

ON GENERAL HYBRID CONTRACTIONS

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Abstract

Khan *et al.* have studied the existence of solutions of functional equations $f_ix \in Sx \cap Tx$ and $x = F_ix \in Sx \cap Tx$ under certain nonlinear hybrid contraction and asymptotic regularity conditions, generalizing many known results on coincidences and fixed points. However, most of their main theorems admit counter examples. In this paper, we rectify these results and obtain many coincidence and fixed point theorems in a more general setting.

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1. Introduction

Nadler's multi-valued contraction theorem [10] has led to a good fixed point theory for multi-valued operators in nonlinear analysis. It has various applications inside and outside mathematics (see, for instance, [1, 19]). Hybrid fixed point theory for nonlinear single-valued and multi-valued operators is a relatively recent development within the ambit of multi-valued fixed point theory (see, for instance, [2, 3, 5, 9, 11–17]). It provides techniques for solving functional inclusions and optimization problems (see [4]). Khan *et al.* [9] have obtained some interesting coincidence and fixed point theorems for nonlinear hybrid contractions in general settings. However, their proofs of the main theorems contain some errors, and the purpose of this paper is to improve these results under suitable conditions in a more general setting. Our results generalize, improve and unify number of results in the existing single-valued, multi-valued and hybrid fixed point theory.

2. Preliminaries

We generally follow the definitions and notations used in [9]. Given a metric space (X, d) , let $(C(X), H)$, $(CB(X), H)$, $(CL(X), H)$ and $(P_X(X), H)$ denote respectively the hyper-spaces of nonempty compact, nonempty closed bounded, nonempty closed and nonempty proximal subsets of X , where H is the Hausdorff metric induced by d . Notice that the hyper-space $CL(X)$ contains the other three spaces. Throughout, $D(x, A)$ will denote the ordinary distance between $x \in X$ and A , a nonempty subset of X . Also, \mathbb{N} and \mathbb{N}_0 will denote respectively the sets of positive and nonnegative integers. Thus $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Further, for $f, g : X \rightarrow X$ and $S, T : X \rightarrow CL(X)$ we shall use the following notations for brevity.

$$(2.1) \quad M(Sx, Ty; f, g) := \max\{D(fx, Sx), D(gy, Ty), D(fx, Ty), \\ D(gy, Sx), d(fx, gy)\}$$

$$(2.2) \quad M(Sx, Ty; f) := M(Sx, Ty; f, g) \quad \text{with } f = g,$$

$$(2.3) \quad M(Sx, y; f) := M(Sx, Ty; f) \quad \text{with } S = T,$$

$$(2.4) \quad m(Sx, Ty; f, g) := \max\{D(fx, Sx), D(gy, Ty), \\ \frac{1}{2}[D(fx, Ty) + D(gy, Sx)], d(fx, gy)\}$$

$$(2.5) \quad m(Sx, Ty; f) := m(Sx, Ty; f, g) \quad \text{with } f = g, \quad \text{and}$$

$$(2.6) \quad m(Tx, y; f) := m(Sx, Ty; f) \quad \text{with } S = T.$$

Recall that (see [6]) mappings $f : X \rightarrow X$ and $S : X \rightarrow CL(X)$ are said to commute at a point $z \in X$ if $fSz \subset Sfz$; f and S are said to commute on X if $fSx \subseteq Sf x$ for all $x \in X$. We remark that weak commutativity (see [9, 14]), compatibility (see [3, 8]) and weak compatibility (see [12]) of maps f and S at a coincidence point z (that is, when $fz \in Sz$) satisfy $fSz = Sfz$ (see [12, lemma]). Therefore, weak commutativity, compatibility, weak compatibility and usual commutativity of two single-valued self-maps on a metric space are equivalent at their coincidence point. However, as the following example shows, the commutativity of f and S at their coincidence point z is indeed more general than compatibility and weak commutativity (weak compatibility) of f and S at z .

EXAMPLE 1. Let $X = [0, \infty)$ with the usual metric, $fx = 3x$ and $Sx = [2+x, \infty)$. Then $f1 \in S1$, $fS1 = [9, \infty) \subset [5, \infty) = Sf1$ and $fS1 \neq Sf1$.

3. Main results

Unless stated otherwise, let Γ denote the family of maps ϕ from the set \mathbb{R}^+ of nonnegative reals to itself such that ϕ is upper semi-continuous, nondecreasing and

$\phi(t) < t$ for all $t > 0$.

First we show that [9, Theorem 3.1] admits a counter example.

