Can. J. Math. Vol. **46** (6), 1994 pp. 1263–1274

EXPONENTIAL BOUNDEDNESS AND AMENABILITY OF OPEN SUBSEMIGROUPS OF LOCALLY COMPACT GROUPS

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ABSTRACT. Let G be a connected amenable locally compact group with left Haar measure λ. In an earlier work Jenkins claimed that exponential boundedness of G is equivalent to each of the following conditions: (a) every open subsemigroup $S \subseteq G$ is amenable; (b) given $\varepsilon > 0$ and a compact $K \subseteq G$ with nonempty interior there exists an integer n such that $\lambda(K^{n+1} \triangle K^n) < \varepsilon\lambda(K^n)$; (c) given a signed measure $\nu \ll \lambda$ of compact support and nonnegative nonzero $f \in L^\infty(G)$, the condition $\nu * f \geq 0$ implies $\nu(G) \geq 0$. However, Jenkins' proof of this equivalence is not complete. We give a complete proof. The crucial part of the argument relies on the following two results: (1) an open σ-compact subsemigroup $S \subseteq G$ is amenable if and only if there exists an absolutely continuous probability measure μ on S such that $\lim_{n\to\infty} \|\delta_s * \mu^n - \mu^n\| = 0$ for every $s \in S$; (2) G is exponentially bounded if and only if $G = \bigcup_{n=1}^\infty U^n U^{-n}$ for every nonempty open subset $U \subseteq G$.

1. **Introduction.** Let S be a topological semigroup. When f is a $\mathbb{C}$-valued function on S the *left translate of f by an element $s \in S$* is defined by $(l_s f)(x) = f(sx)$. A bounded continuous function $f: S \longrightarrow \mathbb{C}$ is said to be *left uniformly continuous* if the mapping $S \ni s \longrightarrow l_s f$ is continuous with respect to the sup norm. We shall denote by LUC(S) the space of bounded left uniformly functions on S. The semigroup S is defined to be amenable if there exists a left invariant mean on LUC(S).

Let S be an open subsemigroup of an amenable locally compact group G. It is well known that amenability of G alone does not guarantee amenability of S. A necessary and sufficient condition for S to be amenable is due in the discrete case to Frey [4] and in the general locally compact case to Jenkins [9]:

THEOREM 1.1. *The following conditions are equivalent for an open subsemigroup S of an amenable locally compact group G:*

(a) S is amenable,

(b) the open right ideals of S have the finite intersection property,

(c) SS^{-1} is a group.

((a) $\Leftrightarrow$ (b) is shown in [9], (b) $\Leftrightarrow$ (c) is elementary [14, Lemma 23.31, p. 330]).

Given a locally compact group G one can ask under what conditions *every* open subsemigroup of G is amenable.

Received by the editors February 18, 1993.

AMS subject classification: 43A07, 22D05, 43A05, 60B15.

Denote by λ the left Haar measure. Recall that G is said to be *exponentially bounded* if $\lim_{n\to\infty} \lambda(V^n)^{1/n} = 1$ for every compact neighbourhood of the identity e; otherwise G is said to have *exponential growth*. The following sufficient condition is implicit in Jenkins' paper [10, first part of the proof of Theorem 3].

THEOREM 1.2. *If G is exponentially bounded then every open semigroup $S \subseteq G$ is amenable.*

Consider for a moment an amenable discrete group acting on a set $\mathcal{X}$. Let $A \subseteq \mathcal{X}$ be a nonempty subset. A subset $Y \subseteq \mathcal{X}$ is said to be *A-bounded* if it can be covered by a finite number of translates of A. Denote by $B_A(\mathcal{X})$ the space of bounded functions $f: \mathcal{X} \longrightarrow \mathbb{C}$ whose supports are A-bounded. A necessary and sufficient condition for the existence of an invariant positive linear functional φ_A on $B_A(\mathcal{X})$ such that $\varphi_A(\chi_A) = 1$ is the following *translate property* for the system $(G, \mathcal{X}, A)$: for every finite sequence $\{g_i\}_{i=1}^n \subseteq G$ and $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{R}$ if $\sum_{i=1}^n \alpha_i \chi_A(g_i x) \geq 0$ for all $x \in \mathcal{X}$, then $\sum_{i=1}^n \alpha_i \geq 0$ [15, Corollary 1.2]. Applying this to the case that $\mathcal{X} = G$ and $A = S$ is a subsemigroup of G, with the use of Theorem 1.1 one easily obtains that S is amenable whenever for every signed measure ν of finite support if $\nu(Sg) \geq 0$ for all $g \in G$ then $\nu(G) \geq 0$. Clearly, if the translate property holds for every system (G, G, A) then every semigroup in G is amenable. The translate property generalizes to the case of an arbitrary locally compact group. We shall say that a locally compact group G has the translate property if for every Borel set $A \subseteq G$ that is not locally null and every signed measure $\nu \in L^1(G)$ of compact support the condition that $\nu(Ag) \geq 0$ for all $g \in G$ implies $\nu(G) \geq 0$. The following two results are implicit in the work of Jenkins [10, proof of Theorem 4] (see also [13, p. 243]).

THEOREM 1.3. *If an amenable locally compact group has the translate property then every open semigroup $S \subseteq G$ is amenable.*

THEOREM 1.4. *An exponentially bounded locally compact group has the translate property.*

Rosenblatt [15, Corollary 3.5] proved that for an exponentially bounded discrete group the translate property holds for every system $(G, \mathcal{X}, A)$.

The next condition that we wish to consider is a strong version of the classical Følner condition which is equivalent to the amenability of a locally compact group [13, Theorems 4.10 and 4.13]. We shall say that a compact set $K \subseteq G$ satisfies the *strong Følner condition* if for every $\varepsilon > 0$ there exists a positive integer n such that $\lambda(K^{n+1} \bigtriangleup K^n) < \varepsilon\lambda(K^n)$. The following result is implicit in an argument of Paterson [13, p. 245].

