N}}, N, \overline{\omega}, G\}$ which in turn produces the lattice L_k in Diagram 1.

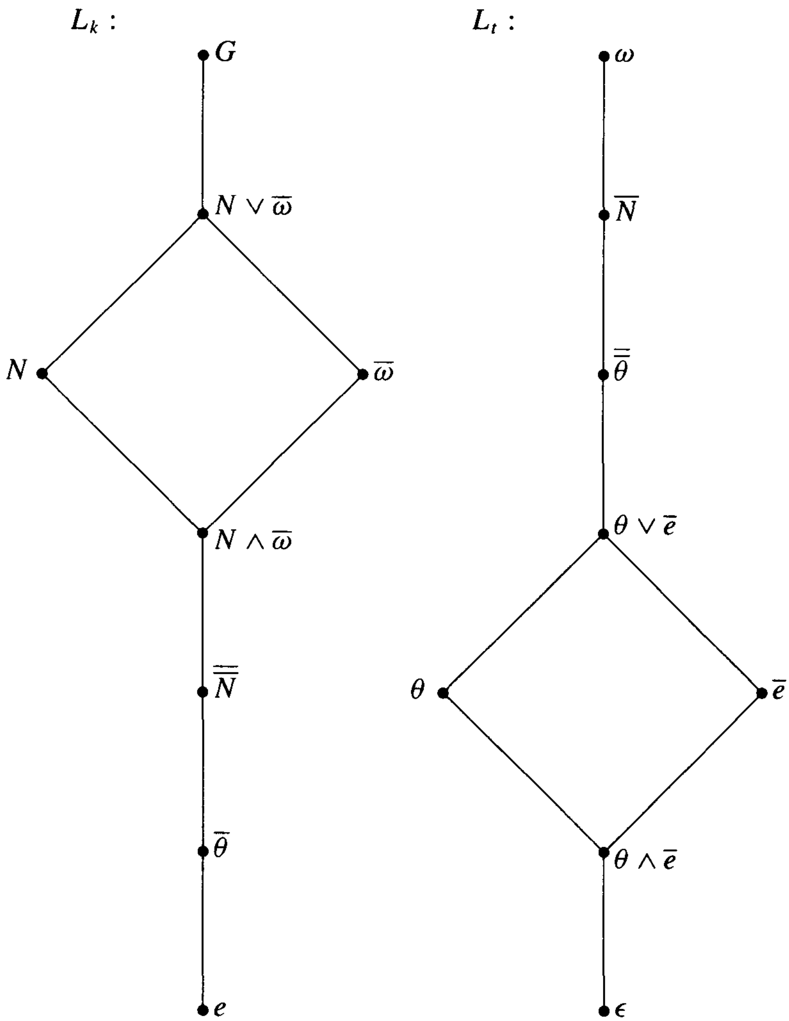


DIAGRAM 1

The discussion for L_t runs analogously. The projection into $I \times \Lambda$ equals $\{\epsilon, \theta, \overline{e}, \overline{\overline{\theta}}, \overline{N}, \omega\}$. First $\overline{e} \subseteq \overline{\overline{\theta}}$ by Lemma 3(iv) since $e \subseteq \overline{\theta}$. Also $\theta \subseteq \overline{\overline{\theta}}$ by Lemma 3(vi). Finally $\theta \subseteq \overline{N}$ implies $\overline{\overline{\theta}} \subseteq \overline{\overline{\overline{N}}} = \overline{N}$ by Lemma 3(iii) - (v). This establishes the order in the set $\{\epsilon, \theta, \overline{e}, \overline{\overline{\theta}}, \overline{N}, \omega\}$ which in turn produces the lattice L_t in Diagram 1.

5. An example

The purpose of the example below is to exhibit an instance of a completely simple semigroup S and a congruence ρ on S for which the values of ρ and ρw, as w runs over the set Ω in Lemma 7, and their joins and meets, are all distinct. This statement will be used later several times in crucial situations. The example represents an expansion of ([4, Example 5.4]).

