Bull. Aust. Math. Soc. **94** (2016), 208–216
doi:10.1017/S0004972715001847

COUNTING POINTS ON DWORK HYPERSURFACES AND p-ADIC HYPERGEOMETRIC FUNCTIONS

RUPAM BARMAN✉, HASANUR RAHMAN and NEELAM SAIKIA

(Received 29 November 2015; accepted 5 December 2015; first published online 17 February 2016)

Abstract

We express the number of points on the Dwork hypersurface $X_\lambda^d : x_1^d + x_2^d + \cdots + x_d^d = d\lambda x_1 x_2 \cdots x_d$ over a finite field of order $q \not\equiv 1 \pmod d$ in terms of McCarthy's p-adic hypergeometric function for any odd prime d.

2010 *Mathematics subject classification*: primary 11G25; secondary 11S80, 11T24, 33C20, 33E50.

Keywords and phrases: characters of finite fields, hypergeometric series, Teichmüller character, p-adic gamma function, Dwork hypersurfaces.

1. Introduction and statement of results

Let p be an odd prime and let $\mathbb{F}_q$ denote the finite field with q elements, where $q = p^r, r \geq 1$. Let $\mathbb{Z}_p$ denote the ring of p-adic integers. Let $\Gamma_p(\cdot)$ denote Morita's p-adic gamma function and let ω denote the Teichmüller character of $\mathbb{F}_q$. We denote the inverse of ω by $\overline{\omega}$. For $x \in \mathbb{Q}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\langle x \rangle$ the fractional part of x, that is, $x - \lfloor x \rfloor$. Also, $\mathbb{Z}^+$ and $\mathbb{Z}_{\geq 0}$ denote the sets of positive integers and nonnegative integers, respectively. We now define McCarthy's p-adic hypergeometric series ${}_nG_n[\cdots]$.

Definition 1.1 [13, Definition 5.1]. Let $q = p^r$, where p is an odd prime and $r \in \mathbb{Z}^+$, and let $t \in \mathbb{F}_q$. For $n \in \mathbb{Z}^+$ and $1 \leq i \leq n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Define ${}_nG_n[\cdots]$ by

$$
{}_nG_n\left[\begin{matrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{matrix} \middle| \, t \right]_q := \frac{-1}{q-1} \sum_{j=0}^{q-2} (-1)^{jn}\, \overline{\omega}^j(t)
\times \prod_{i=1}^{n} \prod_{k=0}^{r-1} (-p)^{-\lfloor \langle a_i p^k \rangle - (jp^k/(q-1)) \rfloor - \lfloor \langle -b_i p^k \rangle + (jp^k/(q-1)) \rfloor}
\times \frac{\Gamma_p(\langle (a_i - \frac{j}{q-1}) p^k \rangle)}{\Gamma_p(\langle a_i p^k \rangle)} \frac{\Gamma_p(\langle (-b_i + \frac{j}{q-1}) p^k \rangle)}{\Gamma_p(\langle -b_i p^k \rangle)}.
$$

This work is partially supported by the IRD project MI01189 of the first author awarded by Indian Institute of Technology, Delhi. The third author acknowledges the financial support of the Department of Science and Technology, Government of India, for supporting a part of this work under an INSPIRE Fellowship.

Koblitz [11] developed a formula for the number of points on diagonal hypersurfaces in the Dwork family in terms of Gauss sums. In [6], Goodson specialises Koblitz's formula to the family of Dwork K3 surfaces. She gives an expression for the number of points on this family, $X_\lambda^4 : x_1^4 + x_2^4 + x_3^4 + x_4^4 = 4\lambda x_1x_2x_3x_4$, in the projective plane $\mathbb{P}^4(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ in terms of Greene's finite field hypergeometric functions [7] under the condition that $q \equiv 1 \pmod 4$. She then considers the higher dimensional Dwork hypersurfaces

$$X_\lambda^d : x_1^d + x_2^d + \cdots + x_d^d = d\lambda x_1x_2\cdots x_d$$

and gives a formula for the number of points on X_λ^d in terms of Gaussian hypergeometric series and Gauss sums when $q \equiv 1 \pmod d$. For primes $p \not\equiv 1 \pmod d$, she conjectures the following.

