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CLASSIFICATION OF 4- AND 5-ARC-TRANSITIVE CUBIC GRAPHS OF SMALL GIRTH

MARGARET J. MORTON

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Abstract

This paper classifies all finite connected 4- and 5-arc-transitive cubic graphs that contain circuits of length less than or equal to 11, or of length 13, and some of those graphs with circuits of length 12.

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1. Introduction

Let Γ be a finite undirected graph, with no loops or multiple edges. An s-arc in Γ is a sequence $(v_0, v_1, \dots, v_s)$ of vertices of Γ such that $\{v_{i-1}, v_i\}$ is an edge of Γ for $1 \le i \le s$, and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. The graph Γ is said to be s-arc-transitive if its automorphism group G acts transitively on the set of all s-arcs of Γ.

A group G of automorphisms is said to act symmetrically on Γ if it acts transitively on the vertices of Γ, and the stabilizer in G of each vertex v acts transitively on the vertices adjacent to v. In this case G acts transitively on the ordered edges of Γ, and Γ is called a symmetric graph. In particular, a 1-arc-transitive graph is precisely a symmetric graph.

By a result of Tutte [9, 10], a finite symmetric graph of valency 3 can be at most 5-arc-transitive. For cubic graphs, if Γ is s-arc transitive and not $(s+1)$-arc transitive then G acts regularly on the s-arcs. Djokovic and

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Miller [7] showed there are seven types of symmetric graph of valency 3. These were later described in a unified way by Conder and Lorimer [6] in terms of generators and relations for their automorphism groups.

In the case where G is regular on the 4-arcs of Γ, the group G has to be a homomorphic image of one of the two groups, G_4^1 and G_4^2, where

(i) G_4^1 is generated by three elements h, a, p, which satisfy $h^3 = a^2 = p^2 = 1$; the other relations to be satisfied are $a^{-1}pa = p$, $h^{-1}ph = q$, $a^{-1}qa = r$, $r^{-1}hr = h^{-1}$, $h^{-1}qh = pq$, $pq = qp$, $pr = rp$, $rq = pqr$;

(ii) G_4^2 is generated by three elements h, a, p, which satisfy $h^3 = p^2 = 1$ and $a^2 = p$; the other relations are the same as for G_4^1 except that $a^{-1}pa = p$ is removed since it becomes redundant.

In the case where G is regular on the 5-arcs of Γ, the group G has to be a homomorphic image of the group G_5, where G_5 is generated by the elements h, a, p, which satisfy $h^3 = a^2 = p^2 = 1$. The other relations to be satisfied are $h^{-1}ph = p$, $a^{-1}pa = q$, $h^{-1}qh = r$, $a^{-1}ra = s$, $s^{-1}hs = h^{-1}$, $h^{-1}rh = pqr$, $pq = qp$, $pr = rp$, $qr = rq$, $ps = sp$, $qs = sq$, $sr = pqrs$.

The object of this paper is to classify 4- and 5-arc-transitive cubic graphs of small girth, using the information given above. This then results in the following theorem.

THEOREM. *All finite 4- and 5-arc-transitive cubic graphs with girth less than or equal to 13 (except for the graphs corresponding to the relator $(ha)^{12}$) are isomorphic to either Heawood's graph, Tutte's 8-cage or a triple cover of Tutte's 8-cage, the sextet graph $S(17)$ or a double cover of $S(17)$, Wong's graph or a double cover of Wong's graph, the sextet graph $S(5)$ or the sextet graph $S(79)$.*

2. Examples of 4-arc-transitive cubic graphs

Some general families of 4-arc-transitive cubic graphs are as follows.

(1) The family of sextet graphs [1]. For each odd prime p there is a sextet graph $S(p)$, constructible using linear fractional transformations of the projective line over $GF(q)$, where $q = p$ or p^2. The order n of $S(p)$ depends on the congruence class of p modulo 16, as follows:

$$n = \tfrac{1}{48}p(p^2 - 1) \quad \text{when } p \equiv 1 \text{ or } 15 \bmod 16;$$

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$$n = \tfrac{1}{24}p(p^2 - 1) \quad \text{when } p \equiv 7 \text{ or } 9 \bmod 16;$$
$$n = \tfrac{1}{24}p^2(p^4 - 1) \quad \text{when } p \equiv 3, 5, 11 \text{ or } 13 \bmod 16.$$

The full automorphism group of $S(p)$ is

$$PSL(2, p), \quad PGL(2, p), \quad P\Gamma L(2, p^2)$$

in the respective cases.