THEOREOM 1.5. *If every compact set $K \subseteq G$ with nonempty interior satisfies the strong Følner condition then every open semigroup $S \subseteq G$ is amenable.*

For an amenable locally compact group Theorems 1.2–1.5 can be summarized by the diagram:

$$\mathrm{EB} \Rightarrow \mathrm{TP} \Rightarrow \mathrm{AS} \Leftarrow \mathrm{SFC}$$

where EB = exponential boundedness, TP = translate property, AS = amenability of every open subsemigroup, SFC = strong Følner condition.

It follows from Rosenblatt's work [15, Corollary 4.14] that for a discrete solvable group TP $\Leftrightarrow$ AS, while for a finitely generated solvable G, EB $\Leftrightarrow$ TP $\Leftrightarrow$ AS [15, Corollaries 4.14, 4.13, and Theorem 4.7]. Jenkins [10] claims that for a connected locally compact group EB $\Leftrightarrow$ SFC $\Leftrightarrow$ AS while for an amenable connected locally compact G EB $\Leftrightarrow$ SFC $\Leftrightarrow$ AS $\Leftrightarrow$ TP. However, there are gaps in Jenkin's argument (in the proofs of his Theorems 2 and 3). In the case of a connected solvable Lie group a complete proof of the equivalence EB $\Leftrightarrow$ SFC $\Leftrightarrow$ AS $\Leftrightarrow$ TP can be found in [13, Theorem 6.39 and pp. 243–245]. Our goal is to show that the equivalence is, in fact, valid for an arbitrary connected amenable locally compact group, as originally claimed by Jenkins. By Theorems 1.2–1.5 to achieve this goal we need to prove AS $\Rightarrow$ EB and EB $\Rightarrow$ SFC. Our method will rely on a characterization of amenability in terms of the asymptotic behaviour of convolution powers of a probability measure on G and on a characterization of exponential boundedness in terms of powers of open subsets of G.

Let μ be a regular probability measure on G. A bounded Borel function $h: G \longrightarrow \mathbb{C}$ is said to be *μ-harmonic* if

$$h(g) = \int_G h(gg')\mu(dg'), \quad g \in G.$$

If every bounded μ-harmonic function is constant modulo a locally null set, μ is called a *Choquet-Deny measure*. A Choquet-Deny measure μ is necessarily adapted, *i.e.*, $\mu(H) < 1$ for every proper closed subgroup $H \subseteq G$. The following theorem due to Derriennic [2, Théorème 6] relates the Choquet-Deny property to the asymptotic behaviour of convolution powers μ^n.

THEOREM 1.6. *Let μ be a probability measure on a locally compact group G. Set $\mu_n = \frac{1}{n}\sum_{i=1}^n \mu^i$. It follows that μ is Choquet-Deny if and only if* $\lim_{n\to\infty} \|\nu_1 * \mu_n - \nu_1 * \mu_n\| = 0$ *for every pair of probability measures $\nu_1, \nu_2 \in L^1(G)$.*

It follows immediately from this theorem and the characterization of amenability in terms of convergence to left invariance [5, Theorem 2.4.3] that Choquet-Deny measures can exist only on amenable groups. As shown by Rosenblatt [16] and Kaimanovich and Vershik [12], a locally compact G admits a Choquet-Deny measure if and only if it is σ-compact and amenable. If G is σ-compact and amenable there even exists an absolutely continuous μ such that $\lim_{n\to\infty} \|\delta_g * \mu^n - \mu^n\| = 0$ for every $g \in G$ (this is stronger than the condition of Theorem 1.6). We remark that amenability and σ-compactness of G do not guarantee that every absolutely continuous adapted probability measure on G is Choquet-Deny [12].

Let $S \subseteq G$ be an open subsemigroup. It is easy to see that if G admits a probability measure μ such that $\mu(S) = 1$ and $\lim_{n\to\infty} \|\delta_g * \mu^n - \mu^n\| = 0$ for every $g \in S$, then S is amenable. By a slight modification of the construction of a Choquet-Deny measure on an amenable locally compact σ-compact group [12] we will show that an open σ-compact subsemigroup S of a locally compact group G is amenable if and only if G admits an absolutely continuous probability measure μ carried on S and such that $\lim_{n\to\infty} \|\delta_g * \mu^n - \mu^n\| = 0$ for all $g \in S$. Note that if G is connected then such a measure is Choquet-Deny (for S generates G). Next we will show that in every connected Lie group of exponential growth one can find an open subsemigroup which does not support any Choquet-Deny measure, and consequently is nonamenable. The usual approximation by Lie groups will then prove the implication AS $\Rightarrow$ EB for every connected locally compact G.

The equivalence AS $\Leftrightarrow$ EB will enable us to obtain the following elegant (and we believe new) characterization of exponential boundedness. We will prove that an amenable connected locally compact G is exponentially bounded if and only if for every nonempty open set $U \subseteq G$ one has $G = \bigcup_{n=1}^{\infty} U^n U^{-n}$. Having this characterization, we will give a simple proof of the implication EB $\Rightarrow$ SFC, thus accomplishing our task of establishing the equivalence EB $\Leftrightarrow$ SFC $\Leftrightarrow$ AS $\Leftrightarrow$ TP for amenable connected locally compact groups.

2. **Amenability and convolution powers.** Let S be an open subsemigroup of a locally compact group G. We will write $L^1(S)$ for the space of complex measures on S absolutely continuous with respect to the left Haar measure λ (restricted to S), and $L^1_1(S)$ for the subspace of probability measures in $L^1(S)$. $L^\infty(S)$ will denote the dual of $L^1(S)$, *i.e.*, space of equivalence classes of bounded $\mathbb{C}$-valued Borel functions on S modulo locally null sets. A mean m on $L^\infty(S)$ is called *topologically left invariant* if $m(\mu * f) = m(f)$ for every $\mu \in L^1_1(S)$, where $(\mu * f)(s) = \int \mu(ds')f(s's)$. A topologically left invariant mean is left invariant. The proof of the following proposition is essentially the same as the proof of the corresponding well known result for groups [5] and we omit it. We use the notation $g\mu$ for $\delta_g * \mu$.