EXAMPLE. Let $S = \mathscr{M}(I, G, \Lambda; P)$ where

$$I = \{0, 1, 2, 3, 4\}, \qquad G = \mathbb{Z} \times \mathbb{Z}_4, \qquad \Lambda = \{0, 1, 2, 3\},$$

$$P = \begin{pmatrix} (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (1,0) & (9,0) & (25,0) \\ (0,0) & (0,0) & (17,0) & (25,0) & (41,0) \end{pmatrix}.$$

For $N = \langle(2, 2)\rangle$, we obtain

$$\overline{N} = \left(\left\{\{0, 1\}, \{2, 3, 4\}\right\}, \left\{\{0, 1\}, \{2, 3\}\right\}\right), \qquad \overline{\overline{N}} = \langle(8, 0)\rangle,$$

and for $\theta = \left(\epsilon, \left\{\{0, 1\}, \{2, 3\}\right\}\right)$, we get

$$\overline{\theta} = \langle(16, 0)\rangle, \qquad \overline{\overline{\theta}} = \left(\left\{\{0, 1\}, \{2\}, \{3, 4\}\right\}, \left\{\{0, 1\}, \{2, 3\}\right\}\right).$$

It follows that (N, θ) is an admissible pair for S.

6. The semigroup generated by the operators

Relations Σ in Lemma 6 are valid for the operators in Γ on the congruence lattice of any completely simple semigroup. The set Ω in Lemma 7 is a subset of Γ^+, the free semigroup on Γ, which serves as a set of representatives for the congruence on Γ^+ induced by the relations in Σ.

We now provide the set Ω with the multiplication of representatives thereby obtaining Table 1. In it we use the symbolism introduced after Lemma 4 using the notation of elements of Δ for the words kt, ktK, etcetera. We note that these elements, considered as operators on the congruence lattice, are constants and thus act in the semigroup Ω as right zeros. Hence there is no need for displaying them in the table.

	K	k	T	t	KT	Kt	Ktk	tKT	tK	tk
K	K	k	KT	Kt	KT	Kt	Ktk	KT	Kt	Ktk
k	K	k	μ	ϵ	KT	Kt	Ktk	$\tau \vee \mu$	τ	ϵ
T	ω	μ	T	t	ω	σ	$\sigma \wedge \mu$	tKT	tK	tk
t	tK	tk	T	t	tKT	tK	tk	tKT	tK	tk
KT	ω	μ	KT	Kt	ω	σ	$\sigma \wedge \mu$	KT	Kt	Ktk
Kt	Kt	Ktk	KT	Kt	KT	Kt	Ktk	KT	Kt	Ktk
Ktk	Kt	Ktk	μ	ϵ	KT	Kt	Ktk	$\tau \vee \mu$	τ	ϵ
tKT	ω	μ	tKT	tK	ω	σ	$\sigma \wedge \mu$	tKT	tK	tk
tK	tK	tk	tKT	tK	tKT	tK	tk	tKT	tK	tk
tk	tK	tk	μ	ϵ	tKT	tK	tk	$\tau \vee \mu$	ϵ	ϵ
ϵ	τ	ϵ	μ	ϵ	$\tau \vee \mu$	τ	ϵ	$\tau \vee \mu$	τ	ϵ
τ	τ	ϵ	$\tau \vee \mu$	τ	$\tau \vee \mu$	τ	ϵ	$\tau \vee \mu$	τ	ϵ
$\sigma \wedge \mu$	σ	$\sigma \wedge \mu$	μ	ϵ	ω	σ	$\sigma \wedge \mu$	τ	τ	ϵ
μ	ω	μ	μ	ϵ	ω	σ	$\sigma \wedge \mu$	μ	τ	ϵ
$\tau \vee \mu$	ω	μ	$\tau \vee \mu$	τ	ω	σ	$\sigma \wedge \mu$	$\tau \vee \mu$	τ	ϵ
σ	σ	$\sigma \wedge \mu$	ω	σ	ω	σ	$\sigma \wedge \mu$	ω	σ	$\sigma \wedge \mu$
ω	ω	μ	ω	σ	ω	σ	$\sigma \wedge \mu$	ω	σ	$\sigma \wedge \mu$

TABLE 1

We are now ready for the first principal result of the paper. Recall the notation (6) for elements of Σ, the notation Δ after Lemma 4, Γ in (3) and Σ in Lemma 6.

THEOREM 1. *The set Ω given the multiplication in Table 1 and in which elements of Δ act as right zeros is a semigroup isomorphic to $\langle \Gamma / \Sigma \rangle$.*

PROOF. By Lemma 7, Ω serves as a set of representatives for the congruence on Γ^+ induced by the relations Σ. A direct but long verification shows that the system of representatives Ω admits products as indicated in Table 1; we omit the details. It is obvious that elements of Δ act as right zeros for Ω.