Conjecture 1.2 [6, Conjecture 8.2]. *Let d be an odd prime and p a prime number such that $p \not\equiv 1 \pmod d$. The number of points over $\mathbb{F}_p$ on the Dwork hypersurface is given by*

$$\#X_\lambda^d(\mathbb{F}_p) = \frac{p^{d-1}-1}{p-1} + \frac{1}{p-1} + {}_{d-1}G_{d-1}\left[\begin{matrix}1/d, & 2/d, & \ldots, & (d-1)/d \\ 0, & 0, & \ldots, & 0\end{matrix}\,\middle|\, \lambda^d\right]_p.$$

In this article, we prove that the above conjecture is not correct. We correct the statement of the conjecture and prove it for any finite field of order $q = p^r \not\equiv 1 \pmod d$. The statement of our main result is as follows.

Theorem 1.3. *Let d be an odd prime and $q = p^r$ be a prime power such that $q \not\equiv 1 \pmod d$ and $p \neq d$. Then the number of points on the Dwork hypersurface*

$$X_\lambda^d : x_1^d + x_2^d + \cdots + x_d^d = d\lambda x_1x_2\cdots x_d$$

in $\mathbb{P}^d(\mathbb{F}_q)$ is given by

$$\#X_\lambda^d(\mathbb{F}_q) = \frac{q^{d-1}-1}{q-1} - {}_{d-1}G_{d-1}\left[\begin{matrix}1/d, & 2/d, & \ldots, & (d-1)/d \\ 0, & 0, & \ldots, & 0\end{matrix}\,\middle|\, \lambda^d\right]_q.$$

The case $d = 5$ is dealt with by McCarthy in [12]. We use a similar technique to prove the above theorem. We note that the expression in the above conjecture contains an error term and the sign of the G-function is negative when d is an odd prime.

For any λ and $q = p^r \not\equiv 1 \pmod 3$,

$$\#X_\lambda^3(\mathbb{F}_q) = 1 + \#\{(x, y) \in \mathbb{F}_q^2 : x^3 + y^3 + 1 = 3\lambda xy\}.$$

Now, from Theorem 1.3 and Theorem 3.3 of [2], we have the following transformation for the ${}_2G_2$-function.

Corollary 1.4. *Let $\lambda \neq 0$ and $\lambda^3 \neq 1$. Let $p \geq 5$ be a prime and $q = p^r \not\equiv 1 \pmod 3$. Then*

$${}_2G_2\left[\begin{matrix}1/3, & 2/3 \\ 0, & 0\end{matrix}\,\middle|\, \lambda^3\right]_q = q\phi(-3\lambda)\,{}_2G_2\left[\begin{matrix}1/2, & 1/2 \\ 1/6, & 5/6\end{matrix}\,\middle|\, \frac{1}{\lambda^3}\right]_q,$$

where ϕ is the quadratic character on $\mathbb{F}_q$.

https://doi.org/10.1017/S0004972715001847 Published online by Cambridge University Press

2. Preliminaries

Let $\widehat{\mathbb{F}_q^\times}$ denote the set of all multiplicative characters χ on $\mathbb{F}_q^\times$. It is known that $\widehat{\mathbb{F}_q^\times}$ is a cyclic group of order $q-1$ under the multiplication of characters: $(\chi\psi)(x) = \chi(x)\psi(x)$, $x \in \mathbb{F}_q^\times$. The domain of each $\chi \in \widehat{\mathbb{F}_q^\times}$ is extended to $\mathbb{F}_q$ by setting $\chi(0) := 0$, including the trivial character ε. Multiplicative characters satisfy the following *orthogonality relations*.