PROPOSITION 2.1. *The following conditions are equivalent for an open subsemigroup S of a locally compact group:*

(a) S amenable,
(b) there exists a topological left invariant mean on $L^\infty(S)$,
(c) there exists a left invariant mean on $L^\infty(S)$,
(d) there exists a net μ_α in $L^1_1(S)$ such that for every $s \in S$ $s\mu_\alpha - \mu_\alpha \longrightarrow 0$ weakly,
(e) there exists a net μ_α in $L^1_1(S)$ such that for every $s \in S$ $\|s\mu_\alpha - \mu_\alpha\| \longrightarrow 0$,
(f) there exists a net μ_α in $L^1_1(S)$ such that for every $\mu \in L^1_1(S)$ $\mu * \mu_\alpha - \mu_\alpha \longrightarrow 0$ weakly,
(g) there exists a net μ_α in $L^1_1(S)$ such that for every $\mu \in L^1_1(S)$ $\|\mu * \mu_\alpha - \mu_\alpha\| \longrightarrow 0$.

REMARK 2.2. If m is a left invariant mean on $L^\infty(S)$ and $I \subseteq S$ is a Borel subset and a right ideal of S then $m(\chi_I) = 1$.

PROPOSITION 2.3 (REITER'S CONDITION). *The following conditions are equivalent for an open subsemigroup S of a locally compact group G:*

(a) S is amenable,

(b) *given* $\varepsilon > 0$ *and a finite set* $F \subseteq S$ *there exists* $\mu \in L^1_1(S)$ *such that* $\|g\mu - \mu\| < \varepsilon$ *for all* $g \in F$,

(c) *given* $\varepsilon > 0$ *and a compact* $K \subseteq S$ *there exists* $\mu \in L^1_1(S)$ *such that* $\|g\mu - \mu\| < \varepsilon$ *for all* $g \in K$.

PROOF. (c) $\Rightarrow$ (b) is trivial, (b) $\Rightarrow$ (a) follows trivially from Proposition 2.1. To obtain (a) $\Rightarrow$ (c) one needs to modify the corresponding proof for groups [5, Theorem 3.2.1]. Let U be a neighborhood of e in G such that $UK \subseteq S$ and $KU \subseteq S$. Note that KS is an open right ideal in S. By Proposition 2.1 there exists a net μ_α in $L^1_1(S)$ such that $\|\mu * \mu_\alpha - \mu_\alpha\| \longrightarrow 0$ for every $\mu \in L^1_1(S)$. By passing to a subnet we may assume that μ_α converges weakly$*$ to a topological left invariant mean on $L^\infty(S)$. Then by Remark 2.2. we may assume that $\mu_\alpha(KS) = 1$ for every α.

Let $\tilde{\mu}_\alpha$ denote the extension of μ_α to G. Choose $\rho \in L^1_1(G)$ with $\rho(U) = 1$ and let $V \subseteq U$ be a neighborhood of e in G such that $\|g\rho - \rho\| < \varepsilon/4$ for $g \in V$. By compactness of K there exist $k_1, \ldots, k_n \in K$ with $K \subseteq \bigcup_{i=1}^n k_i V$. Note that $(k\rho)(S) = 1$ for every $k \in K$ because $KU \subseteq S$. Therefore we can find α such that $\|k_i\rho * \tilde{\mu}_\alpha - \tilde{\mu}_\alpha\| < \varepsilon/4$ and $\|k_i\tilde{\mu}_\alpha - \tilde{\mu}_\alpha\| < \varepsilon/4$ for $i = 1, 2, \ldots, n$.

Set $\tilde{\mu} = \rho * \tilde{\mu}_\alpha$. As $\tilde{\mu}_\alpha(KS) = 1$, $\rho(U) = 1$, and $UKS \subseteq S$, we obtain $1 = \tilde{\mu}(UKS) = \tilde{\mu}(S)$. We shall write μ for the restriction of $\tilde{\mu}$ to S. Let us see that μ satisfies (c).

Let $s \in K$. Then $s = k_i v$ for some $i \in \{1, \ldots, n\}$ and $v \in V$. Hence,

$$\begin{aligned}
\|s\mu - \mu\| &= \|s\tilde{\mu} - \tilde{\mu}\| = \|k_i v\rho * \tilde{\mu}_\alpha - \rho * \tilde{\mu}_\alpha\| \\
&\le \|k_i v\rho * \tilde{\mu}_\alpha - k_i\rho * \tilde{\mu}_\alpha\| + \|k_i\rho * \tilde{\mu}_\alpha - \tilde{\mu}_\alpha\| + \|\tilde{\mu}_\alpha - \rho * \tilde{\mu}_\alpha\| \\
&< \|v\rho - \rho\| + \frac{\varepsilon}{4} + \|k_1\tilde{\mu}_\alpha - k_1\rho * \tilde{\mu}_\alpha\| \\
&< \frac{\varepsilon}{2} + \|k_1\tilde{\mu}_\alpha - \tilde{\mu}_\alpha\| + \|\tilde{\mu}_\alpha - k_1\rho * \tilde{\mu}_\alpha\| < \varepsilon.
\end{aligned}$$

■

THEOREM 2.4. *Let* S *be an open* σ*-compact subsemigroup of a locally compact group* G*. It follows that* S *is amenable if and only if there exists a probability measure* $\mu \in L^1_1(S)$ *such that* $\lim_{n\to\infty} \|g\mu^n - \mu^n\| = 0$ *for every* $g \in S$.

PROOF. In view of Proposition 2.1 it remains to prove the only if part. This can be achieved by slightly modifying the construction of Kaimanovich and Vershik [12, Theorem 4.3]. Before proceeding we note that condition (c) of Proposition 2.3 can be easily seen to be equivalent to the condition that given $\varepsilon > 0$ and compact K, $K_0 \subseteq S$ there exists $\mu \in L^1_1(S)$ such that $\|s\mu - \mu\| < \varepsilon$ for $s \in K$, and the support of μ is compact and contains K_0.