The following statement is a consequence of several earlier results.

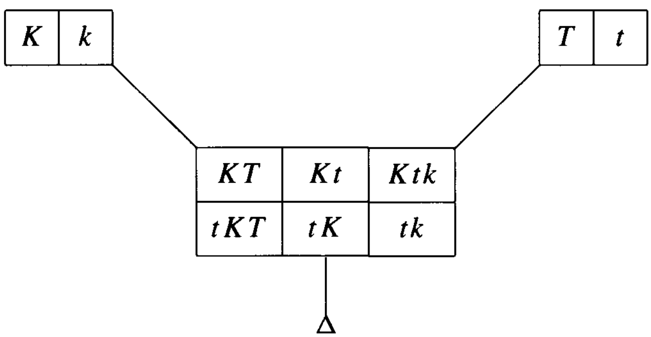
COROLLARY 1. *Let S be a completely simple semigroup. The semigroup $\Omega(S)$ generated by the operators K, k, T and t on the congruence lattice of S is a homomorphic image of Ω. For the semigroup S in the example in Section 5, we have $\Omega(S) \cong \Omega$.*

PROOF. The first assertion follows from Theorem 1 in view of Lemmas 6 and 7. The second assertion is a consequence of Lemma 8 and the fact that all the projections of $\rho\Omega$ into G and $I \times \Gamma$ are distinct in the cited example.

In view of the above theorem and its corollary, we can say that the relations Σ are the defining relations for the semigroup generated by the operators K, k, T and t on congruence lattices of *all* completely simple semigroups. For the relations Σ are valid in the semigroup generated by these operators on the congruence lattice of any completely simple semigroup and, in general, no relation can be added unless it follows from those already in Σ.

The next two propositions contain a few interesting properties of the semigroup Ω.

PROPOSITION 1. *The $\mathscr{D}$-structure of Ω is of the form*



The principal factor of KT is isomorphic to the Rees matrix semigroup

$$\mathscr{M}^0(\{1,2\}, e, \{1,2,3\}; P)$$

where

$$P = \begin{pmatrix} 0 & e \\ e & e \\ e & 0 \end{pmatrix}.$$

PROOF. The standard analysis of the multiplication table shows that the above diagram indeed describes the $\mathscr{D}$-structure of Ω. The same goes for the structure of the principal factor of the element KT. For the latter it only suffices to observe that the above diagram of that $\mathscr{D}$-class is its egg-box picture, namely that the rows represent $\mathscr{R}$-classes and the columns $\mathscr{L}$-classes and that the squares of all its elements remain in the $\mathscr{D}$-class except for $(KT)^2$ and $(tk)^2$ which fall into Δ.

PROPOSITION 2. *For every $w \in \Omega$, $w \neq \mu$, let $\overline{w}$ be the element of Ω obtained from w by the transformation*

$$K \leftrightarrow t, \qquad T \leftrightarrow k, \tag{8}$$

and let $\overline{\mu} = \mu$. Then the mapping $w \to \overline{w}$ is an automorphism of Ω.

PROOF. The proof follows immediately from the presentation of Ω and is omitted. The assertion of the proposition can also be proved directly by analyzing the multiplication table for Ω.

7. A network associated with a congruence

To every congruence ρ on a regular semigroup we may associate the network of congruences

$$P_\rho = \{\, \rho w \mid w \in \Gamma^*\}$$

partially ordered by inclusion. For a Rees matrix semigroup S, it is more convenient to consider admissible pairs rather than congruences. We thus arrive at the second principal result of this paper.