Lemma 2.1 [9, Ch. 8]. *With the notation as above,*

(1) $$\sum_{x\in\mathbb{F}_q} \chi(x) = \begin{cases} q-1 & \text{if } \chi = \varepsilon, \\ 0 & \text{if } \chi \neq \varepsilon. \end{cases}$$

(2) $$\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \chi(x) = \begin{cases} q-1 & \text{if } x = 1, \\ 0 & \text{if } x \neq 1. \end{cases}$$

Let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. Let $\overline{\mathbb{Q}_p}$ be the algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}_p}$. Let $\mathbb{Z}_q$ be the ring of integers in the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. We know that $\chi \in \widehat{\mathbb{F}_q^\times}$ takes values in the group of $(q-1)$th roots of unity in $\mathbb{C}^\times$. Since $\mathbb{Z}_q^\times$ contains all $(q-1)$th roots of unity, we can consider multiplicative characters on $\mathbb{F}_q^\times$ to be maps $\chi : \mathbb{F}_q^\times \to \mathbb{Z}_q^\times$. Let $\omega : \mathbb{F}_q^\times \to \mathbb{Z}_q^\times$ be the Teichmüller character. For $a \in \mathbb{F}_q^\times$, $\omega(a)$ is just the $(q-1)$th root of unity in $\mathbb{Z}_q$ such that $\omega(a) \equiv a \pmod{p}$.

We now introduce some properties of Gauss sums. For further details, see [4]. Let ζ_p be a fixed primitive pth root of unity in $\overline{\mathbb{Q}_p}$. The trace map $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is given by

$$\mathrm{tr}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{r-1}}.$$

Then the additive character $\theta : \mathbb{F}_q \to \mathbb{Q}_p(\zeta_p)$ is defined by

$$\theta(\alpha) = \zeta_p^{\mathrm{tr}(\alpha)}.$$

For $\chi \in \widehat{\mathbb{F}_q^\times}$, the *Gauss sum* is defined by

$$g(\chi) := \sum_{x\in\mathbb{F}_q} \chi(x)\theta(x).$$

Let T denote a fixed generator of $\widehat{\mathbb{F}_q^\times}$. The Gauss sums have two important properties.

Lemma 2.2 [7, Equation 1.12]. *If $k \in \mathbb{Z}$ and $T^k \neq \varepsilon$, then*

$$g(T^k)g(T^{-k}) = qT^k(-1).$$

Lemma 2.3 [5, Lemma 2.2]. *For all $\alpha \in \mathbb{F}_q^\times$,*

$$\theta(\alpha) = \frac{1}{q-1} \sum_{m=0}^{q-2} g(T^{-m})T^m(\alpha).$$

https://doi.org/10.1017/S0004972715001847 Published online by Cambridge University Press

Finally, we recall the p-adic gamma function. For further details, see [10]. For $n \in \mathbb{Z}^+$, the p-adic gamma function $\Gamma_p(n)$ is defined as

$$\Gamma_p(n) := (-1)^n \prod_{0<j<n, p\nmid j} j$$

and one extends it to all $x \in \mathbb{Z}_p$ by setting $\Gamma_p(0) := 1$ and

$$\Gamma_p(x) := \lim_{n\to x} \Gamma_p(n)$$

for $x \neq 0$, where n runs through any sequence of positive integers p-adically approaching x. This limit exists, is independent of how n approaches x and determines a continuous function on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p^\times$. Let $\pi \in \mathbb{C}_p$ be the fixed root of $x^{p-1} + p = 0$ which satisfies $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. Then the Gross–Koblitz formula relates Gauss sums and the p-adic gamma function as follows. (Recall that ω denotes the Teichmüller character of $\mathbb{F}_q$.)

Theorem 2.4 [8, Gross–Koblitz]. *For $a \in \mathbb{Z}$ and $q = p^r$,*

$$g(\overline{\omega}^a) = -\pi^{(p-1)\sum_{i=0}^{r-1}\langle ap^i/(q-1)\rangle} \prod_{i=0}^{r-1} \Gamma_p\Big(\Big\langle \frac{ap^i}{q-1}\Big\rangle\Big).$$

3. Proof of Theorem 1.3

We first state two lemmas which we will use to prove the theorem. The first lemma is a generalisation of [13, Lemma 4.1]. For a proof, see [1].

