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AN INTERMEDIATE VALUE PROPERTY FOR OPERATORS WITH APPLICATIONS TO INTEGRAL AND DIFFERENTIAL EQUATIONS

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1. Introduction.

It is well known that a real valued continuous function f on a closed interval S assumes every value between its maximum and minimum on S, i.e. if ξ is such that $f(\alpha) \leqq \xi \leqq f(\beta)$ then there exists γ between α and β such that $f(\gamma) = \xi$. The purpose of this paper is to develop the existence theory associated with differential and integral inequalities in the context of an intermediate value property for operators on partially ordered spaces. This has the advantage of allowing rather simple proofs of known results while in most cases giving slight improvements, and in some cases substantial improvements, in these results. Classical and recent results from different areas are unified under one principle.

In Section 2 we define the intermediate value property (I) and state a number of examples of operators with the property (I). We also present, in Lemma 2.2, our main device for the elementary extension of these results to more general situations. In Section 3 we give an existence theory for solutions to initial and boundary value problems for functional differential equations based on results for ordinary differential equations due to Perron [**20**], Jackson and Schrader [**13**] and Schmitt [**24**]. Such extensions have hitherto often involved the use of sophisticated fixed point theorems as in the papers of Schmitt. Two types of result are obtained for nth order boundary value problems; Corollary 3.2.1 is a result for certain three point problems based upon a basic result of Jackson and Schrader while Theorem 3.4 is a result for multipoint problems based upon Example 2.4 which is a moderate extension of a known result [**5**, p. 109]. Theorem 3.3 is an existence theorem for periodic solutions of nonlinear functional differential equations, which generalizes a previous result of Schmitt [**27**] obtained only for linear equations. Section 4 treats certain integral equations and inequalities in the context of property (I).

2. The intermediate value property (I).

Definition. Let X and Y be partially ordered spaces and let $H : X \to Y$. If, for each $\alpha, \beta \in X$ and $\xi \in Y$ such that $\alpha \leqq \beta$ and $H\alpha \leqq \xi \leqq H\beta$, there exists $\gamma \in X$ such that $H\gamma = \xi$ and $\alpha \leqq \gamma \leqq \beta$, then H has the intermediate value property (I).

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In what follows let $M(S)$, $L^1(S)$, $AC^{(k)}(S)$, $C^{(k)}(S)$ denote the real valued functions which are measurable, have finite Lebesgue integrals, have absolutely continuous kth derivatives and have continuous kth derivatives on S, respectively. The prefix "loc" means the identifying property needs to hold only locally on the given set. Componentwise partial ordering is assumed throughout for product spaces and the partial ordering of real function spaces is the usual order holding pointwise on the domain of the functions except when specified otherwise.

Example 2.1 (Perron [**20**]). Let $X = \text{loc}\, AC[0, h)$, $Y = [\text{loc}\, L^1(0, h)] \times \mathbf{R}$. If $f(t, x)$ is a real valued function which satisfies the Carathéodory conditions locally in $[0, h) \times \mathbf{R}$, then $H : X \to Y$, where

$$(Hx)(t) = (x'(t) - f(t, x(t)), x(0)),$$

has property (I).

This is a somewhat sharper result than that usually found in textbooks on ordinary differential equations which states that solutions to initial value problems for differential inequalities are bounded by extremal solutions of the corresponding differential equations. A proof, which is essentially due to Perron, can be found in the book of Coppel [**4**, p. 31].

Example 2.2 (Jackson and Schrader [**13**]). Let $X = C[0, 1] \cap \text{loc}\, AC^{(1)}(0, 1)$, $Y = [\text{loc}\, L^1(0, 1)] \times \mathbf{R}^2$. If $f(t, x)$ is continuous on $[0, 1] \times \mathbf{R}$, then $H : X \to Y$, where

$$(Hx)(t) = (f(t, x(t)) - x''(t), x(0), x(1)),$$

has property (I).

This result may be formulated for $f(t, x(t), x'(t)) - x''(t)$ provided $f(t, x, x')$ satisfies an appropriate Nagumo condition in x' (cf. [**13**]), or some other conditions which restrict the growth of the first derivatives of solutions. Also, Erbe [**7**] has shown that the boundary values $(x(0), x(1))$ in Hx may be replaced by expressions of the form $(\mu(x(0), x'(0)), \nu(x(1), x'(1)))$. The general results of Schmitt of the type illustrated by the following example also involve a Nagumo condition and other restrictions on the derivatives of solutions; only the simplest case is mentioned here.

