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OPEN, CONNECTED FUNCTIONS

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1. **Introduction.** Recall that a function $f: X \to Y$ is called connected if $f(C)$ is connected for each connected subset C of X. These functions have been extensively studied. (See Sanderson [6].) A function $f: X \to Y$ is monotone if for each $y \in Y$, $f^{-1}(y)$ is connected. We shall use the techniques of multivalued functions to prove that if $f: X \to Y$ is open and monotone onto Y, then $f^{-1}(C)$ is connected for each connected subset C of Y. This result is used to prove that the product of semilocally connected spaces is semilocally connected and that the image of a maximally connected space under an open, connected, monotone function is maximally connected.

2. **Open, monotone functions.**

2.1 Definition. A multivalued function $F: X \to Y$ is called lower semicontinuous (l.s.c.) if $F^{-1}(V)$ is open for every open subset V of Y. Note that we do not assume $F(x)$ is closed for every $x \in X$.

2.2 Notation. If $GrF = \{(x, y) \mid y \in F(x)\}$ then $p_X: GrF \to X$ will denote the restriction to GrF of the projection.

2.3 Lemma. (Borges, [1]) *If $F: X \to Y$ is a multivalued function, then $F(x) = p_Y p_X^{-1}(x)$ and p_X is open if F is l.s.c.*

The proof of the following theorem follows the techniques of Borges' Theorem 3.4 [1], where he proves it for the special case $E = X$.

2.4 Theorem. *If $F: X \to Y$ is a l.s.c. multivalued function with $F(x)$ connected for each $x \in X$, then $F(E)$ is connected for every connected subset E of X.*

Proof. Let E be a connected subset of X and G the graph multivalued function $G(x) = \{(x, y) \mid y \in F(x)\}$. Then we claim that $G(E)$ is connected. For, if not $G(E) \subseteq U \cup V$ where U and V are open in $X \times Y$ and disjoint. If $U_1 = U \cap G(X)$, $V_1 = V \cap G(X)$, then U_1 and V_1 are open in $G(X) = GrF$, $G(E) \subseteq U_1 \cup V_1$ and $U_1 \cap V_1 = \phi$. Since F has images of points connected $p_X^{-1}(x)$ is connected for all x, so $p_X^{-1}(x) \cap U_1 \neq \phi$ iff $p_X^{-1}(x) \subseteq U_1$ for $x \in E$. Therefore, $E \subseteq p_X(U_1) \cup p_X(V_1)$,

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and $p_X(U_1)$, $p_X(V_1)$ are open since F is l.s.c. If $z \in P_X(U_1) \cap P_X(V_1) \cap E$, then $p_X^{-1}(z) \cap U_1 \neq \phi$ and $p_X^{-1}(z) \cap V_1 \neq \phi$, which is clearly impossible. It follows that $G(E)=p_X^{-1}(E)$ is connected and hence, $F(E)=p_Y p_X^{-1}(E)$ is connected.

2.5 COROLLARY. *If f is an open, monotone (single-valued) function from a space X onto a space Y, then $f^{-1}(A)$ is connected for every connected subset A of Y.*

2.6 DEFINITION. A space X is semilocally connected if it is connected and for every point $x \in X$, if U is a neighborhood of x, then there is a neighborhood V of x such that $V \subseteq U$ and $X-V$ has only finitely many components.

2.7 COROLLARY. *If f is an open, connected, monotone function from a space X onto a semilocally connected T_1 space Y, then f is continuous.*

Proof. The proof follows from Corollary 2.5 and Theorems 3 and 9 of Sanderson [6].

2.8 REMARK. Corollary 2.7 improves a result which appears in [5]. Corollary 2.5 improves a result due to Hagan [3]. Using Corollary 2.5, the condition of first countability may be removed in Hagan's Theorem 3. We state it here. The proof is exactly the same.