Lemma 3.1 [1, Lemma 3.1]. *Let p be a prime and $q = p^r$. For $0 \le j \le q-2$ and $t \in \mathbb{Z}^+$ with $p \nmid t$,*

$$\omega(t^{tj}) \prod_{i=0}^{r-1} \Gamma_p\Big(\Big\langle \frac{tp^i j}{q-1}\Big\rangle\Big) \prod_{h=1}^{t-1} \Gamma_p\Big(\Big\langle \frac{hp^i}{t}\Big\rangle\Big) = \prod_{i=0}^{r-1}\prod_{h=0}^{t-1} \Gamma_p\Big(\Big\langle \frac{p^i h}{t} + \frac{p^i j}{q-1}\Big\rangle\Big),$$

$$\omega(t^{-tj}) \prod_{i=0}^{r-1} \Gamma_p\Big(\Big\langle \frac{-tp^i j}{q-1}\Big\rangle\Big) \prod_{h=1}^{t-1} \Gamma_p\Big(\Big\langle \frac{hp^i}{t}\Big\rangle\Big) = \prod_{i=0}^{r-1}\prod_{h=0}^{t-1} \Gamma_p\Big(\Big\langle \frac{p^i(1+h)}{t} - \frac{p^i j}{q-1}\Big\rangle\Big).$$

Lemma 3.2. *Let $d \neq p$ be a prime number such that $q = p^r \not\equiv 1 \pmod d$. Then, for $1 \le a \le q-2$ and $0 \le i \le r-1$,*

$$d\Big\lfloor \frac{ap^i}{q-1}\Big\rfloor + \Big\lfloor \frac{-dap^i}{q-1}\Big\rfloor = (d-1)\Big\lfloor \frac{ap^i}{q-1}\Big\rfloor + \sum_{h=1}^{d-1}\Big\lfloor \Big\langle \frac{hp^i}{d}\Big\rangle - \frac{ap^i}{q-1}\Big\rfloor - 1. \tag{3.1}$$

Proof. Let $\lfloor -dap^i/(q-1)\rfloor = dk + s$ for some $k, s \in \mathbb{Z}$ satisfying $0 \le s \le d-1$. Since $1 \le a \le q-2$ and $(q-1, dp^i) = 1$, we observe that $-dap^i/(q-1)$ is not an integer. This yields

$$dk + s < \frac{-dap^i}{q-1} < dk + s + 1,$$

https://doi.org/10.1017/S0004972715001847 Published online by Cambridge University Press

which implies

$$\frac{s}{d} + k < \frac{-ap^i}{q-1} < k + \frac{s+1}{d}. \tag{3.2}$$

This is equivalent to

$$-\frac{s+1}{d} - k < \frac{ap^i}{q-1} < -k - \frac{s}{d}. \tag{3.3}$$

From (3.3), $\lfloor ap^i/(q-1)\rfloor = -k-1$, and then the left-hand side of (3.1) becomes $s-d$. Again, since d is a prime and $d \neq p$, we observe that

$$\sum_{h=1}^{d-1}\left\lfloor\left\langle\frac{hp^i}{d}\right\rangle - \frac{ap^i}{q-1}\right\rfloor = \sum_{h=1}^{d-1}\left\lfloor\left\langle\frac{h}{d}\right\rangle - \frac{ap^i}{q-1}\right\rfloor.$$

Thus, for $1 \leq h \leq d-s-1$, (3.2) yields

$$\left\lfloor\left\langle\frac{h}{d}\right\rangle - \frac{ap^i}{q-1}\right\rfloor = k$$

so that

$$\sum_{h=1}^{d-s-1}\left\lfloor\left\langle\frac{h}{d}\right\rangle - \frac{ap^i}{q-1}\right\rfloor = (d-s-1)k. \tag{3.4}$$

Also, for $d-s \leq h \leq d-1$, (3.2) yields

$$\left\lfloor\left\langle\frac{h}{d}\right\rangle - \frac{ap^i}{q-1}\right\rfloor = k+1$$

so that

$$\sum_{h=d-s}^{d-1}\left\lfloor\left\langle\frac{h}{d}\right\rangle - \frac{ap^i}{q-1}\right\rfloor = s(k+1). \tag{3.5}$$