(2) A family of 4-arc-transitive graphs each with girth 12 [4]. If p is any prime, and θ is that automorphism of the group $SL(3, p)$ which takes each matrix to the transpose of its inverse, then there exists a connected trivalent graph $\Gamma(p)$ on $\frac{1}{12}p^3(p^3-1)(p^2-1)$ vertices with the split extension $SL(3, p) \cdot \langle\theta\rangle$ as a group of automorphisms acting regularly on its 4-arcs. If $p \neq 3$ then this group is the full automorphism group of $\Gamma(p)$, while the graph $\Gamma(3)$ is 5-arc-transitive with full automorphism group $SL(3, 3) \cdot \langle\theta\rangle \times C_2$. The girth of $\Gamma(p)$ is 12, except in the case $p = 2$ where the girth is 6.

(3) A family of 5-arc-transitive cubic graphs shown to exist by Conder in [5]. For all but finitely many positive integers n, there is a finite connected 5-arc-transitive cubic graph with the alternating group A_n as its full automorphism group, and another with the symmetric group S_n as its full automorphism group.

Amongst those of small girth are the following:

(i) Heawood's graph (also known as the sextet graph $S(7)$). This graph has 14 vertices, and its full automorphism group is $PGL(2, 7)$, obtainable as a homomorphic image of the group G_4^1 via the linear fractional transformations

$$h: z \to \frac{z-1}{z}, \qquad p: z \to \frac{5}{z},$$
$$q: z \to \frac{z+2}{z-1}, \qquad r: z \to \frac{2-z}{z+1}, \qquad a: z \to -z$$

of the projective line over $GF(7)$.

(ii) Tutte's 8-cage (also known as the sextet graph $S(3)$) [9, 10]. This graph has 30 vertices and is the smallest 5-arc-transitive cubic graph. Its full automorphism group is the group $P\Gamma L(2, 9)$, of order 1440. Subgroups of order 720 are obtainable as homomorphic images of the groups G_4^1 and G_4^2 respectively as follows:

(a) via the linear fractional transformations of the projective line over $GF(9)$

$$h: z \to \frac{t^2(z-1)}{z+1}, \qquad p: z \to -z,$$

$$q: z \to \frac{-1}{z}, \qquad r: z \to \frac{t^2}{z}, \qquad a_1: z \to \frac{t^3}{z},$$

where t is a root of the polynomial $t^2 + t - 1$ over $\mathbb{Z}_3$.

(b) via the linear fractional transformations h, p, q, r as in (a), but with a_1 replaced by

$$a_2: z \to t^3 z^3.$$

(iii) A triple cover of Tutte's 8-cage, on 90 vertices. The corresponding full automorphism group has order 4320, and is a non-split extension of C_3 by $P\Gamma L(2, 9)$. Subgroups of order 2160 are obtainable as homomorphic images of the groups G_4^1 and G_4^2 respectively using permutations of degree 36 as follows:

(a)

$$\begin{aligned} h = {} & (2, 3, 8)(4, 5, 6)(7, 20, 19)(9, 30, 10)(11, 31, 12)(13, 35, 14) \\ & (15, 32, 16)(17, 21, 18)(22, 34, 23)(24, 33, 25)(26, 29, 27), \\ p = {} & (1, 6)(2, 7)(3, 20)(4, 5)(12, 32)(13, 35)(14, 36)(15, 31) \\ & (23, 26)(24, 25)(27, 34)(28, 33), \\ q = {} & (1, 4)(3, 20)(5, 6)(8, 19)(11, 16)(12, 32)(13, 36)(14, 35) \\ & (22, 29)(23, 26)(24, 33)(25, 28), \\ r = {} & (2, 3)(4, 5)(7, 20)(9, 18)(10, 17)(12, 31)(13, 35)(15, 32) \\ & (21, 30)(23, 34)(24, 25)(26, 27), \\ a_1 = {} & (1, 2)(3, 4)(5, 20)(6, 7)(8, 9)(10, 11)(12, 13)(14, 15)(16, 17) \\ & (18, 19)(21, 22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 36) \\ & (32, 35)(33, 34). \end{aligned}$$