EXAMPLE 2. Let $X = \mathbb{R}^+$ be endowed with the usual metric. Let $f, g, S, T : X \rightarrow X$ be such that $fx = 2x^2, gx = 2x^3, Sx = x^2 + \frac{1}{4}$ and $Tx = x^3 + \frac{1}{4}$. Let f_i in [9, Theorem 3.1] be such that $f_i = f$ if i is odd and $f_i = g$ if i is even. Evidently, for any $x, y \in X, d(Sx, Ty) = |x^2 - y^3| = \frac{1}{2}d(fx, gy)$, that is, [9, (3.2) and (3.3)] are satisfied with $\phi(t) = qt, q = \frac{1}{2}$. Further, $S(X) = T(X) = [\frac{1}{4}, \infty) \subset X = f(X) = g(X)$, that is, [9, (3.1) and (3.5)] hold, since a complete metric space is always orbitally complete.

To verify [9, (3.4)], construct sequences $\{x_n\}$ and $\{y_n\}$ as considered in [9, Definition 2.1] and in the beginning of the proof of [9, Theorem 3.1]. They are defined as follows. For some $x_0 \in X, y_{2n} = f_{2n}x_{2n-1} \in Sx_{2n}$ for every $n \in \mathbb{N}, y_{2n+1} = f_{2n+1}x_{2n} \in Tx_{2n+1}$ for every $n \in \mathbb{N}_0$.

Accordingly, for $x_0 = \frac{1}{2}$ in our example, $\{x_n\}_{n=1}^\infty$ is given by $x_n = (\frac{1}{4})^{1/3}$ if n is odd and $x_n = (\frac{1}{4})^{1/2}$ if n is even; and $\{y_n\} = \{fx_0, gx_1, fx_2, gx_3, \dots\} = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots\}$ which converges to $\frac{1}{4}$. Thus [9, (3.4)] is satisfied for $x_0 = \frac{1}{2}$. Therefore, all the hypotheses of [9, Theorem 3.1] are satisfied but f, g, S and T do not have a coincidence point. Notice, however, that $f(\frac{1}{2}) = S(\frac{1}{2}) = \frac{1}{2}$ and $g(4^{-1/3}) = T(4^{-1/3}) = \frac{1}{2}$, that is f, S have a coincidence point at $x = \frac{1}{2}$ and g and T have a (different) coincidence at $x = 4^{-1/3}$.

The main problem with [9, Theorem 3.1] is in the construction of the sequence $\{x_n\}$. Since $S(X) \cup T(X) \subset f_i(X)$ for each $i \in \mathbb{N}$, one cannot choose the x_n as described in [9].

With a view of presenting a correct and more general version of [9, Theorem 3.1], we introduce the following definitions for $S, T : Y \rightarrow CL(X)$ and $f_n : Y \rightarrow X, n \in \mathbb{N}$. In all that follows, Y is an arbitrary nonempty set.

DEFINITION 1. If for $x_0 \in Y$, there exist sequences $\{x_n\}$ in Y and $\{y_n\}$ in X such that

$$\begin{aligned} y_{2n+1} &= f_{2n+1}x_{2n+1} \in Sx_{2n} & n \in \mathbb{N}_0, \\ y_{2n} &= f_{2n}x_{2n} \in Tx_{2n-1}, & n \in \mathbb{N}, \end{aligned}$$

then $O_{f_i}(x_0) = \{y_n : n = 1, 2, \dots\}$ is said to be the orbit for $(S, T; f_i)$ at x_0 . Further, $O_{f_i}(x_0)$ is called regular if

$$d(y_n, y_{n+1}) \leq \begin{cases} H(Sx_{n-1}, Tx_n), & \text{if } n \text{ is odd} \\ H(Tx_{n-1}, Sx_n), & \text{if } n \text{ is even.} \end{cases}$$

In the case $\{f_n\} = \{f, g, f, g, \dots\}, O_{f_i}(x_0)$ and $(S, T; f_i)$ may be denoted by $O_{f,g}(x_0)$ and $(S, T; f, g)$ respectively.

The above definition with $Y = X$ is the corrected version of [9, Definition 2.1]. The following definitions with $Y = X$ are [9, Definitions 2.2 and 2.3].

DEFINITION 2. If, for $x_0 \in Y$, there exist sequences $\{x_n\}$ in Y and $\{y_n\}$ in X such that every Cauchy sequence of the form $O_{f_i}(x_0)$ converges in a subset P of X , then P is called (S, T, f_i) -orbitally complete with respect to x_0 , or simply $(S, T; f_i, x_0)$ -orbitally complete.