We start by choosing a decreasing sequence $\{\varepsilon_n\}_{n=1}^\infty \subseteq (0, \infty)$, a sequence $\{t_n\}_{n=1}^\infty \subseteq (0, \infty)$ with $\sum_{i=1}^\infty t_i = 1$, and an increasing sequence of positive integers $\{N_n\}_{n=1}^\infty$ such that $(\sum_{i=1}^n t_i)^{N_n} < \varepsilon_n$ for every n. Since S is σ-compact there is also an increasing sequence $\{K_n\}_{n=1}^\infty$ of compact subsets of S with $\bigcup_{i=1}^\infty K_n = S$. Then using amenability of S and Proposition 2.3 we can inductively find a sequence $\{\nu_n\}_{n=1}^\infty \subseteq L^1_1(S)$ with the following properties:

https://doi.org/10.4153/CJM-1994-071-2 Published online by Cambridge University Press

(1) each ν_n has compact support,

(2) $K_1 \subseteq \operatorname{supp}\nu_1$, $K_{n+1} \cup \operatorname{supp}\nu_n \subseteq \operatorname{supp}\nu_{n+1}$,

(3) $\|g\nu_{n+1} - \nu_{n+1}\| < \varepsilon_{n+1}$ for all $g \in \bigcup_{i=1}^{N_{n+1}} (\operatorname{supp}\nu_n)^i$.

We set $\mu = \sum_{i=1}^{\infty} t_i\nu_i$ and $\mu_n = \sum_{i=1}^{n} t_i\nu_i$.

Let $g \in S$. Then there exists n_g such that $g \in \operatorname{supp}\nu_n$ for all $n \geq n_g$. We will show that for such n, $\|g\mu^{N_{n+1}} - \mu^{N_{n+1}}\| < 6\varepsilon_{n+1}$. Since the sequence $\|g\mu^n - \mu^n\|$ is nonincreasing this will prove that $\lim_{n\to\infty} \|g\mu^n - \mu^n\| = 0$.

Fix $n \geq n_g$ and find M with $\|\mu^{N_{n+1}} - \mu_M^{N_{n+1}}\| < \varepsilon_{n+1}$. Note that

$$\|\mu_{M'}^{N_{n+1}}\| = \left\|\Big(\sum_{i=1}^{M'} t_i\mu^i\Big)^{N_{n+1}}\right\| \leq \Big(\sum_{i=1}^{n+1} t_i\Big)^{N_{n+1}} < \varepsilon_{n+1} \tag{2.1}$$

whenever $M' \leq n+1$. Consider two cases: $M \leq n+1$ and $M > n+1$. In the first case,

$$\|g\mu^{N_{n+1}} - \mu^{N_{n+1}}\| \leq \|g\mu^{N_{n+1}} - g\mu_M^{N_{n+1}}\| + \|g\mu_M^{N_{n+1}} - \mu_M^{N_{n+1}}\| + \|\mu_M^{N_{n+1}} - \mu^{N_{n+1}}\| < 4\varepsilon_{n+1}. \tag{2.2}$$

In the second case we write

$$\mu_M^{N_{n+1}} = \mu_{n+1}^{N_{n+1}} + \sum{}' t_{i_1}\cdots t_{i_{N_{n+1}}} \nu_{i_1} * \cdots * \nu_{i_{N_{n+1}}} \tag{2.3}$$

where the sum $\sum'$ is over all N_{n+1}-tuples $(i_1, \ldots, i_{N_{n+1}})$ with $M \geq \max i_j > n+1$. Consider a convolution $\nu_{i_1} * \cdots * \nu_{i_{N_{n+1}}}$ with $\max i_j > n+1$. Let $1 \leq j \leq N_{n+1}$ be the smallest integer with $i_j > n+1$. When $j > 1$, note that

$$\operatorname{supp}(\nu_{i_1} * \cdots * \nu_{i_{j-1}}) = (\operatorname{supp}\nu_{i_1})\cdots(\operatorname{supp}\nu_{i_{j-1}}) \subseteq (\operatorname{supp}\nu_{i_j-1})^{j-1}, \tag{2.4}$$

and

$$\operatorname{supp}(g\nu_{i_1} * \cdots * \nu_{i_{j-1}}) = g(\operatorname{supp}\nu_{i_1})\cdots(\operatorname{supp}\nu_{i_{j-1}}) \subseteq (\operatorname{supp}\nu_{i_j-1})^{j}. \tag{2.5}$$

Hence, using property (3) of the sequence $\{\nu_n\}_{n=1}^{\infty}$ we obtain

$$\begin{aligned} &\|g\nu_{i_1} * \cdots * \nu_{i_{N_{n+1}}} - \nu_{i_1} * \cdots * \nu_{i_{N_{n+1}}}\| \\ &\quad\leq \|(g\nu_{i_1} * \cdots * \nu_{i_{j-1}}) * \nu_{i_j} - (\nu_{i_1} * \cdots * \nu_{i_{j-1}}) * \nu_{i_j}\| \\ &\quad\leq \|(g\nu_{i_1} * \cdots * \nu_{i_{j-1}}) * \nu_{i_j} - \nu_{i_j}\| + \|\nu_{i_j} - (\nu_{i_1} * \cdots * \nu_{i_{j-1}}) * \nu_{i_j}\| \\ &\quad< 2\varepsilon_{i_j} < 2\varepsilon_{n+1}. \end{aligned} \tag{2.6}$$

When $j = 1$, a similar argument shows that inequality (2.6) remains in force.

