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ON THE HOMOLOGY OF THE n-SPECIAL REDUCED PRODUCT SPACE OF A EUCLIDEAN SPACE

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§1. **Introduction.** In [2] and [3] the homology of reduced product spaces of certain type of polyhedra was studied. Let $X^n = X \times X \times \cdots \times X$ be the Cartesian product of n copies of a topological space X. Let $T = \{1, t, t^2, \ldots, t^{n-1}\}$ be the cyclic group of order n acting on X^n as:

$$t(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, x_1).$$

Let us denote a point in X^n by $\bar{x}$. The *fat diagonal* $F_n(X)$ of X^n is defined as: $F_n(X) = \{\bar{x} \in X^n \mid t^i\bar{x} = \bar{x}$ for some $i \not\equiv 0 \bmod n\}$. The *n-special reduced product space* $X_*^n = X^n - F_n(X)$.

In this paper we investigate the homology of X_*^n, where $X = R^k$, the k-dimensional Euclidean space.

§2. **The main theorem.** Let n be an integer such that $n = P_1P_2 \cdots P_m$ where P's are distinct prime numbers. We consider $R_*^n = R^n - F_n(R)$. Let $t: R^n \to R^n$ be considered as an orthogonal linear transformation. Because t has a cyclic vector [1, p. 199] the minimal polynomial of t is the characteristic polynomial. Hence from the primary decomposition theorem and rational decomposition theorem we may find invariant subspaces $W_0, W_1, \ldots, W_m, V$ such that

$$R^n = W_0 \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_m \oplus V,$$

where W_0 is one-dimensional and $t \mid W_0$ is the identity. W_i is $(P_i - 1)$-dimensional with zero as the only fixed point with respect to $\{t, t^2, \ldots, t^{P_i-1}\}$, and $W_0 \oplus W_{i(1)} \oplus W_{i(2)} \oplus \cdots \oplus W_{i(k)}$, where $\{i(1), i(2), \ldots, i(k)\}$ is a subset of $\{1, 2, \ldots, m\}$, is the set of vectors fixed under $t^{P_{i(1)}P_{i(2)} \cdots P_{i(k)}}$ and finally, V is an invariant subspace of even dimension $2l = n - \sum_{i=1}^m (P_i - 1) - 1$ for which zero is the only fixed point. If P_i is odd, then W_i may be further decomposed into a direct sum of planes $H_{i,j}$, $j = 1, 2, \ldots, m(i)$, on each of which t is a rotation of order P_i, and if P_i is even, W_i is a line. Decompose V into a direct sum of planes K_h, $h = 1, 2, \ldots, l$, on each of which t is a rotation of order n. Then R^n may be mapped homeomorphically into the compact polyhedron C in R^n where each $H_{i,j}$, K_h goes onto the interior of a $(P_i - 1)$-gon $\tilde{H}_{i,j}$ (if P_i is even for some i, let $\tilde{H}_{i,1}$ denote the closed

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[1] **Added in proof.** Using the results in this paper, one of the authors has obtained the homology of R_n^* completely in [M. Wakae, *On the homology of the n-special reduced product space of a Euclidean space, II*, Kaigaku Kinen Ronbunshu, Soka Univ. (1971), 642–645].

interval $I=[-1, 1]$ so that W_i goes into $\tilde{H}_{i,1}$) and an n-gon $\tilde{K}$. Thus, R^n is mapped homeomorphically onto the interior of $C=I\times\prod_{i,j}\tilde{H}_{i,j}\times\tilde{K}^l$. Let $\tilde{H}_i=\prod_{j=1}^{m(i)}\tilde{H}_{i,j}$, so $C=I\times\prod_{i=1}^{m}\tilde{H}_i\times\tilde{K}^l$. The set of fixed points in C with respect to T is denoted by F_c and consists of the vectors of the form $\bar{x}=(x_0, x_1, x_2, \ldots, x_m, 0)$ where $x_0\in[-1, 1]$, $x_i\in\tilde{H}_i$, $i=1, 2, \ldots, m$, with at least one $x_i=0$, and $0\in\tilde{K}^l$. Notice that R^n_* is mapped homeomorphically into $C-F_c$ and that

$$H_*(R^n_*; G)\cong H_*(C-F_c; G).$$

Let C^* be the cell complex formed from C by taking those faces of C which do not intersect F_c.