DEFINITION 3. A pair (S, T) is said to be *asymptotically regular* at $x_0 \in Y$ if for any sequence $\{x_n\}$ in Y and each sequence $\{y_n\}$ in X such that $y_n \in Sx_{n-1} \cup Tx_{n-1}$, $\lim_n(y_n, y_{n-1}) = 0$.

In all that follows, $C(A, f)$ stands for the set of coincidence points of the mappings $A : C \rightarrow CL(X)$ and $f : X \rightarrow X$, that is $C(A, f) = \{u : fu \in Au\}$.

The following theorem is the corrected version of [9, Theorem 3.1].

THEOREM 1. Let $S, T : Y \rightarrow CL(X)$ and $f_n : Y \rightarrow X$ ($n \in \mathbb{N}$) be such that

$$(3.1) \quad S(Y) \subseteq f_{2n-1}(Y), \quad T(Y) \subseteq f_{2n}(Y), \quad n \in \mathbb{N},$$

$$(3.2) \quad H(Sx, Ty) \leq \phi(M(Sx, Ty; f_{2n}, f_{2n-1})) \text{ for each } x, y \in Y \text{ and } \phi \in \Gamma,$$

$$(3.3) \quad \phi(t) \leq qt \text{ for each } t > 0 \text{ and for some fixed } q \in (0, 1),$$

(3.4) *there exists a point $x_0 \in Y$ such that the pair (S, T) is asymptotically regular at x_0 , and*

(3.5) *one of $S(Y), T(Y), f_{2i-1}(Y)$ or $f_{2i}(Y), i \in \mathbb{N}$, is $(S, T; f_i, x_0)$ -orbitally complete.*

Then:

(1a) *S and f_{2j} have a coincidence point for $j \in \mathbb{N}$;*

(1b) *T and f_{2j-1} have a coincidence point for $j \in \mathbb{N}$.*

Further, if $Y = X$, then

(2a) *S and f_{2j} have a common fixed point $f_{2j}v$, provided $f_{2j}(f_{2j}v) = f_{2j}v$ and S and f_{2j} commute at $v \in C(S, f_{2j}), j \in \mathbb{N}$;*

(2b) *T and f_{2j-1} have a common fixed point $f_{2j-1}w$, provided $f_{2j-1}(f_{2j-1}w) = f_{2j-1}w$ and T and f_{2j-1} commute at $w \in C(T, f_{2j-1}), j \in \mathbb{N}$;*

(3) *S, T and f_n ($n \in \mathbb{N}$) have a common fixed point provided (2a) and (2b) both are true.*

PROOF. In view of (3.1) and (3.4), for a point $x_0 \in Y$, we can construct sequences $\{x_n\} \subseteq Y$ and $\{y_n\} \subseteq X$ such that, for each $n \in \mathbb{N}$,

$$y_{2n-1} = f_{2n-1}x_{2n-1} \in Sx_{2n-2}, \quad y_{2n} = f_{2n}x_{2n} \in Tx_{2n-1}, \quad \text{and}$$

$$d(y_{2n-1}, y_{2n}) \leq q^{-1/2} H(Sx_{2n-2}, Tx_{2n-1}),$$

$$d(y_{2n}, y_{2n+1}) \leq q^{-1/2} H(Tx_{2n-1}, Sx_{2n}).$$

Then it can be easily verified that $\{y_n\}$ is a Cauchy sequence (see [9, proof of Theorem 3.1], [12, 13]).

Now suppose that $f_{2i}(Y)$ is orbitally complete, $i \in \mathbb{N}$. Then the subsequence $\{y_{2n}\}$, being contained in $f_{2i}(Y)$, has a limit, say u , in $f_{2i}(Y)$. Let $v \in f_{2i}^{-1}u$. Thus there is a $v \in Y$ such that $f_{2i}v = u$, and this is true for each $i \in \mathbb{N}$. Note that the subsequence $\{y_{2n-1}\}$ also converges to u . Putting $x = v$ and $y = x_{2n-1}$ in (3.2),

$$D(Sv, y_{2n}) \leq H(Sv, Tx_{2n-1})$$

$$\leq \phi(\max\{D(f_{2n}v, Sv), D(f_{2n-1}x_{2n-1}, Tx_{2n-1}), D(f_{2n}v, Tx_{2n-1}),$$

$$D(f_{2n-1}x_{2n-1}, Sv), d(f_{2n}v, f_{2n-1}x_{2n-1})\})$$

$$\leq \phi(\max\{D(u, Sv), d(y_{2n-1}, y_{2n}), d(u, y_{2n}), D(y_{2n-1}, Sv), d(u, y_{2n-1})\}).$$

Making $n \rightarrow \infty$, we obtain $D(Sv, u) \leq \phi(D(u, Sv))$, proving $u \in Sv$. Thus for each $i \in \mathbb{N}$, $f_i v = u \in Sv$, and this proves (1a).