Now, (2.3), (2.1), and (2.6) produce

$$\begin{aligned} \|g\mu_M^{N_{n+1}} - \mu_M^{N_{n+1}}\| &\leq \|g\mu_{n+1}^{N_{n+1}} - \mu_{n+1}^{N_{n+1}}\| \\ &\quad+ \sum{}' t_{i_1}\cdots t_{i_{N_{n+1}}} \|g\nu_{i_1} * \cdots * \nu_{i_{N_{n+1}}} - \nu_{i_1} * \cdots * \nu_{i_{N_{n+1}}}\| < 4\varepsilon_{n+1}. \end{aligned}$$

Consequently,

$$\|g\mu^{N_{n+1}} - \mu^{N_{n+1}}\| \leq \|g\mu^{N_{n+1}} - g\mu_M^{N_{n+1}}\| + \|g\mu_M^{N_{n+1}} - \mu_M^{N_{n+1}}\| + \|\mu_M^{N_{n+1}} - \mu^{N_{n+1}}\| < 6\varepsilon_{n+1},$$

as claimed. ∎

COROLLARY 2.5. *If S is an open amenable σ-compact subsemigroup of a connected locally compact group G, then S supports an absolutely continuous Choquet-Deny measure.*

3. **Exponential boundedness and amenability of open subsemigroups.** Recall that a (real) Lie algebra $\mathfrak{g}$ is said to be of type R if for every $X \in \mathfrak{g}$ the eigenvalues of $\operatorname{ad} X$ are imaginary. As shown by Jenkins [11], a connected Lie group G is exponentially bounded if and only if its Lie algebra is of type R. We shall say that a Lie algebra $\mathfrak{g}$ is of type R_0 if for every X in the radical $\operatorname{rad}(\mathfrak{g})$ the eigenvalues of $\operatorname{ad} X$ are imaginary. The following lemma is implicit in an argument of Azencott [1, pp. 106–111] and is explicitly stated and proven in [8, Lemma 3.14].

LEMMA 3.1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ not of type R_0. Then there exists a G-space $\mathcal{X}$ such that:*

(1) $\mathcal{X}$ is a finite dimensional vector space,
(2) the function $G \times \mathcal{X} \ni (g, x) \longrightarrow gx$ is (real) analytic,
(3) for every compact $K \subseteq \mathcal{X}$ and every neighbourhood U of 0 in $\mathcal{X}$ there exists $g \in G$ with $gK \subseteq U$,
(4) there exists $g \in G$ and $v \in \mathcal{X} - \{0\}$ such that $gx = x + v$ for all $x \in \mathcal{X}$.

THEOREM 3.2. *An open subsemigroup of an exponentially bounded locally compact group is amenable. A connected locally compact group is exponentially bounded if and only if every open subsemigroup is amenable.*

PROOF. Recall that an exponentially bounded group is amenable [13, Proposition 6.8]. Hence, by Theorem 1.1, if S were an open nonamenable subsemigroup in G, S would contain two disjoint right ideals I, J. It is easy to see that if $a \in I$, $b \in J$, then a and b are free generators of a uniformly discrete subsemigroup T, *i.e.*, there exists a neighbourhood U of e such that $tU \cap sU = \emptyset$ whenever $t, s \in T$ and $t \neq s$. But then $V^n \supseteq \{a, b\}^{n-1}(V \cap U)$ for every compact neighbourhood V of e containing a and b. Consequently, $\limsup_{n\to\infty} \lambda(V^n)^{1/n} \geq 2$, *i.e.*, G has exponential growth, a contradiction. (This argument is due to Jenkins [10, first part of the proof of Theorem 3].)

It remains to prove that for a connected G, the fact that every open subsemigroup is amenable implies exponential boundedness. It is clear that in this case G is itself amenable. We will again argue by contradiction. Suppose that G has exponential growth. Let K be a compact normal subgroup such that $\tilde{G} = G/K$ is a Lie group. Then $\tilde{G}$ is amenable and has exponential growth [6, Théorème 1.4]. If $\tilde{S} \subseteq \tilde{G}$ is an open semigroup then $\pi^{-1}(\tilde{S})$, where $\pi: G \longrightarrow \tilde{G}$ is the canonical homomorphism, is an open, hence, amenable semigroup in G. It is clear that this implies that $\tilde{S}$ is amenable.

Since $\tilde{G}$ is an amenable Lie group, $\tilde{G}/\operatorname{rad}(\tilde{G})$ is compact [13, Theorem 3.8]. Hence, $\operatorname{rad}(\tilde{G})$ has exponential growth [6, Théorème 1.4]. So the Lie algebra of $\operatorname{rad}(\tilde{G})$ is not of type R. But then the Lie algebra of $\tilde{G}$ cannot be of type R_0. Let $\mathcal{X}$ be the vector space described in Lemma 3.1, and let $\|\cdot\|$ denote any norm on $\mathcal{X}$. Set $U = \{x \in \mathcal{X} ; \|x\| < 1\}$, $M = \{x \in \mathcal{X} ; \|x\| \leq 1\}$, and $\tilde{S} = \{g \in \tilde{G} ; gM \subseteq U\}$. By Lemma 3.1, $\tilde{S}$ is a (nonempty) open semigroup in $\tilde{G}$. Let μ be any regular probability measure on G with $\mu(\tilde{S}) = 1$.

Let δ_0 be the point measure concentrated in $0 \in \mathcal{X}$, and set $\rho_n = \frac{1}{n}\sum_{i=1}^{n} \mu^i * \delta_0$. It is clear that $\rho_n(M) = 1$ for all n. Thus using Prohorov's theorem [7, Theorem 1.1.11], $\{\rho_n\}_{n=1}^{\infty}$ is a weakly relatively compact set of probability measures. If ρ is its cluster point then $\rho(M) = 1$ and $\mu * \rho = \rho$. Let $f: \mathcal{X} \longrightarrow [0,1]$ be a continuous function of compact support such that $f(M) = \{1\}$. Then $h(g) = \int f(gx)\rho(dx)$ is a bounded continuous μ-harmonic function. Using (4) of Lemma 3.1 and the fact that ρ has compact support we can see that h is not constant.