LEMMA 1. *C^* is a deformation retract of $C-F_c$.*

Proof. Assume that $\tilde{H}_{i,j}$, and $\tilde{K}$ are given obvious simplicial structure with 0 as a vertex in each except I. The vertices of I are $v_{11}=(-1, 0, \ldots, 0)$ and $v_{1,2}=(1, 0, \ldots, 0)$ and index the vertices of C in F_c as follows: Firstly, we divide the set of the vertices of C in F_c into m categories. In the following sentences Cat stands for category. $\mathrm{Cat}(1)=\{v_{11}, v_{12}\}$. The vertices of $\mathrm{Cat}(k)$, $k=2, 3, \ldots, m$, are those vertices $\bar{x}$ in F_c (i.e., $\bar{x}=(x_0, x_1, x_2, \ldots, x_m, 0)$, where $x_0=-1$ or 1, $x_i\in\tilde{H}_i$ for $i>0$, $0\in\tilde{K}^l$) which have exactly k components x_i nonzero in the decomposition $I\times\tilde{H}_1\times\tilde{H}_2\times\cdots\times\tilde{H}_m\times\tilde{K}^l$. Order the vertices in F_c such that the vertices of $\mathrm{Cat}(k)$ preceed those of $\mathrm{Cat}(k+1)$. Let $\{\bar{v}_{k1}<\cdots<\bar{v}_{ks(k)}\}$ be the vertices of $\mathrm{Cat}(k)$. Define for each $\bar{x}\in C-F_c$ the cell $C(\bar{x})$ containing $\bar{x}$ which is minimal with respect to this property of containing $\bar{x}$, that is, if D is a cell containing $\bar{x}$, then $C(\bar{x})$ is a face of D. Let $P_{k,q}\colon(C-F_c)\to(C-F_c)$ be defined as follows:

(i) if $\bar{v}_{kq}\notin C(\bar{x})$, then $P_{k,q}(\bar{x})=\bar{x}$

(ii) if $\bar{v}_{kq}\in C(\bar{x})$, then $P_{k,q}(x)$ is the foot of the projection of $\bar{x}$ along the radius ray $|\bar{x}, \bar{v}_{kq}|$ into the face of $C(\bar{x})$ not containing $\bar{v}_{kq}$.

Let $r_k=P_{k,s(k)}\circ\cdots\circ P_{k,1}$ and $r=r_m\circ\cdots\circ r_1$. By the construction of $P_{k,q}$, $C(r_k\circ r_{k-1}\circ\cdots\circ r_1(\bar{x}))$ does not contain the vertices of $\mathrm{Cat}(j)$ for $j\leq k$. Hence r takes $C-F_c$ into C^*. Notice that if $C(P_{k+1,j}\circ\cdots\circ P_{k+1,1}\circ r_k\circ\cdots\circ r_1(\bar{x}))$ contains vertices of $\mathrm{Cat}(k+1)$, they are all such that their nonzero components lie in precisely the same $\tilde{H}_i$'s because, if not, the above cell would contain at least one vertices of $\mathrm{Cat}(q)$ where $q\leq k$, which is a contradiction. Hence we may define a homotopy

$$h_{k,q}\colon(C-F_c)\times I\to(C-F_c)$$

by

$$h_{k,q}(\bar{x}, t)=(1-t)(P_{k,q-1}\circ\cdots\circ P_{k,1}\circ r_{k-1}\circ\cdots\circ r_1(\bar{x}) +t(P_{k,q}\circ\cdots\circ P_{k,1}\circ r_{k-1}\circ\cdots\circ r_1)(\bar{x})$$

where $k, q\geq 2$.

$$h_{1,1}(\bar{x}, t)=(1-t)\bar{x}+tP_{1,1}(\bar{x})$$
$$h_{1,2}(\bar{x}, t)=(1-t)P_{1,1}(\bar{x})+t(P_{1,2}\circ P_{1,1})(\bar{x})$$
$$h_{k,1}(\bar{x}, t)=(1-t)(r_{k-1}\circ\cdots\circ r_1)(\bar{x})+t(P_{k,q}\circ r_{k-1}\circ\cdots\circ r_1)(\bar{x})$$

where $k\geq 2$.