2.9 THEOREM. *Let f be an open, connected, monotone function from a unicoherent T_2 continuum X onto a compact T_1 space Y. Then Y is a unicoherent continuum.*

3. Applications. Whyburn [9] showed that the finite product of semilocally connected metric spaces is semilocally connected if one of the factors is. We do not know if this is true for infinite products of T_1 spaces. However, the following is true:

3.1 THEOREM. *If X_α is a semilocally connected T_1 space for all α, then $X=\Pi X_\alpha$ is semilocally connected.*

3.2 DEFINITION. A function f from a space X onto a space Y is semiconnected if $f^{-1}(A)$ is closed and connected for every closed and connected subset A of Y.

3.3 REMARK. It follows from Corollary 2.5 and Theorem 3 of [6] that an open, connected, monotone function onto a T_1 space is semiconnected.

3.4 LEMMA. (Yu-Lee [10]) *A connected T_1 space Y is semilocally connected iff for all spaces X and all semiconnected functions f mapping X onto Y, f is continuous.*

Proof of Theorem 3.1. Let $f:Z\to X$ be a semiconnected function from some space Z onto $X=\Pi X_\alpha$. Since π_α is open, connected, and monotone, it is semiconnected for all α by Remark 3.3. Clearly, the composition of semiconnected functions is semiconnected, so that $\pi_\alpha\circ f$ is semiconnected for all α. But then $\pi_\alpha\circ f$ is continuous for all α since X_α is semilocally connected. It follows that f is continuous and therefore, by Lemma 3.3, X is semilocally connected.

3.5 Definition (Thomas [7]). A topology τ on a set X will be said to be finer than a topology τ_1 on X if $\tau_1\subseteq\tau$. If $\tau_1\neq\tau$ we will say that τ is strictly finer than τ_1. A connected topology τ will be said to be maximally connected if τ_1 strictly finer than τ implies τ_1 is not connected.

3.6 Theorem. *Let f be an open, connected, monotone function from (X, τ_1), where τ_1 is maximally connected, onto the T_1 space (Y, γ_1). Then γ_1 is maximally connected.*

Proof. (Y, γ_1) is connected since f is a connected function. Let γ_2 be a finer topology on Y and $U_0\in\gamma_2-\gamma_1$. Clearly $f^{-1}(U_0)\notin\tau_1$, so let τ_2 be the smallest topology generated by τ_1 and $f^{-1}(U_0)$. τ_2, being strictly finer than τ_1, cannot be a connected topology.

Consider f as a function from (X, τ_2) onto (Y, γ_2) and relabel it f_*. We claim that f_* is open and monotone.

(1) Let W be τ_2-open. Then we may assume that $W=Z\cup(V\cap f^{-1}(U_0))$ for some Z and $V\in\tau_1$ since basic open sets are of this form and arbitrary unions do not change the form. Therefore, $f_*(W)=f(Z)\cup(f(V)\cap U_0)$ which is γ_2-open since f is open, so f_* is open.

(2) Let $y\in Y$. Consider the following two cases:

Case 1. If $y\notin U_0$, then $f_*^{-1}(y)\cap f_*^{-1}(U_0)=\phi$. If $f_*^{-1}(y)\subseteq W_1\cup W_2$ where $W_1, W_2\in\tau_2$, $W_1\cap W_2\cap f_*^{-1}(y)=\phi$, and $W_i\cap f_*^{-1}(y)\neq\phi$ for $i=1, 2$, then $W_1=Z_1\cup(V_1\cap f^{-1}(U_0))$ and $W_2=Z_2\cup(V_2\cap f^{-1}(U_0))$ for some V_1, V_2, Z_1, $Z_2\in\tau_1$. We have $f_*^{-1}(y)\subseteq[Z_1\cup(f^{-1}(U_0)\cap V_1)]\cup[Z_2\cup(f^{-1}(U_0)\cap V_2)]$ and $Z_1\cap Z_2\cap f_*^{-1}(y)=\phi$. We claim that $f_*^{-1}(y)\nsubseteq Z_1\cup Z_2$. If not, then since $f^{-1}(y)$ is τ_1-connected, either $f^{-1}(y)\cap Z_1=\phi$ or $f^{-1}(y)\cap Z_2=\phi$, say $f^{-1}(y)\cap Z_1=\phi$. But, $f_*^{-1}(y)\cap W_1\neq\phi$ and $f^{-1}(y)\cap Z_1=\phi$ imply that $f_*^{-1}(y)\cap f_*^{-1}(U_0)\neq\phi$-contradiction. So, there is some $x\in f_*^{-1}(y)-(Z_1\cup Z_2)$. However, then $x\in f_*^{-1}(y)\cap f_*^{-1}(U_0)$-contradiction. It follows that $f_*^{-1}(y)$ is connected.