We obtained that $\tilde{S}$ does not support any Choquet-Deny measure. As $\tilde{S}$ is amenable, this contradicts Corollary 2.5. ∎

4. **The translate property.** We can now prove that for amenable connected G exponential boundedness is equivalent to the translate property.

THEOREM 4.1. *An exponentially bounded locally compact group has the translate property. If an amenable locally compact group has the translate property then every open subsemigroup is amenable. A connected amenable locally compact group has the translate property if and only if it is exponentially bounded.*

PROOF. Clearly, the last statement follows trivially from the first two and Theorem 3.2. The proof of the first two statements is essentially the argument of Jenkins [10, proof of Theorem 4].

Assume that G is exponentially bounded. Let $\nu \in L^1(G)$ be a signed measure of compact support, and $A \subseteq G$ a Borel set such that for every $g \in G$, $\nu(Ag) \geq 0$ and that A is not locally null. We need to show that $\nu(G) \geq 0$. Choose a compact symmetric neighbourhood U of e with $\lambda(A \cap U) \neq 0$ and $\operatorname{supp}\nu \subseteq U$. Let $\alpha_n = \lambda(A \cap U^n) \leq \lambda(U^n)$. Since G is exponentially bounded, $\lim_{n\to\infty} \alpha_n^{1/n} = 1$. By an elementary result on convergent sequences there exists a subsequence α_{n_k} with $\lim_{k\to\infty} \alpha_{n_k+2}/\alpha_{n_k} = 1$. Now, by the Fubini theorem

$$\int_G \nu(dx)\lambda(A \cap xU^n) = \int_{U^n} \lambda(dg)\nu(Ag^{-1}) \geq 0.$$

But if $x \in U = U^{-1} \supseteq \operatorname{supp}\nu$, then $U^n \subseteq xU^{n+1} \subseteq U^{n+2}$. Therefore $\alpha_n/\alpha_{n+2} \leq \alpha_n/\alpha_{n+1} \leq \lambda(A \cap xU^{n+1})/\alpha_{n+1} \leq \alpha_{n+2}/\alpha_{n+1} \leq \alpha_{n+2}/\alpha_n$. Hence, $\lim_{k\to\infty} \lambda(A \cap xU^{n_k+1})/\alpha_{n_k+1} = 1$ uniformly in $x \in U$. So

$$\nu(G) = \nu(U) = \lim_{k\to\infty} \alpha_{n_k+1}^{-1} \int_U \nu(dx)\lambda(A \cap xU^{n_k+1}) \geq 0.$$

To prove the second statement we argue by contradiction. Suppose G contains an open nonamenable subsemigroup S. By Theorem 1.1 S contains two open disjoint ideals I and J. Let $a \in I$, $b \in J$, and U be a compact symmetric neighbourhood of e such that $UaU \subseteq I$ and $UbU \subseteq J$. Let $d\mu_1 = \chi_{aU}\, d\lambda$, $d\mu_2 = \chi_{abU}\, d\lambda$, $d\mu_3 = \chi_{a^2U}\, d\lambda$, and $\mu = \hat{\mu}_1 - \hat{\mu}_2 - \hat{\mu}_3$, where $\hat{\mu}_i(B) = \mu_i(B^{-1})$. Then $\mu(G) = -\lambda(U) < 0$. We will show that $\mu(Sg) \geq 0$ for all $g \in G$, thus contradicting the translate property. Clearly,

$$\mu(Sg) = \lambda\big(aU \cap (Sg)^{-1}\big) - \lambda\big(abU \cap (Sg)^{-1}\big) - \lambda\big(a^2U \cap (Sg)^{-1}\big),$$

and one can see that if $\mu(Sg) \neq 0$ then $g^{-1} \in aUS \cup abUS \cup a^2US$. If $g^{-1} \notin abU \cup a^2U$ then obviously $\mu(Sg) \geq 0$. It remains to consider the cases that $g^{-1} \in abUS$ and $g^{-1} \in a^2US$.

Since these cases can be treated in the same way, we shall consider only the first one. If $g^{-1} \in abUS$ then $gaU \subseteq (UbUS)^{-1} \subseteq S^{-1}$. Therefore $\lambda\big(aU \cap (Sg)^{-1}\big) = \lambda(gaU \cap S^{-1}) = \lambda(U)$ and $\lambda\big(abU \cap (Sg)^{-1}\big) \leq \lambda(U)$. Moreover, $ga^2U \subseteq S^{-1}U^{-1}b^{-1}aU \subseteq S^{-1}U^{-1}b^{-1}U^{-1}UaU = (UbUS)^{-1}UaU \subseteq J^{-1}I$. Consequently, $ga^2U \cap S^{-1} \subseteq J^{-1}I \cap S^{-1} = 0$ and $\lambda\big(a^2U \cap (Sg)^{-1}\big) = 0$. So $\mu(Sg) \geq 0$. ∎

REMARK 4.2. The second part of the proof shows that an open subsemigroup S of a locally compact group G is amenable whenever for every signed measure $\nu \in L^1(G)$ of compact support, $\nu(Sg) \geq 0$ for all $g \in G$ implies $\nu(G) \geq 0$. Thus in the last statement of the theorem one can substitute for the translate property the weakened translate property with respect to open subsemigroups of G. On the other hand, it is straightforward to adapt the first part of the proof to demonstrate a stronger version of the translate property as considered in [10]. Namely, if G is locally compact and exponentially bounded then for every signed measure $\nu \in L^1(G)$ of compact support and every nonnegative nonzero $f \in L^\infty(G)$ the condition $\nu * f \geq 0$ implies $\nu(G) \geq 0$. However, for connected G the variations of the translate property are all equivalent.

5. Generating an exponentially bounded group.

When A is a nonempty subset of a group G we shall denote by $\mathrm{gp}(A)$ the subgroup generated by A by $\mathrm{ngp}(A)$ the smallest normal subgroup N of $\mathrm{gp}(A)$ such that A is contained in a coset xN. The following lemma is a version of a lemma obtained in [3, p. 4].