(b) h, p, q, r as in (a), but with a_1 replaced by

$$\begin{aligned} a_2 = {} & (1, 2, 6, 7)(4, 3, 5, 20)(8, 10)(19, 17)(9, 11)(30, 22)(21, 29) \\ & (18, 16)(31, 33, 15, 28)(12, 24, 32, 25)(34, 14, 27, 36) \\ & (23, 35, 26, 13). \end{aligned}$$

(iv) The sextet graph $S(17)$ on 102 vertices. Its full automorphism group is $PSL(2, 17)$, obtainable as a homomorphic image of the group G_4^1 via the

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linear fractional transformations

$$h\colon z \to \frac{z+4}{4-z}, \qquad p\colon z \to \frac{-1}{z},$$

$$q\colon z \to \frac{1}{z}, \qquad r\colon z \to \frac{4}{z}, \qquad a\colon z \to \frac{2}{z}$$

of the projective line over $GF(17)$.

(v) A double cover of $S(17)$, on 204 vertices. Its full automorphism group is the direct product $PSL(2, 17) \times C_2$, obtainable as a homomorphic image of G_4^1 using the following permutations of degree 36:

$$\begin{aligned} h = {} & (1, 3, 18)(2, 20, 19)(4, 5, 32)(6, 7, 10)(8, 9, 30)(11, 15, 12) \\ & (13, 14, 35)(16, 25, 17)(21, 36, 22)(23, 34, 24)(26, 31, 27) \\ & (28, 33, 29), \\ p = {} & (3, 20)(4, 9)(5, 27)(6, 14)(7, 34)(8, 26)(10, 28)(11, 17) \\ & (12, 25)(13, 24)(18, 19)(21, 31)(22, 32)(23, 33)(29, 35)(30, 36), \\ q = {} & (1, 2)(4, 21)(5, 30)(6, 33)(7, 35)(8, 22)(9, 31)(10, 24) \\ & (11, 17)(13, 28)(14, 23)(15, 16)(18, 19)(26, 32)(27, 36)(29, 34), \\ r = {} & (1, 2)(3, 19)(5, 32)(6, 29)(7, 33)(8, 30)(10, 28)(11, 25) \\ & (12, 17)(14, 35)(15, 16)(18, 20)(21, 31)(22, 27)(23, 34)(26, 36), \\ a = {} & (1, 2)(3, 4)(5, 6)(7, 26)(8, 34)(9, 20)(10, 11)(12, 13)(14, 27) \\ & (15, 16)(17, 28)(18, 31)(19, 21)(22, 23)(24, 25)(29, 30) \\ & (32, 33)(35, 36). \end{aligned}$$

(vi) Wong's graph on 234 vertices [11]. Its full automorphism group is Aut $PSL(3, 3)$, obtainable as a homomorphic image of the group G_4^2 using permutations of degree 13 as follows:

$$\begin{aligned} h &= (2, 3, 5)(4, 12, 11)(6, 7, 10)(8, 9, 13), \\ p &= (1, 11)(2, 9)(3, 13)(4, 12), \\ q &= (1, 4)(3, 13)(5, 8)(11, 12), \\ r &= (2, 3)(4, 12)(6, 7)(9, 13), \\ a &= (1, 2, 11, 9)(3, 12, 13, 4)(5, 6)(7, 8). \end{aligned}$$

(vii) A double cover of Wong's graph, having 468 vertices. Its full automorphism group has order 11232, and is the group $SL(3, 3)$ extended by an automorphism of order 2. A subgroup of order 5616 is obtainable as a homomorphic image of G_4^2 by taking

$$h = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad p = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

and $a = c\theta$, where

$$c = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

and θ is the 'inverse-transpose' automorphism of $S(3, 3)$, taking every matrix to the transpose of its inverse.