Then we define a homotopy $h:(C-F_c)\times I\to(C-F_c)$ by

$$h = h_{m,s(m)} * \cdots * h_{m,1} * \cdots * h_{2,s(2)} * \cdots * h_{2,1} * h_{1,2} * h_{1,1},$$

where $*$ refers to the path product. It is clear that h is the desired homotopy from the identity to r. This completes the proof of the Lemma.

Let us denote $\tilde{K}^l$ by $\tilde{K}_1\times\cdots\times\tilde{K}_l$ where each $K_j=K$. Since dim $C^*=n-1$ and

$$C^* = \left(\bigcup_{i=1}^{l} I\times\tilde{H}_1\times\cdots\times\tilde{H}_m\times\tilde{K}_1\times\cdots\times\dot{\tilde{K}}_i\times\cdots\times\tilde{K}_l\right)$$
$$\cup\,(I\times\dot{\tilde{H}}_1\times\cdots\times\dot{\tilde{H}}_m\times\tilde{K}^l)$$

where dot stands for the boundary, we have $C^*=I\times C^{**}$, where C^{**} is a subcomplex of $\tilde{H}_1\times\cdots\times\tilde{H}_m\times\tilde{K}^l$ of dimension $n-2$. Hence

$$H_i(C^*) = H_0(I)\otimes H_i(C^{**}) = H_i(C^{**}).$$

Let

$$A = \tilde{H}_1\times\cdots\times\tilde{H}_m\times\dot{\tilde{K}}^l \quad\text{and}\quad B = \dot{\tilde{H}}_1\times\cdots\times\dot{\tilde{H}}_m\times\tilde{K}^l.$$

Then

$$C^{**} = A\cup B \quad\text{and}\quad A\cap B = \dot{\tilde{H}}_1\times\cdots\times\dot{\tilde{H}}_m\times\dot{\tilde{K}}^l.$$

It is clear that $H_i(A\cap B;G)=0$ for $i\geq\sum_{i=1}^m(P_i-2)+2l$ and $H_i(A\cap B)\neq 0$ for $i=\sum_{i=1}^m(P_i-2)+2l-1$. Also we have

$$H_i(A)\cong H_i(\dot{\tilde{K}}^l) \quad\text{and}\quad H_i(B)\cong H_i(\dot{\tilde{H}}_1\times\cdots\times\dot{\tilde{H}}_m).$$

Thus by the Mayer-Vietoris sequence

$$\cdots\to H_i(A\cap B)\to H_i(A)\oplus H_i(B)\to H_i(A\cup B)\to H_{i-1}(A\cap B)\to\cdots$$

we have

$$H_i(A\cap B)\cong H_{i+1}(A\cup B)$$

for $i>\max(2l-1,\sum_{i=1}^m(P_i-2))+1=2l$.

Hence

$$H_i(C^*)\cong H_i(C^{**}) = 0 \quad\text{for}\quad i\geq\sum_{i=1}^m(P_i-2)+2l+1$$

$$H_i(C^*)\cong H_i(C^{**}) \neq 0 \quad\text{for}\quad i=\sum_{i=1}^m(P_i-2)+2l$$

Using $2l=n-\sum_{i=1}^m(P_i-1)-1$, Lemma 1 and the fact $H_*(R^n_*;G)\cong H_*(C-F_c;G)$, we have the following theorem:

THEOREM 1.

$$H_i(R^n_*;G) = 0 \quad\text{for}\quad i\geq n-m$$
$$H_i(R^n_*;G) \neq 0 \quad\text{for}\quad i=n-m-1.$$

§3. **Some remarks.** Let $X = R^k$ and let T act on X^n as in §1 where

$$n = P_1 P_2 \cdots P_m$$

as in §2. Then by a similar argument as that in §2, we may prove the following theorem.