LEMMA 5.1. *Let A be a nonempty subset of a group G. Consider the group $G \times \mathbb{Z}$. It follows that*

$$\mathrm{gp}(A \times \{1\}) = \bigcup_{n \in \mathbf{Z}} \Big[\big(x^n \,\mathrm{ngp}(A)\big) \times \{n\}\Big].$$

where $x \in G$ is such that $A \subseteq x\,\mathrm{ngp}(A)$.

PROOF. Let ψ denote the restriction to $\mathrm{gp}(A \times \{1\})$ of the canonical projection $G \times \mathbb{Z} \ni (g,n) \longrightarrow n \in \mathbb{Z}$ and let a be an element of A. Then

$$\mathrm{gp}(A \times \{1\}) = \bigcup_{n \in \mathbf{Z}} \psi^{-1}(\{n\}) = \bigcup_{n \in \mathbf{Z}} (a^n, n)\psi^{-1}(\{0\}).$$

But $\psi^{-1}(\{0\}) = (G \times \{0\}) \cap \mathrm{gp}(A \times \{1\}) = N \times \{0\}$ where N is a subgroup of $\mathrm{gp}(A)$. Thus $\mathrm{gp}(A \times \{1\}) = \bigcup_{n \in \mathbf{Z}}\big((a^n N) \times \{n\}\big)$.

Note that $a\,\mathrm{ngp}(A) = x\,\mathrm{ngp}(A)$. Since $\bigcup_{n \in \mathbf{Z}}\Big[\big(a^n \,\mathrm{ngp}(A)\big) \times \{n\}\Big]$ is a group containing $A \times \{1\}$, it suffices to show that $N \supseteq \mathrm{ngp}(A)$. As $A \subseteq aN$ this will follow if we show that N is normal in $\mathrm{gp}(A)$.

Let $H = \{g \in \mathrm{gp}(A) \ ; \ gN = Ng\}$ be the normalizer of N in $\mathrm{gp}(A)$. Let $g \in A$. Then $gN \times \{1\} = \psi^{-1}(\{1\})$, $Ng^{-1} \times \{-1\} = \psi^{-1}(\{-1\})$. Hence, $gNg^{-1} \times \{0\} = \psi^{-1}(\{1\})\psi^{-1}(\{-1\}) = \psi^{-1}(\{0\}) = N \times \{0\}$. Thus $A \subseteq H$. By the definition of H we then have $H = \mathrm{gp}(A)$ and thus N is indeed normal in $\mathrm{gp}(A)$. ∎

REMARK 5.2. If $A \subseteq x\,\mathrm{ngp}(A)$ then $\mathrm{gp}(A) = \bigcup_{n \in \mathbf{Z}}\big(x^n \,\mathrm{ngp}(A)\big)$.

The next lemma is a modification of a lemma proven in [1, p. 97].

LEMMA 5.3. *Let G_1 and G_2 be groups, $p\colon G_1 \longrightarrow G_2$ a group homomorphism, and $S \subseteq G_1$ a semigroup. Assume that*

(i) $S^{-1}S \cap \ker p \subseteq SS^{-1} \cap \ker p$,

(ii) $p(S^{-1}S) \subseteq p(SS^{-1})$.

Then $S^{-1}S \subseteq SS^{-1}$ and SS^{-1} is a group.

PROOF. Let $s_1, s_2 \in S$. Then by (ii) there exist $s_3, s_4 \in S$ and $g \in \ker p$ such that $s_1^{-1}s_2 = s_3s_4^{-1}g$, or $s_3^{-1}s_1^{-1}s_2s_4 = s_4^{-1}gs_4 \in \ker p$. Using (i) we have $s_3^{-1}s_1^{-1}s_2s_4 = s_5s_6^{-1}$ for some $s_5, s_6 \in S$. Hence, $s_1^{-1}s_2 \in SS^{-1}$. So $S^{-1}S \subseteq SS^{-1}$ and this obviously implies that SS^{-1} is a group. ∎

LEMMA 5.4. *Let A be a nonempty subset of a group G. Set $S = \bigcup_{n=1}^{\infty} A^n$ and $\hat{S} = \bigcup_{n=1}^{\infty}(A^n \times \{n\}) \subseteq G \times \mathbb{Z}$. If SS^{-1} is a group then $\hat{S}\hat{S}^{-1}$ is also a group.*

PROOF. A straightforward application of Lemma 5.3. ∎

LEMMA 5.5. *Let A be a nonempty subset of a group G and $S = \bigcup_{n=1}^{\infty} A^n$. It follows that* $\operatorname{ngp}(A) = \bigcup_{n=1}^{\infty} A^nA^{-n}$ *if and only if* $\operatorname{gp}(A) = SS^{-1}$.

PROOF. Suppose that $N = \operatorname{ngp}(A) = \bigcup_{n=1}^{\infty} A^nA^{-n}$ and let x be any element of A. By Remark 5.2 $\operatorname{gp}(A) = \bigcup_{n\in\mathbf{Z}} x^nN$. Thus $\operatorname{gp}(A) = N \cup \bigcup_{n=1}^{\infty}(x^nN \cup Nx^{-n}) = \bigcup_{n=1}^{\infty} A^nA^{-n} \cup \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}(x^nA^mA^{-m} \cup A^mA^{-m}x^{-n}) \subseteq SS^{-1}$. Consequently, $\operatorname{gp}(A) = SS^{-1}$.