Since $S(Y) \subseteq f_{2i-1}(Y)$, there exists a $w \in Y$ such that $f_{2i-1}w = u$, and this is true for each $i \in \mathbb{N}$, therefore, as previously, for each odd $i \in \mathbb{N}$, $f_i w = u \in Tw$. This proves (1b).

If we suppose $f_{2i-1}(Y)$ is orbitally complete ($i \in \mathbb{N}$), then analogous argument establishes (1a) and (1b). If $T(Y)$ (respectively $S(Y)$) is orbitally complete, then $u \in T(Y) \subseteq f_{2i}(Y)$ (respectively $u \in S(Y) \subseteq f_{2i-1}(Y)$), $i \in \mathbb{N}$ and the above argument establishes (1a) and (1b).

To prove (2a), note that $Y = X$, $v \in C(S, f_{2j})$ and $u = f_{2j}v$, $j \in \mathbb{N}$. From this, $f_{2j}u = u$, and the commutativity of S and f_{2j} ($j \in \mathbb{N}$), we derive

$$u = f_{2j}u = f_{2j}(f_{2j}v) \in f_{2j}(Sv) \subseteq S(f_{2j}v) = Su$$

Thus u is a common fixed point of S and f_{2j} ($j \in \mathbb{N}$). Similar argument yields $u = f_{2j-1}u \in Tu$ ($j \in \mathbb{N}$, proving (2b). Now (3) is immediate. □

REMARK 1. Example 2 shows that [9, Theorem 3.2] (in which $f_i(X)$ is misprinted as $f(X)$, see [9, (3.7)] and [9, Theorem 3.3] are incorrect. The following are the corrected and substantially improved versions of these results.

THEOREM 2. Let $S, T : Y \rightarrow CL(X)$ and $f_n : Y \rightarrow X$ ($n \in \mathbb{N}$) be such that (3.2) and the following hold.

$$(3.6) \quad \phi(t) < t \text{ for each } t > 0 \text{ and some } \phi \in \Gamma,$$

(3.7) *there exists a point $x_0 \in Y$ such that the orbit $O_{f_i}(x_0)$ for $(S, T; f_i)$ is regular, the pair (S, T) is asymptotically regular at x_0 and one of $S(Y)$, $T(Y)$, $f_{2i-1}(Y)$ or $f_{2i}(Y)$ ($i \in \mathbb{N}$) is $(S, T; f_i, x_0)$ -orbitally complete.*

Then all the conclusions of Theorem 1 are true.

THEOREM 3. *Let $S, T : Y \rightarrow C(X)$ and $f_n : Y \rightarrow X$ ($i \in \mathbb{N}$) be such that the conditions (3.1), (3.2) and (3.4)–(3.6) hold. Then all the conclusions of Theorem 1 are true.*

PROOF. An appropriate blend of the proof of Theorem 1 works for Theorems 2 and 3. See also remarks [9, p. 376]. □

REMARK 2. We now discuss another theorem of Khan *et al.* [9, Theorem 3.4]. Notice that [9, Theorem 3.4, condition (3.8)] and [9, Theorem 3.1, condition (3.2)] are the same. Further, [9, Theorem 3.4] is [9, Theorem 3.1] without [9, (3.3) and (3.4)]. So by Example 2, [9, Theorem 3.4] is also incorrect. Even certain special cases of [9, Theorem 3.4] admit counter examples. For instance, it can be seen using an example of Totik [18, p. 241] that [9, Theorem 3.4] with $f_i x = x$ ($i \in \mathbb{N}$), $S, T : X \rightarrow X$ and $\phi(t) = qt$, ($\frac{1}{2} < q < 1$) is not true, that is S and T do not have a common fixed point. It seems that the authors' intention in [9, Theorem 3.4] was to present [9, Theorem 3.1] with [9, (3.2)] replaced by the following (*), since [9, (3.4)] is not needed in such a situation.