THEOREM 2. *Let (N, θ) be an admissible pair for S. The lattice $L_{(N,\theta)}$ generated by the set*

$$P_{(N,\theta)} = \{(N, \theta)w \mid w \in \Gamma^*\}$$

is depicted in Diagram 2 wih some possible coincidence of vertices. The labels on the sides of this diagram represent the $\mathscr{K}$- and $\mathscr{T}$-classes of the congruences in $P_{(N,\theta)}$. For the semigroup S in the example in Section 5, all the vertices in Diagram 2 are distinct. Moreover,

$$L_{(N,\theta)} = \mathscr{A}\mathscr{P} \cap (L_k \times L_t).$$

PROOF. According to Lemma 7, we have

$$P_{(N,\theta)} = \{(N, \theta)w \mid w = 1 \text{ or } w \in \Omega\}.$$

We have seen in Lemma 8 that $P_{(N,\theta)}$ is contained in the Cartesian product of the lattices L_k and L_t in Diagram 1. Now drawing the vertices corresponding to elements of $P_{(N,\theta)}$ in $L_k \times L_t$, we easily obtain part of Diagram 2. The second assertion of the theorem now clearly holds.

Let $w \in \Gamma^+$. Trivially $(N, \theta)w \in \mathscr{A}\mathscr{P}$. In view of Lemma 7, we may suppose that $w \in \Omega$ which by the above gives that $(N, \theta)w \in L_k \times L_t$. Therefore $L_{(N,\theta)} \subseteq \mathscr{A}\mathscr{P} \cap (L_k \times L_t)$.

Conversely, we may represent each element of $\mathscr{A}\mathscr{P} \cap (L_k \times L_t)$ by means of an expression in terms of meets and joins of elements of $P_{(N,\theta)}$. These expressions can be extracted from Diagram 2. The long verification of this assertion is omitted.

This establishes the last assertion of the theorem. Now plotting the elements of $\mathscr{A}\mathscr{P} \cap (L_k \times L_t)$ into the diagram started above, we easily obtain Diagram 2. In view of Lemma 8 and the fact that all vertices in the example in Section 5 are distinct, all the vertices of $L_{(N,\theta)}$ within $L_k \times L_t$ are also distinct, as asserted.