Conversely, suppose that $\operatorname{gp}(A) = SS^{-1}$. By Lemma 5.1, if $N = \operatorname{ngp}(A)$ and $A \subseteq xN$, then $\operatorname{gp}(A \times \{1\}) = \bigcup_{n\in\mathbf{Z}}(x^nN \times \{n\})$. By Lemma 5.4 $\operatorname{gp}(A \times \{1\}) = \hat{S}\hat{S}^{-1}$. Thus if $g \in N$, then $(g, 0) \in \hat{S}\hat{S}^{-1}$. Since $\hat{S} = \bigcup_{n=1}^{\infty}(A^n \times \{n\})$ this implies that $g \in A^nA^{-n}$ for some $n = 1, 2, \ldots$. Thus $N \subseteq \bigcup_{n=1}^{\infty} A^nA^{-n}$. On the other hand, it is clear that $\bigcup_{n=1}^{\infty} A^nA^{-n} \subseteq N$. ∎

THEOREM 5.6. *Let U be a nonempty open subset of an exponentially bounded locally compact group and let $S = \bigcup_{n=1}^{\infty} U^n$. It follows that* $\operatorname{gp}(U) = SS^{-1}$ *and* $\operatorname{ngp}(U) = \bigcup_{n=1}^{\infty} U^nU^{-n}$.

PROOF. From Theorem 3.2 S is an open amenable semigroup. From Theorem 1.1 SS^{-1} is a group; clearly, $SS^{-1} = \operatorname{gp}(U)$. The second equality results from Lemma 5.5. ∎

THEOREM 5.7. *The following conditions are equivalent for a connected amenable locally compact group G:*

(a) G is exponentially bounded,

(b) for every nonempty open $U \subseteq G$, $G = (\bigcup_{n=1}^{\infty} U^n)(\bigcup_{n=1}^{\infty} U^n)^{-1}$,

(c) for every nonempty open $U \subseteq G$, $G = \bigcup_{n=1}^{\infty} U^nU^{-n}$.

PROOF. Since G is connected, for every open U we have $\operatorname{gp}(U) = G = \operatorname{ngp}(U)$. Thus (b) $\Leftrightarrow$ (c) by Lemma 5.5 and (a) $\Rightarrow$ (b) is contained in Theorem 5.6. Finally, when (b) is true and $S \subseteq G$ is an open semigroup, then $SS^{-1} = G$. Hence, S is amenable by Theorem 1.1. Theorem 3.2 then shows that G is exponentially bounded. ∎

LEMMA 5.8. *Let A be a Borel subset of a locally compact group. If A is not locally null then A^2 has nonempty interior.*

PROOF. It suffices to consider the case that A is compact and $\lambda(A) \neq 0$. But then the function $G \ni g \longrightarrow f(g) = \lambda(gA^{-1} \cap A)$ is continuous. Furthermore, by the Fubini theorem $\int \lambda(dg) f(g) = \lambda(A)^2 \neq 0$. The nonempty open set $\{g \in G \,;\, f(g) \neq 0\}$ is contained in A^2. ■

COROLLARY 5.9. *Let G be a connected exponentially bounded locally compact group and let $A \subseteq G$ be a Borel subset. If A is not locally null then $G = (\bigcup_{n=1}^{\infty} A^n)(\bigcup_{n=1}^{\infty} A^n)^{-1} = \bigcup_{n=1}^{\infty} A^n A^{-n}$.*

6. The strong Følner condition.

THEOREM 6.1. *A connected locally compact group is exponentially bounded if and only if every compact set $K \subseteq G$ with $\lambda(K) \neq 0$ satisfies the strong Følner condition.*

PROOF. $\Rightarrow$. By Corollary 5.9 $G = \bigcup_{n=1}^{\infty} K^n K^{-n}$. Consequently, $K^{n+1} \cap K^n \neq \emptyset$ for some n. Note that if $x \in K^{n+1} \cap K^n$ then for every $r = 1, 2, \ldots$, $\lambda(K^{r+n+1} \cap K^{r+n}) \geq \lambda(xK^r) = \lambda(K^r)$. Hence,

$$
\begin{aligned}
\lambda(K^{r+n+1} \mathbin{\triangle} K^{r+n}) &= \lambda(K^{r+n+1}) + \lambda(K^{r+n}) - 2\lambda(K^{r+n+1} \cap K^{r+n}) \\
&\leq 2\lambda(K^{r+n+1}) - 2\lambda(K^r) = 2\lambda(K^r)\Big(\frac{\lambda(K^{r+n+1})}{\lambda(K^r)} - 1\Big).
\end{aligned} \tag{6.1}
$$

But exponential boundedness implies that $\lim_{r\to\infty} \lambda(K^r)^{1/r} = 1$. Then by an elementary result $\liminf_{r\to\infty} \lambda(K^{r+n+1})/\lambda(K^r) = 1$. This together with (6.1) implies the strong Følner condition.

$\Leftarrow$. Suppose that G is not exponentially bounded. Then by Theorem 3.2 it contains an open nonamenable semigroup. But the strong Følner condition implies the classical Følner condition; hence, G is amenable. We now use an argument of Paterson [13, p. 245]. By Theorem 1.1, S contains two open disjoint right ideals I and J. Let $A \subseteq I$, $B \subseteq J$ be compact subsets with nonempty interiors. Set $K = A \cup B$. Write $K^n = A_n \cup B_n$, $A_n = K^n \cap I$, $B_n = K^n \cap J$. Then $A^n \subseteq A_n$ and $B^n \subseteq B_n$. Furthermore, if $x \in A$ then $xA_n \cup xB_n \subseteq A_{n+1}$. So $\lambda(A_{n+1}) \geq \lambda(A_n) + \lambda(B_n)$. Similarly, $\lambda(B_{n+1}) \geq \lambda(A_n) + \lambda(B_n)$. Therefore $\lambda(K^{n+1}) \geq 2\lambda(K^n)$, and, hence, $\lambda(K^{n+1} \mathbin{\triangle} K^n) \geq \lambda(K^{n+1}) - \lambda(K^n) \geq \lambda(K^n)$. This holds for every n and contradicts the strong Følner condition. ■

REMARK 6.2. For a not necessary connected G the second part of the proof shows that the strong Følner condition for compact sets with nonempty interiors implies that every open subsemigroup is amenable, *i.e.*, we obtain the proof of Theorem 1.5.

ACKNOWLEDGEMENT. The author wishes to thank David Handelman for his support at the University of Ottawa.

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https://doi.org/10.4153/CJM-1994-071-2 Published online by Cambridge University Press