Example 2.3 (Schmitt [**24**]). Let $T > 0$,

$$X = \{x \in C^2(-\infty, \infty) : x(t + T) = x(t)\},$$
$$Y = \{x \in C(-\infty, \infty) : x(t + T) = x(t)\}.$$

If $f(t, x)$ is continuous on $\mathbf{R}^2$ and T-periodic in t for each x, then $H : X \to Y$ has property (I) where

$$(Hx)(t) = f(t, x(t)) - x''(t).$$

Example 2.4. Suppose $\{t_0, \ldots, t_m\}$ is a subset of $m + 1$ distinct points of $[0, 1]$ with $t_0 = 0$ or 1 when $m = 0$ and $t_0 = 0$, $t_m = 1$ when $m \geqq 1$, $r_0, \ldots, r_m$ are positive integers such that $r_0 + \ldots + r_m = n$ and

$$P(t) = (t - t_0)^{r_0} \ldots (t - t_m)^{r_m}.$$

Let $X_{\mu m}$ be that subset of $AC^{(n-1)}[0, 1]$ such that

$$(2.1) \qquad x^{(k)}(t_j) = \mu_{jk};\ k = 0, \ldots, r_j - 2;\ j = 0, m;\ \text{and}$$
$$k = 0, \ldots, r_j - 1,\ j = 1, \ldots, m - 1 \quad (m > 1).$$

Let $Y_0 = L^1[0, 1] \times \mathbf{R}$ and $Y_m = L^1[0, 1] \times \mathbf{R}^2$ if $m \geqq 1$. The partial order on $X_{\mu m}$ is $x \succ 0$ if $P(t)x(t) \geqq 0$. If $a_1, \ldots, a_m \in L^1[0, 1]$,

$$Lx = x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x,$$

$Lx = 0$ is disconjugate on $[0, 1]$ and $H_m : X_{\mu m} \to Y_m$ is defined by

$$(H_0x)(t) = ((Lx)(t), (-1)^{t_0}x^{(n-1)}(t_0))$$
$$(H_mx)(t) = ((Lx)(t), P^{(r_0)}(t_0)x^{(r_0-1)}(t_0), -x^{(r_m-1)}(t_m)), \quad m \geqq 1,$$

then H_m has property (I).

The proof is based on the following lemma.

LEMMA 2.1. *If $Lz = f \geqq 0$, $f \in L^1[0, 1]$, $z \in X_{om}(\mu_{jk} = 0$ in (2.1)) and*

$$(2.2) \qquad \begin{cases} (-1)^{t_0}z^{(n-1)}(t_0) \geqq 0, \text{ if } m = 0, \\ P^{(r_0)}(0)z^{(r_0-1)}(0) \geqq 0,\ z^{(r_m-1)}(1) \leqq 0 \text{ if } m \geqq 1, \end{cases}$$

then $P(t)z(t) \geqq 0$ (i.e., $z \succ 0$).

In the special case that the inequalities in (2.2) are replaced by equalities this result is well-known [**2**, p. 143] ($m = 0$), [**5**, p. 109] ($m \geqq 1$).

Proof. The result follows from

$$z(t) = u(t) + \int_0^1 k(t, \tau)f(\tau)d\tau$$

where $x = u(t)$ is that solution of $Lx = 0$ such that $x^{(k)}(t_j) = z^{(k)}(t_j)$, $k = 0, \ldots, r_j - 1$, $j = 0, \ldots, m$ and $k(t, \tau)$ is the Green's function for the operator L with homogeneous boundary conditions

$$x^{(k)}(t_j) = 0, \quad k = 0, \ldots, r_j - 1; \quad j = 0, \ldots, m.$$

In case $m = 0$, $k(t, \tau) = 0$ for $\tau > t$ and k is usually referred to as the Cauchy function. In any case, $P(t)u(t) \geqq 0$ since if $P(s)u(s) < 0$ for some $s \in [0, 1]$ then u is a nontrivial solution with n zeros, counting multiplicities, in $[0, 1]$ contradicting the disconjugacy of $Lx = 0$ on $[0, 1]$. Furthermore,

$$P(t)k(t, \tau) \geqq 0, \quad 0 \leqq t, \quad \tau \leqq 1,$$

by [**5**, p. 106, Lemma 14]. Thus,

$$P(t)z(t) = P(t)u(t) + P(t)\int_0^1 k(t,\tau)f(\tau)d\tau \geqq 0.$$

Now suppose $L\alpha \leqq \xi \leqq L\beta, \alpha, \beta \in X_{\mu m}, \xi \in L^1[0, 1], \alpha \prec \beta$ (i.e., $P(t)(\beta(t) - \alpha(t)) \geqq 0)$ and