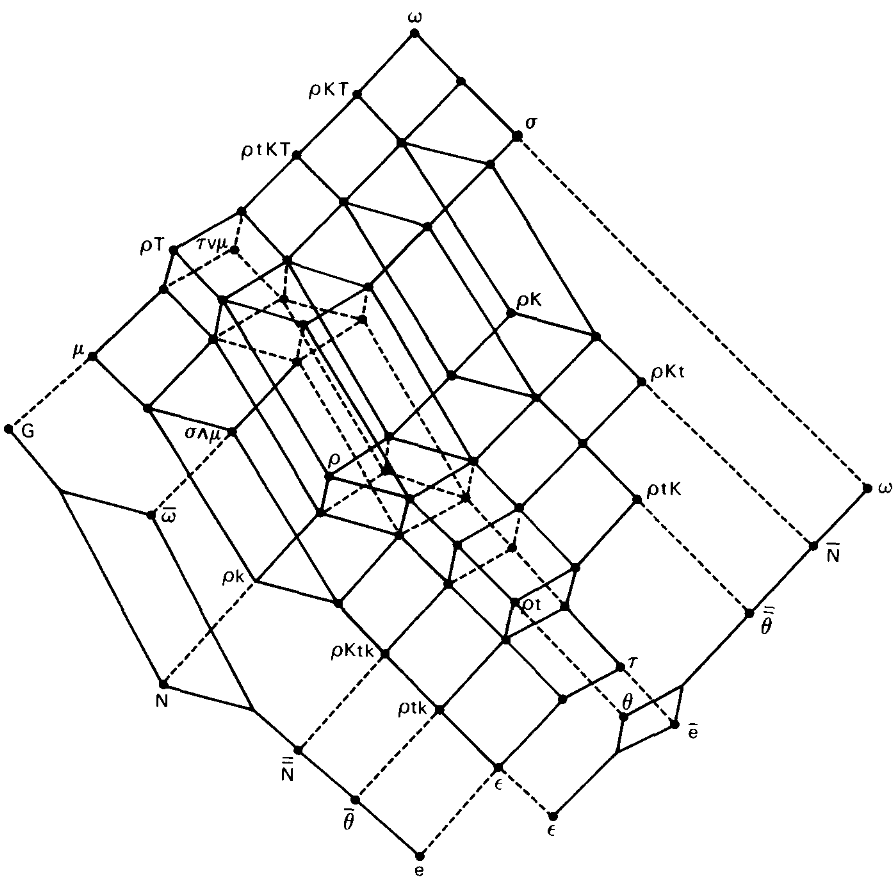


DIAGRAM 2

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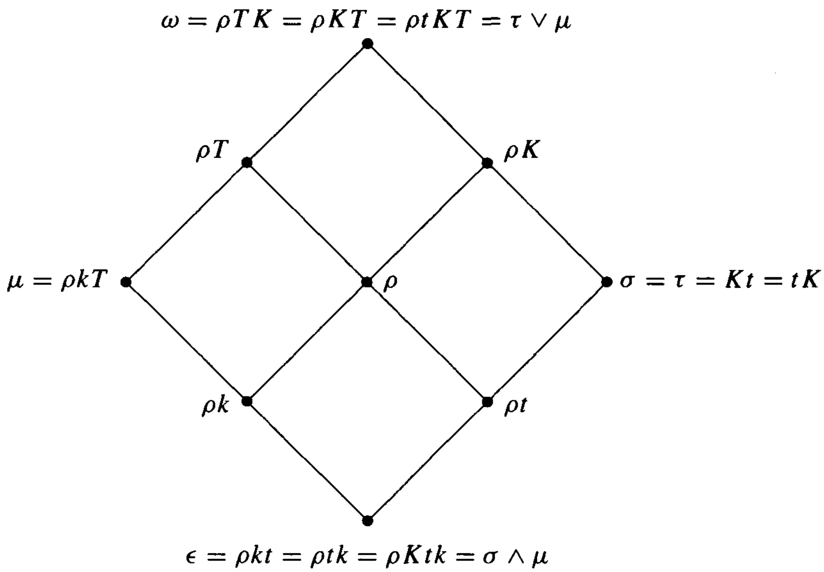

DIAGRAM 3

Recall that a *rectangular group* is a completely simple semigroup S in which idempotents form a subsemigroup. In a Rees matrix semigroup $S = \mathscr{M}(I, G, \Lambda; P)$ with P normalized, this condition is equivalent to all entries of P being equal to the identity of G. It is then clear that in such a case, any pair (N, θ), where N is a normal subgroup of G and θ is a normal partition of $I \times \Lambda$, is admissible. Now letting $\rho \sim (N, \theta)$, ([4, Proposition 5.5]) asserts that Diagram 3 represents P_ρ for any congruence ρ on a rectangular group S.

If for $u, v \in \Gamma^*$, we have $\rho u = \rho v$ for all congruences ρ on a completely simple semigroup S, we say that the congruence lattice of S *agrees with the identity* $u = v$.

The next result indicates how close-knit the semigroup $\langle \Gamma^+ / \Sigma \rangle$ with regard to relations Σ is. For adding one more relation to Σ may result in much collapsing as we will now see.

PROPOSITION 3. *The following statements are equivalent for any completely simple semigroup* S.

(i) *S is a rectangular group.*

(ii) *The congruence μ has a complement in $\mathscr{C}$.*

(iii) *$\mathscr{C}$ agrees with any single identity $u = v$ in Diagram 3 which is not in Σ.*

PROOF. (i) implies (ii). By the remarks preceding the proposition, (e, ω) is an admissible pair for S. But $\mu \sim (G, e)$ and trivially (e, ω) is a complement of (G, e) in the lattice .

(ii) implies (i). Let (N, θ) be a complement of (G, ϵ) in $\mathscr{A}\mathscr{P}$. It follows at once that $N = e$ and $\theta = \omega$ so that (e, ω) is an admissible pair. But this means that $p_{\lambda i} p_{\lambda j}^{-1} = e$ for all $i, j \in I$ and $\lambda \in \Lambda$, and for $j = 1$, we get $p_{\lambda i} = e$ and S is a rectangular group.

(i) implies (iii). By ([4, Proposition 5.5]), for any $\rho \in \mathscr{C}$, P_ρ is as in Diagram 6. Now it is easy to see that all the equalities as indicated at the vertices of the diagram hold.

(iii) implies (i). There are 18 equalities with the property indicated. For example, it is well known that $\sigma = \tau$ implies that S is a rectangular group. As a sample, we prove that $Ktk = \epsilon$ implies that S is a rectangular group. For any $(N, \theta) \in \mathscr{A}\mathscr{P}$, by hypothesis we have $(N, \theta)Ktk = (e, \epsilon)$ which gives $\overline{\overline{N}} = e$. Since this holds for all normal subgroups N of G, we may let $N = G$ getting $\overline{\overline{G}} = e$. By Lemma 3(v), we obtain $\overline{G} = \overline{e}$ so by Lemma 3(i), $\omega = \overline{e}$. It follows that $p_{\lambda i} p_{\lambda j}^{-1} = e$ for all $i, j \in I$ and $\lambda \in \Lambda$. Now letting $j = 1$, we get $p_{\lambda i} = e$ for all $i \in I$ and $\lambda \in \Lambda$ and S is a rectangular group. The remaining cases are treated similarly.