Case 2. If $y\in U_0$, then $f_*^{-1}(y)\subseteq f_*^{-1}(U_0)$. Now, by Remark 3.3, $f^{-1}(y)$ is τ_1-connected and τ_1-closed, and so τ_2-closed (i.e., $f_*^{-1}(y)$ is closed). We assume that $f_*^{-1}(y)$ is not connected. Let $f_*^{-1}(y)=G\cup H$, where G and H are disjoint, non-empty and closed in $f_*^{-1}(y)$ (and thus in (X, τ_2)). We may assume that $G=[Z_1\cup(f^{-1}(U_0)\cap V_1)]^c$ and $H=[Z_2\cup(f^{-1}(U_0)\cap V_2)]^c$ for some V_1, V_2, Z_1,

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$Z_2 \in \tau_1$. Then, $G = Z_1^c \cap [(f^{-1}(U_0))^c \cup V_1^c] \supseteq V_1^c \cap Z_1^c$. If $x \in G$, $x \in G \cup H = f_*^{-1}(y) \subseteq f_*^{-1}(U_0)$. On the other hand, $x \in G$ implies $x \in [Z_1^c \cap V_1^c] \cup [Z_1^c \cap (f^{-1}(U_0))^c]$. Therefore, $x \in Z_1^c \cap V_1^c$ and $G = V_1^c \cap Z_1^c$. It follows that G, and similarly H, is τ_1-closed. But this contradicts the τ_1-connectedness of $f^{-1}(y)$. Therefore, $f_*^{-1}(y)$ is connected.

If (Y, γ_2) were connected, then by Corollary 2.5, (X, τ_2) would be connected, which cannot happen. Therefore, any topology strictly finer than γ_1 is not connected and so γ_1 is maximally connected.

3.7 Corollary. *If (X_α, τ_α) are T_1 spaces for all α and ΠX_α is maximally connected, then so is X_α for all α.*

The converse to the above corollary is not true. Let X be the real line and let $F = \{F \subseteq X \mid F^c \text{ is finite}\}$. Then F is a filter and hence is contained in an ultrafilter F^*. Thomas [7] has shown that X with the topology induced by F^* is T_1 and maximally connected. However, $X \times X$ with the product topology, $\tau_{X\times X}$, is not maximally connected. For, if $G = \{F \subseteq X \times X \mid F^c \text{ is finite}\}$ then $G \subseteq \tau_{X\times X} \cdot \tau_{X\times X}$ is a filter base so $\tau_{X\times X}$ is contained in an ultrafilter G^*. Consider $H = (X - \{x_0\}) \times (X - \{x_0\}) \cup \{(x_0, x_0)\}$, where x_0 is an arbitrary element of X. Clearly, $H \in G^* - \tau_{X\times X}$. Since $(X \times X, G^*)$ is maximally connected, it follows that $(X \times X, \tau_{X\times X})$ is not maximally connected.

Added in Proof. See M. R. Hagan, *Conditions for continuity of certain open monotone functions*, Proc. Amer. Math. Soc. **30** (1971), 175–178, for recent related results.

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