(*)
$$H(Sx, Ty) \leq \phi(m(Sx, Ty; f_i, f_j)) \quad \text{for } x, y \in X, i, j \in \mathbb{N}, i \neq j, \text{ and } \phi \in \Gamma.$$

However, in view of Example 2, [9, Theorem 3.4] with [9, (3.2)] being replaced by (*) remains incorrect.

The theorem we want to prove is the following:

THEOREM 4. *Let $S, T : Y \rightarrow CL(X)$ and $f_n : Y \rightarrow X$ be such that (3.1), (3.2) with M replaced by m , (3.3) and (3.5) hold. Then all the conclusions of Theorem 1 are true.*

PROOF. An appropriate blend of the proof of Theorem 1 works. □

THEOREM 4 BIS. *Theorem 4 without (3.3) and $CL(X)$ replaced by P_X .*

REMARK 3. The following example shows that [9, Corollaries 3.5 and 3.6] which are intended to present important and sharpened versions of special cases of [9, Theorems 3.1 and 3.3], need some modifications in their conclusions.

EXAMPLE 3. ([14, Example 2.4]). Let $X = [0, 1]$ with the absolute value metric, and let $Sx = Tx = \{0, 1\}$, $fx = 1 - x$ for all $x \in X$. Then $Sfx = fSx$ for any $x \in X$, and $H(Sx, Sy) = 0 \leq qd(fx, fy)$ for each $x, y \in X$ and any $q \in (0, 1)$. Now it is easy to see that all the hypotheses of [9, Corollaries 3.5 and 3.6] with $S = T$ are satisfied. Evidently, S and f have no common fixed point. However, S and f have a coincidence. It may also be mentioned that this example proves that [9, Theorem 3.7] in its present form is not true.

Now we derive certain corollaries from our theorems, which contain and improve a number of coincidence and fixed point theorems for single-valued and multi-valued mappings in metric spaces. Corollaries 2, 2 bis and 3, 3 bis present corrected, improved and sharpened versions of [9, Corollaries 3.5 and 3.6] respectively.

COROLLARY 1. Let $S, T : Y \rightarrow CL(X)$ and $f, g : Y \rightarrow X$ be such that

$$(3.8) \quad S(Y) \subseteq g(Y) \text{ and } T(Y) \subseteq f(Y),$$

$$(3.9) \quad H(Sx, Ty) \leq qm(Sx, Ty; f, g) \text{ for each } x, y \in Y \text{ and some } q \in (0, 1), \text{ and}$$

$$(3.10) \quad \text{one of } S(Y), T(Y), f(Y) \text{ or } g(Y) \text{ is a complete subspace of } X.$$

Then

$$(1a^*) \quad S \text{ and } f \text{ have a coincidence point } v \text{ in } Y,$$

$$(1b^*) \quad T \text{ and } g \text{ have a coincidence point } w \text{ in } Y.$$

Further, if $Y = X$, then

$$(2a^*) \quad S \text{ and } f \text{ have a common fixed point } fv \text{ provided } fv \text{ is a fixed point of } f, \text{ and } f \text{ and } S \text{ commute at } v,$$

$$(2b^*) \quad T \text{ and } g \text{ have a common fixed point } gw \text{ provided } gw \text{ is a fixed point of } g \text{ and } T \text{ and } g \text{ commute at } w,$$

$$(3^*) \quad S, T, f \text{ and } g \text{ have a common fixed point provided } (2a^*) \text{ and } (2b^*) \text{ both are true.}$$

The above corollary follows from Theorem 1 by noting that (3.9) implies (3.2) and (3.3), while (3.4) becomes redundant. In view of these facts, the following is a slightly more general and sharpened version of the above corollary with $f = g$.

COROLLARY 2. Let $S, T : Y \rightarrow CL(X)$ and $f : Y \rightarrow X$ be such that (3.4) (with $f = g$) and the following hold.

$$(3.11) \quad S(Y) \cup T(Y) \subseteq f(Y),$$

$$(3.12) \quad H(Sx, Ty) \leq qM(Sx, Ty; f) \text{ for each } x, y \in Y \text{ and some } q \in (0, 1), \text{ and}$$

$$(3.13) \quad \text{one of } S(Y), T(Y) \text{ or } f(Y) \text{ is } (S, T; f)\text{-orbitally complete with respect to a point } x_0.$$

Then S, T and f have a coincidence point v in Y . Further, if $Y = X$, then S, T and

f have a common fixed point provided *f v* is a fixed point of *f*, and *f* commutes with each of *S* and *T* at *v*.