THEOREM 2.

$$H_i(X^n_*; G) = 0 \quad \text{for} \quad i \geq kn - k + 1 - m.$$

$$H_i(X^n_*; G) \neq 0 \quad \text{for} \quad i = kn - k - m.$$

Let $\quad n = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_m^{\alpha_m}, \quad n'' = P_1 P_2 \cdots P_m, \quad$ and $\quad n' = n/n''.$

Let $T = \{1, t, \ldots, t^{n-1}\}$ and $T' = \{1, t^{n'}, t^{2n'}, \ldots, t^{(n''-1)n'}\}$. If T acts on X^n as in §1, then T' acts on $Y^{n''}$ where $Y = X^{n'}$.

LEMMA 2. *Let α be the smallest positive integer in $\{1, 2, \ldots, n-1\}$ such that $t^\alpha(\bar{x}) = \bar{x}$ for some $\bar{x} \in X^n$, then α divides n.*

Proof. Suppose α does not divide n. Then there exists positive integers a and b with $\alpha > b$ such that $n = a\alpha + b$. Then $\bar{x} = t^n(\bar{x}) = t^b(\bar{x})$. This is a contradiction to the assumption that α is the smallest positive integer such that $t^\alpha(\bar{x}) = \bar{x}$.

LEMMA 3. *$F_n(X) = F_{n''}(X^{n'})$ as subspaces of X^n.*

Proof. That $F_n(X) \supseteq F_{n''}(X^{n'})$ is trivial since $T' \subset T$.

Let $\bar{x} \in F_n(X)$. Then there exists α in $\{1, 2, \ldots, n-1\}$ such that $t^\alpha(\bar{x}) = \bar{x}$. We may assume that α is the smallest integer satisfying the above condition. Then by Lemma 2, $\alpha = P_1^{b_1}, \ldots, p_m^{b_m}$ where $b_i \leq \alpha_i$ for $i = 1, \ldots, m$. If $b_i \leq \alpha_i - 1$ for $i = 1, 2, \ldots, m$, then α divides n'. Thus $t^{n'}(\bar{x}) = \bar{x}$. Hence $\bar{x} \in F_{n''}(X^{n'})$. In the other case, by rearranging the order of $P_1, \ldots, P_m$ if necessary, we may assume that

$$b_j = \alpha_j \qquad j = 1, \ldots, l$$

$$b_j \leq \alpha_j - 1 \qquad j = l+1, \ldots, m.$$

where l is strictly less than m since $\alpha \leq n-1$. Thus

$$\alpha = P_1^{\alpha_1} \cdots P_l^{\alpha_l} P_{l+1}^{b_{l+1}} \cdots P_m^{b_m}$$

Therefore $t^{P_1 \cdots P_l n'}(\bar{x}) = \bar{x}$. Since l is strictly less than m,

$$t^{P_1 \cdots P_l n'} \neq 1 \quad \text{and} \quad t^{P_1 \cdots P_l n'} \in T'.$$

Hence $x \in F_{n''}(X^{n'})$.

THEOREM 3. *If* $n=P_1^{\alpha_1}\cdots P_m^{\alpha_m}$, *then*

$$H_i(R_*^n; G) = 0 \quad \textit{for} \quad i \geq n-n'+1-m$$
$$H_i(R_*^n; G) \neq 0 \quad \textit{for} \quad i = n-n'-m.$$

Proof. Let $Y=N^{n'}$. Then by Lemma 2, $R_*^n = Y_*^{n''}$. Hence by Theorem 2

$$H_i(R_*^n; G) = H_i(Y_*^{n''}; G) = 0 \quad \text{for} \quad i \geqq n'n''-n'+1-m = n-n'+1-m$$

and

$$H_i(R_*^n; G) = H_i(Y_*^{n''}; G) \neq 0 \quad \text{for} \quad i = n'n''-n'-m = n-n'-m$$

COROLLARY 1. *If* $X=R^k$, $n=P_1^{\alpha_1}\cdots P_m^{\alpha_m}$, $n''=P_1\cdots P_m$, *and* $n'=n/n''$ *then*

$$H_i(X_*^n; G) = 0 \quad \textit{for} \quad i \geq nk-n'k-m+1$$

and

$$H_i(X_*^n; G) \neq 0 \quad \textit{for} \quad i = nk-n'k-m$$

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