$$(-1)^{t_0}\alpha^{(n-1)}(t_0) \leqq (-1)^{t_0}\nu_0 \leqq (-1)^{t_0}\beta^{(n-1)}(t_0), \text{ if } m = 0;$$
$$P^{(r_0)}(t_0)\alpha^{(r_0-1)}(t_0) \leqq P^{(r_0)}(t_0)\nu_0 \leqq P^{(r_0)}(t_0)\beta^{(r_0-1)}(t_0), \quad \text{and}$$
$$\alpha^{(r_m-1)}(t_m) \geqq \nu_m \geqq \beta^{(r_m-1)}(t_m), \text{ if } m \geqq 1.$$

Let γ be the unique solution in $X_{\mu m}$ of $Lx = \xi$ such that

$$x^{(n-1)}(t_0) = \nu_0, \text{ if } m = 0$$
$$x^{(r_j-1)}(t_j) = \nu_j, \text{ if } m \geqq 1, \ (j = 0, m).$$

Then $z = \beta - \gamma$ satisfies $Lz \geqq 0, z \in X_{om}$, and (2.2), so that Lemma 2.1 implies $\beta - \gamma \succ 0$. A similar argument proves that $\gamma - \alpha \succ 0$.

The sign of the Green's function $k(t, \tau)$ for multi-point boundary value problems for $Lx = 0$ was determined by Levin [**17**] and Pokornyi [**21**].

Example 2.5 (Kantorovich [**15**]). Let X be a partially ordered topological space and $T : X \to X$ be nondecreasing. If

$$\alpha \leqq T\alpha \leqq T\beta \leqq \beta$$

then

$$T\alpha \leqq T^n\alpha \leqq T^{n+1}\alpha \leqq T\beta, \qquad n = 1, 2, \ldots .$$

Furthermore, if bounded monotone nondecreasing sequences in X are convergent and T is continuous with respect to such sequences then $\gamma = \lim_{n\to\infty} T^n\alpha$ satisfies

$$\alpha \leqq \gamma \leqq \beta \quad \text{and} \quad \gamma = T\gamma.$$

Under these circumstances, if X is a linear space and J is the identity map, $J - T$ has the property (I).

Example 2.5 includes most of the results on integral inequalities of the Gronwall type. Special cases of this result have had independent rediscoveries. In the case that X is the space of Lebesgue measurable functions on an interval S it has been shown by Hanson and Waltman [**31**] that the operator $J - T$ of Example 2.5 has the intermediate value property even without the continuity hypothesis. This result also includes most of the known results on Gronwall-type inequalities and is in fact more general than Example 2.5 in its application to this type of problem. For example, suppose $g(t, x)$ and $f(t, \tau, x)$ are nondecreasing in x for each (t, τ) and

$$g(t, \int_S f(t, \tau, x(\tau))d\tau)$$

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exists, is measurable and satisfies

$$\psi(t) \leqq g\left(t, \int_S f(t, \tau, x(\tau))d\tau\right) \leqq \omega(t) \tag{2.3}$$

whenever $x(t)$ is measurable and $\psi(t) \leqq x(t) \leqq \omega(t)$ ($\psi(t)$ and $\omega(t)$ are measurable); then there exists a measurable function $\phi(t)$, $\psi(t) \leqq \phi(t) \leqq \omega(t)$ such that

$$\phi(t) = g\left(t, \int_S f(t, \tau, \phi(\tau))d\tau\right), \quad \text{a.e. in } S.$$

A special case covered by the preceding remarks which is useful for the discussion of the classical integral inequalities is due to Wajewski.

Example 2.6 (Wajewski [**29**]). Let $f(t, x)$, $g(t, x)$ be real valued functions on $\mathbf{R}^2$ such that $f(t, g(t, x))$ satisfies the Carathéodory conditions locally and, together with $f(t, x)$, is nondecreasing in x. If $x(t)$ is a measurable function such that

$$x(t) \leqq g\left(t, \int_0^t f(\tau, x(\tau))d\tau\right) < \infty, \quad t \in [0, T), \tag{2.4}$$

and $\theta^*(t)$ is the maximal solution of

$$\theta' = f(t, g(t, \theta)), \qquad \theta(0) = 0, \tag{2.5}$$

then

$$x(t) \leqq g(t, \theta^*(t))$$

on the intersection of the domains of α and θ^*.