Combining (3.4) and (3.5), and using the fact that $\lfloor ap^i/(q-1)\rfloor = -k-1$, we see that the right-hand side of (3.1) also becomes $s-d$. This completes the proof of the lemma. □

Proof of Theorem 1.3. Let $N_q^A(\lambda)$ denote the number of points on the Dwork hypersurface X_λ^d in $\mathbb{A}^d(\mathbb{F}_q)$. Then

$$\#X_\lambda^d(\mathbb{F}_q) = \frac{N_q^A(\lambda)-1}{q-1}. \tag{3.6}$$

Letting $\overline{x} = (x_1, x_2, \ldots, x_d)$ and $f(\overline{x}) = x_1^d + x_2^d + \cdots + x_d^d - d\lambda x_1 x_2 \cdots x_d$ and using the identity

$$\sum_{z\in\mathbb{F}_q}\theta(zf(\overline{x})) = \begin{cases} q & \text{if } f(\overline{x}) = 0,\\ 0 & \text{if } f(\overline{x}) \neq 0,\end{cases}$$

we can write

$$\begin{aligned} q \cdot N_q^A(\lambda) &= q^d + \sum_{z \in \mathbb{F}_q^\times} \sum_{x_i \in \mathbb{F}_q} \theta(z f(\overline{x})) \\ &= q^d + \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z f(\overline{x})) + \sum_{\substack{z \in \mathbb{F}_q^\times \\ \text{some } x_i = 0}} \theta(z f(\overline{x})). \end{aligned} \tag{3.7}$$

We now rewrite the second summation: let $f_1(\overline{x}) = x_1^d + x_2^d + \cdots + x_d^d$ and let N_q' be the number of solutions to $f_1(\overline{x}) = 0$ in $\mathbb{A}^d(\mathbb{F}_q)$. Since $x \mapsto x^d$ is an automorphism of $\mathbb{F}_q^\times$ when d is prime and $q \not\equiv 1 \pmod d$, we have $N_q' = q^{d-1}$. Also, proceeding as above,

$$q \cdot N_q' = q^d + \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z f_1(\overline{x})) + \sum_{\substack{z \in \mathbb{F}_q^\times \\ \text{some } x_i = 0}} \theta(z f_1(\overline{x})).$$

Thus,

$$\sum_{\substack{z \in \mathbb{F}_q^\times \\ \text{some } x_i = 0}} \theta(z f_1(\overline{x})) = - \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z f_1(\overline{x})). \tag{3.8}$$

Since

$$\sum_{\substack{z \in \mathbb{F}_q^\times \\ \text{some } x_i = 0}} \theta(z f(\overline{x})) = \sum_{\substack{z \in \mathbb{F}_q^\times \\ \text{some } x_i = 0}} \theta(z f_1(\overline{x})), \tag{3.9}$$

by using (3.8) and (3.9), we can rewrite (3.7) as

$$\begin{aligned} q \cdot N_q^A(\lambda) &= q^d + \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z f(\overline{x})) - \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z f_1(\overline{x})) \\ &= q^d + A - B, \end{aligned} \tag{3.10}$$

where $A = \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z f(\overline{x}))$ and $B = \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z f_1(\overline{x}))$. First we evaluate

$$B = \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z f_1(\overline{x})) = \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(z x_1^d) \theta(z x_2^d) \cdots \theta(z x_d^d).$$

Lemma 2.3 gives

$$\begin{aligned} B &= \frac{1}{(q-1)^d} \sum_{a_1, a_2, \ldots, a_d = 0}^{q-2} g(T^{-a_1}) g(T^{-a_2}) \cdots g(T^{-a_d}) \\ &\quad \times \sum_{z, x_i \in \mathbb{F}_q^\times} T^{a_1}(z x_1^d) T^{a_2}(z x_2^d) \cdots T(z x_d^d) \\ &= \frac{1}{(q-1)^d} \sum_{a_1, a_2, \ldots, a_d = 0}^{q-2} g(T^{-a_1}) g(T^{-a_2}) \cdots g(T^{-a_d}) \sum_{x_1 \in \mathbb{F}_q^\times} T^{d a_1}(x_1) \\ &\quad \times \sum_{x_2 \in \mathbb{F}_q^\times} T^{d a_2}(x_2) \cdots \sum_{x_d \in \mathbb{F}_q^\times} T^{d a_d}(x_d) \sum_{z \in \mathbb{F}_q^\times} T^{a_1 + a_2 + \cdots + a_d}(z). \end{aligned}$$

https://doi.org/10.1017/S0004972715001847 Published online by Cambridge University Press