The lattice $L_{(N,\theta)}$ has 54 elements and is a subdirect product of the distributive lattices L_k and L_t and is thus distributive. For any congruence ρ on a regular semigroup, we have $\rho K \wedge \rho T = \rho = \rho k \vee \rho t$ so that our L_p satisfies the relation

$$K \wedge T = k \vee t = 1. \tag{9}$$

It is easy to verify that

$$\sigma \vee \mu = \omega, \qquad \tau \wedge \mu = \epsilon. \tag{10}$$

Hence in the set $\Omega \cup \{1\}$ of generators of $L_{(N,\theta)}$ we may drop $1, \sigma \wedge \mu, \omega, \tau \vee \mu, \epsilon$, so $L_{(N,\theta)}$ is generated by the remaining 13 elements and satisfies relations (9) and (10).

The invariance of the lattice $L_{(N,\theta)}$ under the set of operators Γ^* is the content of the next result.

COROLLARY 2. *We have* $L_{(N,\theta)}\Gamma^* \subseteq L_{(N,\theta)}$

PROOF. Clearly $L_{(N,\theta)}\Gamma \subseteq \mathscr{A}\mathscr{P}$ and thus $L_{(N,\theta)}\Gamma^* \subseteq \mathscr{A}\mathscr{P}$. According to Theorem 3, it remains to prove that $L_{(N,\theta)}\Gamma^* \subseteq L_k \times L_t$ and for this, it suffices to show that $L_{(N,\theta)}\Gamma \subseteq L_k \times L_t$. The latter further reduces to verifying that the bar functions map L_k into L_t and L_t into L_k. We shall use Lemma 3 freely.

It follows easily that $\overline{\overline{\omega}} = \omega = \overline{G}$. Since $\overline{\omega} \subseteq N \vee \overline{\omega} \subseteq G$, by monotonicity, we get $\overline{\overline{\omega}} \subseteq \overline{N \vee \overline{\omega}} \subseteq \overline{G}$ and thus $\overline{N \vee \overline{\omega}} = \omega$. We also have $\overline{\overline{\overline{N}}} = \overline{N}$. Since $\overline{\overline{N}} \subseteq N \wedge \overline{\omega} \subseteq N$, by monotonicity, we have $\overline{\overline{\overline{N}}} \subseteq \overline{N \wedge \overline{\omega}} \subseteq \overline{N}$ and hence $\overline{N \wedge \overline{\omega}} = \overline{N}$. In addition, $\overline{\overline{\theta}}$ and $\overline{e}$ are in L_t. Therefore $\overline{L_k} \subseteq L_t$.

We proceed similarly with L_t. It follows easily that $\overline{\epsilon} = e = \overline{\overline{e}}$. Since $\epsilon \subseteq \theta \wedge \overline{e} \subseteq \overline{e}$, by monotonicity, we get $\overline{\epsilon} \subseteq \overline{\theta \wedge \overline{e}} \subseteq \overline{\overline{e}}$ and thus $\overline{\theta \wedge \overline{e}} = e$. We also have $\overline{\theta} = \overline{\overline{\overline{\theta}}}$. Since $\theta \subseteq \theta \vee \overline{e} \subseteq \overline{\overline{\theta}}$, by monotonicity, we have $\overline{\theta} \subseteq \overline{\theta \vee \overline{e}} \subseteq \overline{\overline{\overline{\theta}}}$ and hence $\overline{\theta \vee \overline{e}} = \overline{\theta}$. In addition, $\overline{\overline{N}}$ and $\overline{\omega}$ are in L_k. Therefore $\overline{L_t} \subseteq L_k$.