(viii) The sextet graph $S(5)$ on 650 vertices. Its full automorphism group is the group $P\Gamma L(2, 25)$ of order 31200, and subgroups of order 15600 are obtainable as a homomorphic image of the groups G_4^1 and G_4^2 respectively via the linear fractional transformations of the projective line over $GF(25)$ as follows:

(a)

$$h\colon z \to \frac{2z+2}{z-1}, \qquad p\colon z \to \frac{-1}{z},$$

$$q\colon z \to \frac{1}{z}, \qquad r\colon z \to \frac{1-z}{1+z}, \qquad a_1\colon z \to \frac{t^2 - z}{1 + t^2 z}$$

where t is a root of the polynomial $t^2 - t + 2 = 0$ over $\mathbb{Z}_5$;

(b) via the linear fractional transformations h, p, q, r as in (a), but with a_1 replaced by

$$a_2\colon z \to \frac{t^2 z^5 + 1}{z^5 + t^{14}}.$$

(ix) The sextet graph $S(79)$ on 10270 vertices. Its full automorphism group is $PSL(2, 79)$, obtainable as a homomorphic image of the group G_4^1 via the linear fractional transformations

$$h\colon z \to \frac{-23z}{z+24}, \qquad p\colon z \to \frac{-1}{z},$$

$$q\colon z \to \frac{24-23z}{23+24z}, \qquad r\colon z \to \frac{14-35z}{35+14z}, \qquad a\colon z \to \frac{32z+9}{9z-32}$$

of the projective line over $GF(79)$.

It will be shown that every 4- or 5-arc-transitive cubic graph of girth less than 12 is one of the nine examples given above.

3. Method

Following the construction described by Lorimer in [8], for each finite homomorphic image G of G_4^1 or G_4^2 having order greater than 2, we may define a graph Γ as follows: the vertices of Γ are the left cosets of H in G, where H is the image of $\langle h, p, q, r\rangle$ in G, with cosets xH and yH being

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adjacent if and only if $x^{-1}y \in HaH$. Defined in this way Γ is a connected finite cubic graph on which the group G acts faithfully and symmetrically as a group of automorphisms under the action $g : xH \to gxH$ for each $g \in G$ and each vertex xH of Γ. The vertex H is adjacent to the vertices aH, haH, h^2aH, and its stabilizer is the subgroup H itself.

Let $S = \{a, ha, h^2a\}$. Call any word of the form $w_1w_2\cdots w_n$ where each $w_i \in S$, $1 \le i \le n$, n a positive integer, a *base word*. Then each coset of H in G contains a base word, and the coset containing the base word w is adjacent to those containing wa, wha, and wh^2a, all of which are base words.

Since the graph Γ is connected, any two cosets xH and yH are connected by a path, corresponding to a base word t such that $xtH = yH$. Consequently there is a circuit (closed path) from the coset xH back to itself whenever there is a base word t such that $xtH = xH$, that is, $t \in H$.

We define the girth of a graph to be the length of its shortest circuit.

Examining G_4^1. Since a cubic graph obtained from G_4^1 in the above manner is 4-arc-transitive, it may be assumed that a path of length $n \ge 4$ begins at the vertex corresponding to H and the next four vertices along the path are aH, $ahaH$, $a(ha)^2H$ and $a(ha)^3H$ respectively. The graph will have a circuit of length n, where $6 \le n \le 13$, if there is a base word of the form $a(ha)^3h^{f_1}ah^{f_2}\cdots ah^{f_m}a$, $f_i = \pm 1$, $2 \le m \le 9$, in H. For a circuit of length n there are 2^{n-4} possible base words of this form, and consequently $(2^{n-4})(24)$ possible relations of the type $wu^{-1} = 1$, where w is a base word of the above form, and u is one of the 24 elements in H.

Now if any of these new relations is adjoined to the standard relations for G_4^1, this replaces G_4^1 by a new group G which is a homomorphic image of G_4^1. After each such replacement, the index of H in G can be determined, and if this is greater than 2 then it will be the number of vertices of some 4-arc-transitive cubic graph. Instances of $[G : H]$ equalling 1 or 2 may be ignored since the corresponding graphs are trivial.