The inner sums in the above expression are nonzero only if $da_1, da_2, \ldots, da_d \equiv 0 \pmod{q-1}$ and $a_1 + a_2 + \cdots + a_d \equiv 0 \pmod{q-1}$. Since $q \not\equiv 1 \pmod{d}$, these congruences hold simultaneously only if $a_1 = a_2 = \cdots = a_d = 0$. Finally, using the fact that $g(\varepsilon) = -1$, we obtain $B = 1 - q$. Next,

$$A = \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(zf(\overline{x})) = \sum_{z, x_i \in \mathbb{F}_q^\times} \theta(zx_1^d)\theta(zx_2^d)\cdots\theta(zx_d^d)\theta(-d\lambda z x_1 x_2 \cdots x_d).$$

By using Lemma 2.3, we see that

$$A = \frac{1}{(q-1)^{d+1}} \sum_{a_1, a_2, \ldots, a_d, a_{d+1}=0}^{q-2} g(T^{-a_1})g(T^{-a_2})\cdots g(T^{-a_d})g(T^{-a_{d+1}})T^{a_{d+1}}(-d\lambda)$$
$$\times \sum_{x_1 \in \mathbb{F}_q^\times} T^{da_1 + a_{d+1}}(x_1) \sum_{x_2 \in \mathbb{F}_q^\times} T^{da_2 + a_{d+1}}(x_2) \cdots \sum_{x_d \in \mathbb{F}_q^\times} T^{da_d + a_{d+1}}(x_d) \sum_{z \in \mathbb{F}_q^\times} T^{a_1 + a_2 + \cdots + a_d + a_{d+1}}(z).$$

The inner sums here are nonzero only when all the following congruences hold: $da_1 + a_{d+1}, da_2 + a_{d+1}, \ldots, da_d + a_{d+1} \equiv 0 \pmod{q-1}$ and $a_1 + a_2 + \cdots + a_d + a_{d+1} \equiv 0 \pmod{q-1}$, giving $a_1 = a_2 = \cdots = a_d = a$ (say) and $a_{d+1} = -da$ as $q \not\equiv 1 \pmod{d}$. Thus

$$A = \sum_{a=0}^{q-2} g^d(T^{-a})g(T^{da})T^{-da}(-d\lambda).$$

Taking $T = \omega$ and then using the Gross–Koblitz formula, we obtain

$$A = \sum_{a=0}^{q-2} \pi^{(p-1)\sum_{i=0}^{r-1}\{d\langle ap^i/(q-1)\rangle + \langle -dap^i/(q-1)\rangle\}}\, \overline{\omega}^{da}(-d\lambda) \prod_{i=0}^{r-1} \Gamma_p^d\Big(\Big\langle \frac{ap^i}{q-1}\Big\rangle\Big)\Gamma_p\Big(\Big\langle \frac{-dap^i}{q-1}\Big\rangle\Big). \tag{3.11}$$

Applying Lemma 3.1 for $t = d$ and $j = a$ gives

$$\prod_{i=0}^{r-1} \Gamma_p\Big(\Big\langle \frac{-dap^i}{q-1}\Big\rangle\Big) = \omega^{da}(d) \prod_{i=0}^{r-1} \frac{\prod_{h=1}^{d} \Gamma_p(\langle(\frac{h}{d} - \frac{a}{q-1})p^i\rangle)}{\prod_{h=1}^{d-1} \Gamma_p(\langle \frac{hp^i}{d}\rangle)}.$$