We shall now represent the lattice $L_{(N,\theta)}$ depicted in Diagram 2 in terms of generators and relations. By $\mathscr{F}\mathscr{D}\mathscr{L}(a, b, \ldots)$ we denote the free distributive lattice on the generators $a, b, \ldots$. If $\{\rho_\alpha\}_{\alpha \in A}$ is a family of relations on $L = \mathscr{F}\mathscr{D}\mathscr{L}(a, b, \ldots)$, we denote the quotient lattice L divided by the congruence generated by $\{\rho_\alpha\}_{\alpha \in A}$ by $\langle \mathscr{F}\mathscr{D}\mathscr{L}(a, b, \ldots), \{\rho_\alpha\}_{\alpha \in A}\rangle$.

THEOREM 3. *The lattice $L_{(N,\theta)}$ depicted in Diagram 2 is isomorphic to the lattice*

$$\langle \mathscr{F}\mathscr{D}\mathscr{L}(\tau, \mu, \sigma, \rho t, \rho t K, \rho K t, \rho K), \mathscr{R}\rangle \tag{11}$$

where

$$\mathscr{R} = (\rho t K \wedge \mu \leq \rho t \leq \rho t K \leq \rho K t \leq \rho K \leq \rho K t \vee \mu,\ \tau \leq \rho t K, \rho K t \leq \sigma)$$

and none of these relations may be omitted.

PROOF. To facilitate the writing, we introduce the notation

$$a = \rho t, \qquad b = \rho t K, \qquad c = \rho K t, \qquad d = \rho K. \tag{12}$$

1. We verify first that $L_{(N,\theta)}$ is generated by the set

$$\tau, \mu, \sigma, a, b, c, d. \tag{13}$$

Recall that $L_{(N,\theta)}$ is the lattice generated by the partially ordered set $P_{(N,\theta)}$. According to Theorem 2, the former is depicted in Diagram 2. It follows easily from Lemma 2 that

$$\lambda k = \lambda \wedge \mu, \qquad \lambda T = \lambda \vee \mu, \qquad (\lambda \in \mathscr{C}).$$

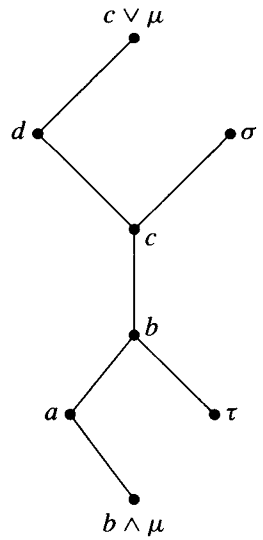

DIAGRAM 4

Using this and notation (12), we obtain

$$\begin{aligned} \rho k &= d \wedge \mu, & \rho t k &= b \wedge \mu, & \rho K t k &= c \wedge \mu, \\ \rho T &= a \vee \mu, & \rho K T &= c \vee \mu, & \rho t K T &= b \vee \mu, \end{aligned}$$

and using simple arguments, we get

$$\epsilon = \tau \wedge \mu, \qquad \omega = \sigma \vee \mu, \qquad \rho = \rho K \wedge \rho T = d \wedge (a \vee \mu).$$

It follows that the set (13) generates the partially ordered set $P_{(N,\theta)}$ and thus also the lattice $L_{(N,\theta)}$

2. A simple inspection of Diagram 2 shows that all the relations in $\mathscr{R}$ are satisfied in the lattice L and therefore hold in $L_{(N,\theta)}$. In the notation introduced in (12), these relations take on the form

$$(14) \qquad b \wedge \mu \leq a \leq b \leq c \leq d \leq c \vee \mu, \qquad \tau \leq b, \qquad c \leq \sigma$$

which can be represented as the partially ordered set in Diagram 4.

3. It follows from Theorem 2 that the lattice L is a subdirect product of the distributive lattices L_k and L_t and is thus itself distributive. Therefore, by the above, L is a homomorphic image of the lattice D in (11).

4. By simple counting of elements of the lattice L in Diagram 2 or from Theorem 2, we get that L has 54 elements. To prove the first assertion of the theorem, it now suffices to show that D has exactly 54 elements.

5. The lattice D in (11) is distributive and is generated by the seven elements in (13). Hence we may consider it as a subdirect product of the lattice Y, where $Y = \{0, 1\}$ is the nontrivial subdirectly irreducible distributive lattice. We thus must find all septuples of elements of Y which satisfy relations (14).