This procedure was performed for all possible additional relators, except that for circuits of length 12 (that is, $m = 8$) the case $f_1 = \cdots = f_8 = 1$ was omitted, Conder [4] having shown that in this case there is an infinite family of 4-arc-transitive cubic graphs each with girth 12. Remarkably, all cases other than this gave rise to finite groups.

Finally all the graphs (produced by the above procedure) with the same number of vertices were examined to determine if they were isomorphic. An example showing how this was done is given after the results.

Examining G_4^2. The same method was used as in G_4^1 with the appropriate relations being modified to allow for the fact that $a^2 = p$ instead of $a^2 = 1$.

Examining G_5. Since every graph which is 5-arc-transitive is also 4-arc-transitive, no new information will be gained by repeating the above process for G_5.

A similar approach to the above was adopted by Biggs [2], but he only considered the addition of extra relators of the form $(ha)^m$, for $5 \leq m \leq 16$.

4. Results

These were obtained using coset enumeration in the CAYLEY group system [3]. First the relators which led to nontrivial graphs from G_4^1 are listed.

Circuit length	Relator	Index of $\langle h, p, q, r \rangle$
6	$(ha)^6$	14
8	$(ha)^8$	14
8	$a(ha)^4h^2a(ha)^2h^2q$	30
8	$a(ha)^4(h^2a)^2hapr$	14
9	$(ha)^9$	102
10	$(ha)^{10}$	90
10	$a(ha)^5h^2a(ha)^3p$	30
10	$a(ha)^5h^2a(ha)^2h^2ahr$	14
11	$a(ha)^4(h^2a)^3ha(h^2a)^2rq$	102
10	$a(ha)^5(h^2a)^2(ha)^2rq$	14
10	$a(ha)^4h^2a(ha)^4h^2$	14
10	$a(ha)^4h^2a(ha)^2h^2ahahpq$	14
10	$a(ha)^4(h^2aha)^2h^2ahqr$	14
10	$a(ha)^4(h^2aha)^2h^2a$	14
10	$a(ha)^4(h^2a)^2ha(h^2a)^2h^2$	14
10	$a(ha)^4(h^2a)^3(ha)^2$	14
12	$a(ha)^8h^2ahah^2ah^2$	14
12	$a(ha)^7h^2aha(h^2a)^2h$	14
12	$a(ha)^7(h^2a)^2(ha)^2h^2$	14
12	$a(ha)^6h^2a(ha)^3h^2apr$	30
12	$a(ha)^6h^2a(ha)^2h^2ahah^2$	14
12	$a(ha)^6h^2a(ha)^2(h^2a)^2q$	14
12	$a(ha)^6h^2aha(h^2a)^2hah$	14
12	$a(ha)^6h^2aha(h^2a)^3h^2pq$	14
12	$a(ha)^6(h^2a)^2(ha)^3pq$	14