On substituting this into (3.11),

$$\begin{aligned}
A &= \sum_{a=0}^{q-2} \pi^{(p-1)\sum_{i=0}^{r-1}\{-d\lfloor ap^i/(q-1)\rfloor - \lfloor -dap^i/(q-1)\rfloor\}}\, \overline{\omega}^{da}(-\lambda) \\
&\quad \times \prod_{i=0}^{r-1} \Gamma_p^d\Big(\Big\langle \frac{ap^i}{q-1}\Big\rangle\Big) \frac{\prod_{h=1}^{d} \Gamma_p(\langle(\frac{h}{d} - \frac{a}{q-1})p^i\rangle)}{\prod_{h=1}^{d-1} \Gamma_p(\langle \frac{hp^i}{d}\rangle)} \\
&= \sum_{a=0}^{q-2} \pi^{(p-1)\sum_{i=0}^{r-1}\{-d\lfloor ap^i/(q-1)\rfloor - \lfloor -dap^i/(q-1)\rfloor\}}\, \overline{\omega}^{da}(-\lambda) \\
&\quad \times \prod_{i=0}^{r-1} \Gamma_p\Big(\Big\langle \frac{ap^i}{q-1}\Big\rangle\Big)\Gamma_p\Big(\Big\langle\Big(1 - \frac{a}{q-1}\Big)p^i\Big\rangle\Big)\Gamma_p^{d-1}\Big(\Big\langle \frac{ap^i}{q-1}\Big\rangle\Big) \prod_{h=1}^{d-1} \frac{\Gamma_p(\langle(\frac{h}{d} - \frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle \frac{hp^i}{d}\rangle)}
\end{aligned}$$

https://doi.org/10.1017/S0004972715001847 Published online by Cambridge University Press

$$= 1 + \sum_{a=1}^{q-2} (-p)^{\sum_{i=0}^{r-1}\{-d\lfloor ap^i/(q-1)\rfloor - \lfloor -dap^i/(q-1)\rfloor\}}\, \overline{\omega}^{da}(-\lambda)$$
$$\times \prod_{i=0}^{r-1} \Gamma_p\Big(\Big\langle \frac{ap^i}{q-1}\Big\rangle\Big)\Gamma_p\Big(\Big\langle\Big(1-\frac{a}{q-1}\Big)p^i\Big\rangle\Big)\Gamma_p^{d-1}\Big(\Big\langle\frac{ap^i}{q-1}\Big\rangle\Big)\prod_{h=1}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}-\frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}.$$

From [3, Lemma 3.4], for $0 < a \le q-2$,

$$\prod_{i=0}^{r-1} \Gamma_p\Big(\Big\langle \frac{ap^i}{q-1}\Big\rangle\Big)\Gamma_p\Big(\Big\langle\Big(1-\frac{a}{q-1}\Big)p^i\Big\rangle\Big) = (-1)^r\overline{\omega}^a(-1).$$

Applying this in the above expression gives

$$A = 1 + \sum_{a=1}^{q-2}(-1)^r(-p)^{\sum_{i=0}^{r-1}\{-d\lfloor ap^i/(q-1)\rfloor - \lfloor -dap^i/(q-1)\rfloor\}}\, \overline{\omega}^{da}(\lambda)$$
$$\times \prod_{i=0}^{r-1}\Gamma_p^{d-1}\Big(\Big\langle\frac{ap^i}{q-1}\Big\rangle\Big)\prod_{h=1}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}-\frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}.$$

Now, Lemma 3.2 yields

$$A = 1 + \sum_{a=1}^{q-2}(-1)^r(-p)^{\sum_{i=0}^{r-1}\{1-(d-1)\lfloor ap^i/(q-1)\rfloor - \sum_{h=1}^{d-1}\lfloor\langle hp^i/d\rangle - ap^i/(q-1)\rfloor\}}\, \overline{\omega}^{da}(\lambda)$$
$$\times \prod_{i=0}^{r-1}\Gamma_p^{d-1}\Big(\Big\langle\frac{ap^i}{q-1}\Big\rangle\Big)\prod_{h=1}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}-\frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}.$$

