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109.12 A natural occurrence of the 3:4:5 triangle in the truncated regular octahedron

The central angle of any Platonic or Archimedean solid can be defined as the angle subtended at its centre by an edge. We will show that the central angle of the truncated octahedron is the most acute angle of the 3:4:5 triangle.

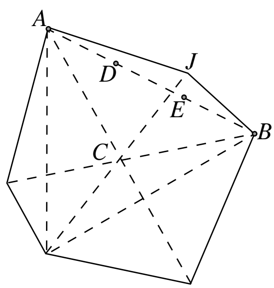

FIGURE 1

The regular octahedron has eight faces and six vertices. Its faces are equilateral triangles, four of which meet at each vertex. The octahedron can be seen as eight congruent triangular pyramids whose apexes come together at its centre. In Figure 1, $ABCJ$ is one such example. As $\angle ACB$ is a right angle and given a unit radius, the ratio of the lengths of the sides, $AC : BC : AB$ is $1 : 1 : \sqrt{2}$.

In Figure 2, the triangle *ABJ* from Figure 1 appears horizontal. To truncate an octahedron, each edge must be trisected. The length *DE* is one third of *AB*. The two points such as *D* and *E* on each edge become vertices of the truncated solid. Then the six pyramids such as *EFGHB* at the vertices of the octahedron are cut off. The angle $\angle DCE$, subtended by the edge *DE* at the centre *C*, is an example of the central angle of the truncated octahedron.

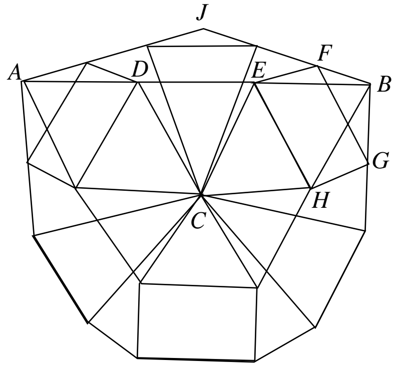

FIGURE 2

Consider triangle *ABC* in Figure 1. The perpendicular from *C* to the midpoint of *AB* will have a length of

$$\sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2}.$$

The triangle *ABC* shares its perpendicular with triangle *DEC*. *DE* has a length of $\sqrt{2}/3$. We can use Pythagoras' again to find the length of *DC*. We need half of *DE*, $\sqrt{2}/6$, and the perpendicular, $\sqrt{2}/2$. Then

$$DC = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{6}\right)^2} = \frac{\sqrt{5}}{3}.$$

So the sides of the isosceles triangle *DEC* are in the ratio $\frac{\sqrt{5}}{3} : \frac{\sqrt{5}}{3} : \frac{\sqrt{2}}{3}$ or $5 : 5 : \sqrt{10}$.

In general, given any isosceles triangle with sides in the ratio a, a, b, we may drop a perpendicular from a base vertex to the opposite side and form a right-angled triangle whose hypotenuse will have length a. Let the height of the perpendicular be h and the length of the other side be s. Figure 3 shows this procedure applied to triangle *DEC*, with the foot of the perpendicular denoted by *K*. Note that $\angle DCK$ is the same as $\angle DCE$, the central angle of the truncated octahedron. By applying Pythagoras' Theorem in the two right angled triangles, we may determine h and s in terms of a and b.

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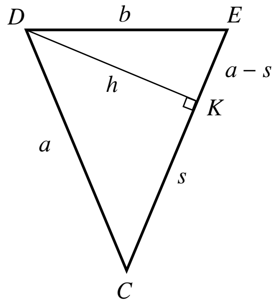

FIGURE 3

We have $h^2 + s^2 = a^2$ and $h^2 + (a - s)^2 = b^2$ which yields $h = \frac{b}{2a}\sqrt{4a^2 - b^2}$ and $s = \frac{1}{2a}\left(2a^2 - b^2\right)$.

Substituting $a = 5$, $b = \sqrt{10}$, we obtain $h = 3$, $s = 4$, giving $3 : 4 : 5$ as the ratio of the lengths of the sides of the right-angle triangle DKC.

It follows that the central angle of the truncated octahedron, DCE, is the smallest angle of the $3 : 4 : 5$ triangle.

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109.13 Apollonius' theorem for *n*-gons

Let ABC be a triangle and M be the midpoint of BC as shown.

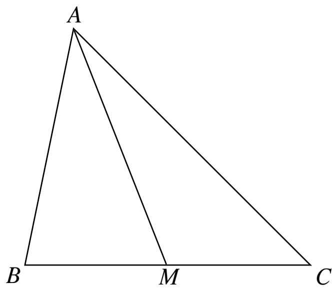

FIGURE 1

Apollonius' theorem states that $|AB|^2 + |AC|^2 = 2\left(|AM|^2 + |AM|^2\right)$. Substituting $|BM| = \frac{1}{2}|BC|$, we can derive an alternative form of Apollonius' theorem:

$$4|AM|^2 - 2\left(|AB|^2 + |AC|^2\right) + |BC|^2 = 0.$$