Circuit length	Relator	Index of $\langle h, p, q, r\rangle$
12	$a(ha)^6(h^2a)^2)(ha)^2h^2ah$	14
12	$a(ha)^6(h^2a)^4hahr$	14
12	$a(ha)^6(h^2a)^5h^2$	14
12	$a(ha)^6(h^2a)^5h^2p$	30
12	$a(ha)^5h^2a(ha)^5$	14
12	$a(ha)^5h^2a(ha)^2h^2ahah^2ah$	14
12	$a(ha)^5(h^2aha)^2(h^2a)^2h^2pq$	204
12	$a(ha)^5(h^2aha)^2(h^2a)^2$	14
12	$a(ha)^5(h^2aha)^2(h^2a)^2qr$	14
12	$a(ha)^5h^2aha(h^2a)^2(ha)^2h$	14
12	$a(ha)^5h^2aha(h^2a)^4h^2rq$	14
12	$a(ha)^5(h^2a)^2(ha)^3h^2ahrq$	90
12	$a(ha)^5(h^2a)^2(ha)^2h^2ahah$	14
12	$a(ha)^5(h^2a)^2ha(h^2a)^2ha$	14
12	$a(ha)^5(h^2a)^3(ha)^2h^2ah^2q$	14
12	$a(ha)^5(h^2a)^3(ha)^2h^2a$	14
12	$a(ha)^5(h^2a)^4(ha)^2hpq$	14
12	$a(ha)^5(h^2a)^6h$	14
12	$a(ha)^5(h^2a)^6hq$	30
12	$a(ha)^4h^2a(ha)^4h^2aharq$	14
12	$a(ha)^4h^2a(ha)^3h^2ahah^2ah^2q$	14
12	$a(ha)^4h^2a(ha)^2(h^2aha)^2h$	14
12	$a(ha)^4h^2a(ha)^2(h^2a)^2hah^2ah^2pq$	650
12	$a(ha)^4h^2a(ha)^2(h^2a)^3ha$	14
12	$a(ha)^4(h^2aha)^2(h^2a)^3h^2rq$	204
12	$a(ha)^4h^2aha(h^2a)^2(ha)^3h$	14
12	$a(ha)^4h^2aha(h^2a)^2ha(h^2a)^2r$	14
12	$a(ha)^4h^2aha(h^2a)^3hah^2ah^2$	14
12	$a(ha)^4h^2aha(h^2a)^5h^2pr$	14
12	$a(ha)^4(h^2a)^2(ha)^3h^2ahah^2pr$	90
12	$a(ha)^4(h^2a)^2(ha)^3(h^2a)^2$	14
12	$a(ha)^4(h^2a)^2(ha)^2h^2a(ha)^2h$	14
12	$a(ha)^4(h^2a)^2(hah^2a)^2hahqr$	204
12	$a(ha)^4(h^2a)^2ha(h^2a)^2hah^2apr$	30
12	$a(ha)^4(h^2a)^3(ha)^2(h^2a)^2h^2r$	14

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Circuit length	Relator	Index of $\langle h, p, q, r\rangle$
12	$a(ha)^4(h^2a)^3(hah^2a)^2q$	90
12	$a(ha)^4(h^2a)^3ha(h^2a)^3hrq$	14
12	$a(ha)^4(h^2a)^4(ha)^3h^2rq$	14
12	$a(ha)^4(h^2a)^4hah^2ahah^2$	14
12	$a(ha)^4(h^2a)^6hah$	14
12	$a(ha)^4(h^2a)^6hah^2p$	14
12	$a(ha)^4(h^2a)^6hah^2r$	30
13	$(ha)^{13}$	10270
13	$a(ha)^5(h^2a)^3ha(h^2a)^2hapq$	102
13	$a(ha)^4h^2aha(h^2a)^2(ha)^4rq$	102

Now the relators which led to nontrivial graphs from G_4^2 are listed.

Circuit length	Relator	Index of $\langle h, p, q, r\rangle$
8	$(ha)^8$	30
10	$a(ha)^4h^2a(ha)^2(h^2a)^2hqr$	90
10	$a(ha)^4(h^2a)^3(ha)^2p$	30
12	$(ha)^{12}$	650
12	$a(ha)^5h^2a(ha)^2(h^2a)^3rq$	90
12	$a(ha)^5(h^2aha)^2(h^2a)^2h^2pq$	30
12	$a(ha)^5(h^2a)^2(ha)^3h^2ahrq$	90
12	$a(ha)^5(h^2a)^3(ha)^2h^2ar$	30
12	$a(ha)^4h^2a(ha)^2(h^2a)^2hah^2ah^2pq$	468
12	$a(ha)^4(h^2aha)^2(h^2a)^3h^2rq$	30
12	$a(ha)^4h^2aha(h^2a)^2ha(h^2a)^2$	30
12	$a(ha)^4(h^2a)^2(ha)^3h^2ahah^2pr$	90
12	$a(ha)^4(h^2a)^2ha(h^2aha)^2hqr$	30
13	$(ha)^{13}$	234

The preceding listing contains only half of the relators that led to nontrivial graphs. This is because if there is a circuit corresponding to a relator which begins $a(ha)^4$, then there is a matching relator which beings $a(ha)^3h^2a$ and gives the same group and therefore the same graph. Note that the defining relations for G_4^1 and G_4^2 are equally valid if a is replaced by ap. Also note that in a 4-arc-transitive cubic graph the vertex xH is joined to the three vertices xaH, $xhaH$ and xh^2aH, and then since $p \in H$ we have $x(ap)H = xH$, $xh(ap)H = xhaH$ and $xh^2(ap)H = x^2haH$; consequently using ap instead of a gives the same graph. Thus if a relator begins with

$a(ha)^3h^2a$ then replacement of a by ap gives $(ap)(hap)^3h^2(ap)$, and then conjugation of this by r gives $a(ha)^4qr$. An example of a pair of relators of this type is $a(ha)^4h^2a(ha)^2h^2q$ and $a(ha)^3h^2a(ha)^2h^2ah^2r$, corresponding to circuits of length 8.