Adding and subtracting the term under the summation for $a = 0$,

$$A = 1 - q + q\sum_{a=0}^{q-2}(-p)^{\sum_{i=0}^{r-1}\{-(d-1)\lfloor ap^i/(q-1)\rfloor - \sum_{h=1}^{d-1}\lfloor\langle hp^i/d\rangle - ap^i/(q-1)\rfloor\}}\, \overline{\omega}^{da}(\lambda)$$
$$\times \prod_{i=0}^{r-1}\Gamma_p^{d-1}\Big(\Big\langle\frac{ap^i}{q-1}\Big\rangle\Big)\prod_{h=1}^{d-1}\frac{\Gamma_p(\langle(\frac{h}{d}-\frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{hp^i}{d}\rangle)}$$
$$= 1 - q - q(q-1)_{d-1}G_{d-1}\left[\begin{matrix}1/d, & 2/d, & \dots, & (d-1)/d\\ 0, & 0, & \dots, & 0\end{matrix}\,\Big|\,\lambda^d\right]_q.$$

Finally, substituting the values of A and B into (3.10) and then using (3.6), we deduce the result. This completes the proof of Theorem 1.3. □

Acknowledgements

We are grateful to Wadim Zudilin and Dermot McCarthy for many helpful comments on an initial draft of the manuscript.

References

[1] R. Barman and N. Saikia, 'p-Adic gamma function and the trace of Frobenius of elliptic curves', *J. Number Theory* **140**(7) (2014), 181–195.

[2] R. Barman and N. Saikia, 'Certain transformations for hypergeometric series in the p-adic setting', *Int. J. Number Theory* **11**(2) (2015), 645–660.

[3] R. Barman, N. Saikia and D. McCarthy, 'Summation identities and special values of hypergeometric series in the p-adic setting', *J. Number Theory* **153** (2015), 63–84.

[4] B. Berndt, R. Evans and K. Williams, *Gauss and Jacobi Sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts (A Wiley-Interscience Publication, John Wiley and Sons, Inc., New York, 1998).

[5] J. Fuselier, 'Hypergeometric functions over $\mathbb{F}_p$ and relations to elliptic curve and modular forms', *Proc. Amer. Math. Soc.* **138** (2010), 109–123.

[6] H. Goodson, 'Hypergeometric functions and relations to Dwork hypersurfaces'. arXiv:1510.07661v1.

[7] J. Greene, 'Hypergeometric functions over finite fields', *Trans. Amer. Math. Soc.* **301**(1) (1987), 77–101.

[8] B. H. Gross and N. Koblitz, 'Gauss sum and the p-adic Γ-function', *Ann. of Math.* (2) **109** (1979), 569–581.

[9] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer International Edition (Springer, New York, 2005).

[10] N. Koblitz, *p-Adic Analysis: A Short Course on Recent Work*, London Mathematical Society Lecture Note Series, 46 (Cambridge University Press, Cambridge–New York, 1980).

[11] N. Koblitz, 'The number of points on certain families of hypersurfaces over finite fields', *Compositio Math.* **48**(1) (1983), 3–23.

[12] D. McCarthy, 'On a supercongruence conjecture of Rodriguez-Villegas', *Proc. Amer. Math. Soc.* **140** (2012), 2241–2254.

[13] D. McCarthy, 'The trace of Frobenius of elliptic curves and the p-adic gamma function', *Pacific J. Math.* **261**(1) (2013), 219–236.

RUPAM BARMAN, Department of Mathematics,
Indian Institute of Technology, Hauz Khas, New Delhi-110016, India
e-mail: rupam@maths.iitd.ac.in

HASANUR RAHMAN, Department of Mathematics,
Indian Institute of Technology, Hauz Khas, New Delhi-110016, India
e-mail: hasrah93@gmail.com

NEELAM SAIKIA, Department of Mathematics,
Indian Institute of Technology, Hauz Khas, New Delhi-110016, India
e-mail: nlmsaikia1@gmail.com

https://doi.org/10.1017/S0004972715001847 Published online by Cambridge University Press