Proof. Let

$$\alpha(t) = \int_0^t f(\tau, x(\tau))d\tau.$$

Then (2.4) implies

$$\alpha(t) \leqq \int_0^t f(\tau, g(\tau, \alpha(\tau))d\tau. \tag{2.6}$$

The hypotheses imply the existence of a constant $\beta \geqq \alpha(t)$ such that

$$\int_0^t f(\tau, g(\tau, \beta))d\tau \leqq \beta$$

on an interval $[0, T_\beta]$. From the result of Kantorovich (Example 2.5) or the result of Hanson and Waltman for (2.3) there exists a function $\gamma(t)$, $\alpha \leqq \gamma \leqq \beta$ such that

$$\gamma(t) = \int_0^t f(\tau, g(\tau, \gamma(\tau))d\tau, \quad t \in [0, T_\beta] \cap [0, T).$$

Clearly γ is locally absolutely continuous on its domain and $\gamma' = f(t, g(t, \gamma))$, $\gamma(0) = 0$; thus $\alpha \leqq \gamma \leqq \theta^*$. By a standard continuation argument $\alpha \leqq \theta^*$ on the intersection of the domains of these functions and so, from (2.4), $x(t) \leqq g(t, \theta^*(t))$.

As partially indicated by Wajewski, many of the classical integral inequalities such as those due to Gronwall [**11**], Reid [**22**], Bihari [**3**], Maroni [**19**], Li [**18**] and Gollwitzer [**8**] can be derived from Example 2.6. The actual upper bounds obtained in these inequalities are derived from equation (2.5) either by solving (2.5) for the maximal solution or by obtaining upper bounds for all solutions. Obtaining these bounds is actually a complete problem in itself and usually requires considerable ingenuity. In this regard, most of the solutions so far offered in the literature can be arrived at directly from the integral inequality (2.4) without resort to equation (2.5) at all. Hence, the solution of (2.5) given by Example 2.6 is not really needed for obtaining upper bounds (or lower bounds in the case of reversed inequalities). We will be considering the problem of obtaining explicit bounds in a subsequent paper.

The following lemma is the basis for extending existence theorems and estimates, such as those contained in Examples (2.1)–(2.4), to more general functional equations and inequalities in terms of property (I).

LEMMA 2.2. *Let X and Y be partially ordered topological spaces and $K : X \times X \to Y$. If, for each $z \in X$, $K(z, x) \leqq K(z, y)$ when $y \leqq x$ and $Hx = K(x, z)$ has property* (I), *then*

$$\alpha \leqq \beta \quad \text{and} \quad K(\alpha, \alpha) \leqq \xi \leqq K(\beta, \beta) \tag{2.7}$$

imply that there exists $\{z_n\} \subset X$ such that

$$\alpha \leqq z_1 \leqq z_2 \leqq \ldots \leqq \beta \quad \text{and} \quad K(z_{n+1}, z_n) = \xi. \tag{2.8}$$

Thus, for example, if bounded nondecreasing sequences $\{z_n\}$ converge in X (so that $z_n \uparrow z$) and K is continuous with respect to $\{(z_{n+1}, z_n)\}$ then $K(z, z) = \xi$ and $\alpha \leqq z \leqq \beta$. Hence $Mx = K(x, x)$ has property (I). Example 2.5 is a special case of Lemma 2.2 with $K(x, y) = Jx - Ty$.

Proof. Let (2.7) hold. Then $\alpha \leqq \beta$ implies $K(\beta, \beta) \leqq K(\beta, \alpha)$; hence

$$K(\alpha, \alpha) \leqq \xi \leqq K(\beta, \alpha),$$

which implies that there exists $z_1 \in X$ such that $K(z_1, \alpha) = \xi$ and $\alpha \leqq z_1 \leqq \beta$. But $K(z_1, \alpha) = \xi$ and $\alpha \leqq z_1$ imply $K(z_1, z_1) \leqq \xi$; hence

$$K(z_1, z_1) \leqq \xi \leqq K(\beta, \beta), \qquad \alpha \leqq z_1 \leqq \beta.$$

The argument can be repeated with z_1 replacing α and (2.8) follows by induction.

3. Applications to functional differential equations.

Let

$$f(t, x, y(\cdot)) : [0, h) \times \mathbf{R} \times X \to \mathbf{R}$$

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where X is the real function space specified in Theorem 3.1 below; X is partially ordered in the usual manner. Assume that, for each $y \in X$, $f(t, x, y(\cdot))$ satisfies the Carathéodory conditions locally in $[0, h) \times \mathbf{R}$ and, for each $(t, x) \in [0, h) \times \mathbf{R}$, $f(t, x, y(\cdot))$ is nondecreasing in y. All relationships are assumed to hold almost everywhere.