It follows from relations (14) that $\tau \wedge \mu$ is the least element of the lattice D in (11), and hence we may require that $\tau \wedge \mu = 0$. Similarly, we may postulate that $\sigma \vee \mu = 1$. With these restrictions and the given $\tau \leq \sigma$, we have the following choices for τ, μ and σ:

	τ	μ	σ
A	0	0	1
B	1	0	1
C	0	1	0
D	0	1	1

Relations (14) now simplify to

$$(15) \qquad a \leq b \leq c \leq d, \qquad \tau \leq b, \qquad c \leq \sigma.$$

With these restrictions, we get the following cases:

$$(16) \qquad \begin{cases} A, D & : \quad a \leq b \leq c \leq d, \\ \quad B & : \quad a \leq b = c = d = 1, \\ \quad C & : \quad 0 = a = b = c \leq d. \end{cases}$$

We can now write all septuples of 0's and 1's according to the above table and satisfying the restrictions (16).

In Table 2, the second row violates the condition $d \leq c \vee \mu$ and the thirteenth row violates the condition $b \wedge \mu \leq a$, and so they must be omitted. The remaining rows satisfy these two conditions. We have thus arrived at the following representation of the generators by means of 12-tuples:

$$\begin{aligned}
\tau &= (0000\ 1100\ 0000) \\
\mu &= (0000\ 0011\ 1111), \\
\sigma &= (1111\ 1100\ 1111), \\
a &= (0001\ 0100\ 0001), \\
b &= (0011\ 1100\ 0001), \\
c &= (0111\ 1100\ 0011), \\
d &= (0111\ 1101\ 0111).
\end{aligned}$$

	τ	μ	σ	a	b	c	d
A:	0	0	1	0	0	0	0
	0	0	1	0	0	0	1
	0	0	1	0	0	1	1
	0	0	1	0	1	1	1
	0	0	1	1	1	1	1
B:	1	0	1	0	1	1	1
	1	0	1	1	1	1	1
C:	0	1	0	0	0	0	0
	0	1	0	0	0	0	1
D:	0	1	1	0	0	0	0
	0	1	1	0	0	0	1
	0	1	1	0	0	1	1
	0	1	1	0	1	1	1
	0	1	1	1	1	1	1

TABLE 2

With these data, Barry Wolk of the University of Manitoba computed on a machine that the lattice generated by the above 12-tuples $\tau, \mu, \sigma, a, b, c, d$ has 54 elements. Therefore the lattices L and D are isomorphic which establishes the first assertion of the theorem.

In order to prove the second assertion of the theorem, that is that the relations $\mathscr{R}$ are independent, we construct in Table 3 for each relation r in $\mathscr{R}$, a septuple which satisfies all the relations in $\mathscr{R}$ except r.

	τ	μ	σ	a	b	c	d
$b \wedge \mu \leq a$	0	1	1	0	1	1	1
$a \leq b$	1	0	1	1	0	1	1
$b \leq c$	1	1	1	1	1	0	1
$c \leq d$	1	1	1	1	1	1	0
$d \leq c \vee \mu$	0	0	1	0	0	0	1
$\tau \leq b$	1	0	0	0	0	0	0
$c \leq \sigma$	1	1	0	1	1	1	1

TABLE 3

References

[1] J. M. Howie, *An introduction to semigroup theory* (Academic Press, London, 1976).
[2] F. Pastijn and M. Petrich, 'Congruences on a regular semigroup', *Trans. Amer. Math. Soc.* **295** (1986), 607–633.
[3] F. Pastijn and P. G. Trotter, 'Lattices of completely regular semigroup varieties', *Pacific J. Math* **119** (1985), 191–214.
[4] M. Petrich, 'A special sublattice of the congruence lattice of a regular semigroup', preprint.
[5] ———, 'Congruence lattices of regular semigroups related to kernels and traces', *Proc. Edinburgh Math. Soc.* **34** (1991), 179–203.
[6] M. Petrich and N. R. Reilly, 'The kernel-trace approach to congruences on completely simple semigroups', *Studia Sci. Math. Hungar.* **16** (1981), 103–114.
[7] ———, 'A network of congruences on an inverse semigroup', *Trans. Amer. Math. Soc.* **270** (1982), 309–325.
[8] ———, 'Semigroups generated by certain operators on varieties of completely regular semigroups', *Pacific J. Math.* **132** (1988), 151–175.

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