The following example illustrates how graphs with the same number of vertices, constructed as described in the preceding section on method, were shown to be isomorphic. The relation $(ha)^9 = 1$ when adjoined to the usual relations in G_4^1 gave a graph with 102 vertices. CAYLEY was used to construct the automorphism group for this graph. It had the generators

$$h = (1, 3, 6)(2, 8, 7)(4, 5, 11)(9, 15, 18)(10, 16, 13)(12, 17, 14),$$
$$p = (3, 8)(4, 9)(5, 17)(6, 7)(10, 12)(11, 13)(14, 18)(15, 16),$$
$$q = (1, 2)(4, 10)(5, 15)(6, 7)(9, 12)(11, 14)(13, 18)(16, 17),$$
$$r = (1, 2)(3, 7)(5, 11)(6, 8)(10, 12)(13, 17)(14, 16)(15, 18),$$
$$a = (1, 2)(3, 4)(6, 12)(7, 10)(8, 9)(11, 15)(13, 16)(14, 18).$$

The three other relators which had given a graph with 102 vertices were

$$a(ha)^4(h^2a)^3ha(h^2a)^2rq, \quad a(ha)^5(h^2a)^3ha(h^2a)^2hapq$$

and

$$a(ha)^4h^2aha(h^2a)^2(ha)^4rq.$$

In each instance, substitution into these relators by h, p, q, r and a (or ap) gave the result that the relator was equal to the identity element. Consequently all the graphs with 102 vertices were isomorphic to the sextet graph $S(17)$.

In a similar way graphs with 14-vertices were all shown to be isomorphic to Heawood's graph; those with 30 vertices to Tutte's 8-cage; those with 90 vertices to the same triple cover of Tutte's 8-cage; those with 204 vertices to the same double cover of $S(17)$; those with 234 vertices to Wong's graph; those with 468 vertices to the same double cover of Wong's graph; those with 650 vertices to $S(5)$; and the graph with 10270 vertices to $S(79)$.

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References

[1] N. Biggs and M. Hoare, 'The sextet construction for cubic graphs', *Combinatorica* **3** (1983), 153–165.
[2] N. Biggs, 'Presentations for cubic graphs', *Computational Group Theory*, edited by M. Atkinson, pp. 57–63 (Academic Press, London, 1984).
[3] J. Cannon, 'Software tools for group theory', *Proc. Sympos. Pure Math.* **37** (1980), 495–502.
[4] M. Conder, 'An infinite family of 4-arc-transitive cubic graphs each with girth 12', *Bull. London Math. Soc.* **21** (1989), 375–380.
[5] M. Conder, 'An infinite family of 5-arc-transitive cubic graphs', *Ars Combin.* **25A** (1988), 95–108.
[6] M. Conder and P. Lorimer, 'Automorphism groups of symmetric graphs of valency 3', *J. Combin. Theory Ser. B*, to appear.
[7] D. Djokovic and G. Miller, 'Regular groups of automorphisms of cubic graphs', *J. Combin. Theory Ser. B* **29** (1980), 195–230.
[8] P. Lorimer, 'Vertex transitive graphs: symmetric graphs of prime valency', *J. Graph Theory* **8** (1984), 55–68.
[9] W. Tutte, 'A family of cubical graphs', *Proc. Cambridge Philos. Soc.* **43** (1947), 459–474.
[10] W. Tutte, 'On the symmetry of cubic graphs', *Canad. J. Math.* **11** (1959), 621–624.
[11] W. Wong, 'Determination of a class of primitive permutation groups', *Math. Z.* **99** (1967), 235–246.

University of Auckland
Private Bag
Auckland
New Zealand

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