COROLLARY 2 BIS. *Corollary 2 with $Y = X$, f continuous, and (3.13) replaced by X is $(S, T; f)$ -orbitally complete with respect to a point x_0 .*

COROLLARY 3. *Corollary 2 without (3.4) and M replaced by m .*

COROLLARY 3 BIS. *Corollary 2 bis without (3.4) and M replaced by m .*

COROLLARY 4. *Let $T : Y \rightarrow CL(X)$ and $f : Y \rightarrow X$ be such that $T(Y) \subseteq f(Y)$ and $H(Tx, Ty) \leq qm(Tx, y; f)$ for each $x, y \in Y$ and some $q \in (0, 1)$. If one of $T(Y)$ or $f(Y)$ is (T, f) -orbitally complete, then there exists a $v \in Y$ such that $f v \in Tv$. Further, if $Y = X$, $f(f v) = f v$ and T and f commute at v , then $f v \in Tf v$.*

PROOF. The proof follows from Corollary 3 when $S = T$. □

We remark that when all the mappings are single-valued in the above results, then, as expected, we have a unique common fixed point under relaxed conditions. For example, we present one such result which may be derived from Theorem 4 bis or can be proved on the lines of the proof of Theorem 1. Let J be the family of maps ϕ from \mathbb{R}^+ to itself such that ϕ is semi-continuous from the right and $\phi(t) < t$ ($t > 0$), or nondecreasing and $\lim_n \phi^n(t) = 0$ ($t \geq 0$) (refer to Jachymski [7] for an excellent discussion on the equivalency of these two conditions on ϕ).

COROLLARY 5. *Let $S, T, f, g : Y \rightarrow X$ be such that (3.8), (3.10) and the following hold:*

(3.14)

$$d(Sx, Ty) \leq \phi \left(\max \left\{ d(fx, Sx), d(gy, Ty), \frac{1}{2} [d(fx, Ty) + d(gy, Sx)], d(fx, gy) \right\} \right)$$

for all $x, y \in Y$ and some $\phi \in J$.

Then:

- (i) *S and f have a coincidence,*
- (ii) *T and g have a coincidence.*

Further, if $Y = X$, $Sf u = f Su$, $u \in C(S, f)$ and $Tgv = gTv$, $v \in C(T, g)$, then S, T, f and g have a unique common fixed point.

Our results (theorems and corollaries) extend, generalize, improve and unify a number of results in the existing single-valued, multi-valued and hybrid fixed point theory (see for instance, [3, 5, 7, 8, 10–15, 17] and references thereof). We mention a few here. [3, Theorem 3.1] is substantially improved by Corollary 3 with $Y = X$ and $f =$ the identity mapping on X . Corollary 5 is a substantially improved version of a major theorem of Jachymski [7, Theorem 3.3], which in turn, improves a theorem of Jungck (see [8, Theorem 3.1]), Corollary 4 presents a slightly more general version of the main results of Singh and Mishra [17, Theorem 2 and Corollary 1], which, in turn, represent an improvement over Pathak's main result [12, Theorem 2], Corollary 2 is a slight improvement over the main result of Singh *et al.* [14, Theorem 2.1]. Several coincidence and fixed point theorems surveyed in [5] may be obtained as special cases of our results.

Finally, the following example shows that [9, Corollary 4.3] in its present form (wherein the space X is complete) cannot be true unless one of the mappings in question is continuous.

EXAMPLE 4. Let $X = [-\frac{1}{4}, \frac{1}{4}]$ with the absolute value metric, and $Tx = x/3$ for all $x \in X$,

$$fx = \begin{cases} (x - 2)/9, & x \leq 0 \\ (x + 2)/9, & x > 0. \end{cases}$$

Then $|Tx - Ty| \leq \frac{1}{3}|fx - fy|$ for each $x, y \in X$ and $|Tfx - fTx| \leq |Tx - fx|$ for all $x \in X$, that is T and f are weakly commuting on X . Therefore, conditions [9, (4.2) and (4.3)] of the corollary in [9] are satisfied for $S = T$, $g = f$, $\phi(t) < t/3$ but T and f do not have a common fixed point in X . Notice also that all the conditions of Corollary 5 with $Y = X$, $S = T$ and $g = f$ are satisfied except (3.8), since $T(X)$ is not contained in $f(X)$. Besides Corollary 5 with $Y = X$ (wherein, among other improvements, the completeness of the space has been replaced by a set of weaker conditions), several correct versions of [9, Corollary 3.4] exist in the literature (see for instance, [7, Theorem 3.3] and [8, Theorem 3.1]).

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