THEOREM 3.1. *Let*

$$\begin{aligned} X &= M(a, b) \cap C(a, h) \cap [\mathrm{loc}\, AC[0, h)], \\ Y &= M(a, b) \cap C(a, 0] \cap [\mathrm{loc}.L^1[0, h)], \\ -\infty &\leqq a \leqq 0 < h \leqq b \leqq \infty. \end{aligned}$$

Assume that X is topologized by pointwise convergence on $[h, b)$ and uniform convergence on compact subsets of (a, h). If $y_n, y \in X$ and $y_n \uparrow y$ implies $f(t, x, y_n(\cdot)) \to f(t, x, y(\cdot))$ uniformly in compact subsets of $[0, h) \times \mathbf{R}$ and

$$(Mx)(t) = \begin{cases} x(t), & t \in (a, 0], \\ x'(t) - f(t, x(t), x(\cdot)), & t \in (0, h), \\ x(t), & t \in [h, b), \end{cases}$$

then M has property (I) *on X.*

Proof. Let $\alpha, \beta \in X$ and $\xi \in Y$ satisfy

$$\alpha(t) \leqq \beta(t)$$
$$(M\alpha)(t) \leqq \xi(t) \leqq (M\beta)(t), \qquad t \in (a, b).$$

For $y \in X$ let

$$K(x, y)(t) = \begin{cases} x(t), & t \in (a, 0], \\ x'(t) - f(t, x(t), y(\cdot)), & t \in (0, h), \\ x(t), & t \in [h, b). \end{cases}$$

Example 2.1 implies that, for each $y \in X$, $Hx \equiv K(x, y)$ has property (I) over X; hence Lemma 2.2 implies that there exists a sequence $\{z_n\} \subset X$ such that

$$\alpha \leqq z_n \leqq z_{n+1} \leqq \beta \quad \text{and} \quad K(z_{n+1}, z_n) = \xi.$$

So there exists $z \in M(a, b)$ such that $z_n \uparrow z$, $z(t) = \xi(t)$, $t \in (a, 0] \cup [h, b)$ and

$$(3.1) \qquad z_{n+1}(t) = \xi(0) + \int_0^t [\xi(\tau) + f(\tau, z_{n+1}(\tau), z_n(\cdot))] d\tau, \quad t \in [0, h).$$

For each fixed $t \in [0, h)$

$$f(s, z_{n+1}(\tau), \alpha(\cdot)) \leqq f(\tau, z_{n+1}(\tau), z_n(\cdot)) \leqq f(\tau, z_{n+1}(\tau), \beta(\cdot))$$

so that $|f(\tau, z_{n+1}(\tau), z_n(\cdot))| \leqq m(\tau)$, $n = 1, 2, \ldots,$ where $m \in L^1[0, t]$; also

$\xi \in L^1[0, t]$. Thus $z_n(s)$ is uniformly convergent on $[0, t]$ and our hypotheses imply

$$\lim_{n\to\infty} f(\tau, z_{n+1}(\tau), z_n(\cdot)) = f(\tau, z(\tau), z(\cdot)).$$

From the Lebesgue dominated convergence theorem and (3.1) we conclude that

$$z(t) = \xi(0) + \int_0^t [\xi(\tau) + f(\tau, z(\tau), z(\cdot))]ds, \quad t \in [0, h).$$

Hence $z \in \operatorname{loc} AC[0, h)$ and

$$z'(t) = f(t, z(t), z(\cdot)) + \xi(t), \qquad t \in [0, h), z(0) = \xi(0).$$

Since $z(0) = \xi(0)$ we have $z \in C(a, h)$ and so $z \in X$ and $Mz = \xi$.

In case $f(t, x, y(\cdot)) = f(t, x, y(t - d))$, $d > 0$, Theorem 3.1 is an existence theorem for solutions to initial data problems for delay differential equations which is more restrictive than need be since $f(t, x, y(\cdot))$ is monotone in $y(\cdot)$. However, it is less restrictive in that $f(t, x, y(\cdot))$ is not required to be continuous in $y(\cdot)$. Thus, for example

$$z' = f(t, z) + g(\sup_{0\leqq s\leqq t} z(s)), \quad z(0) = z_0 \tag{3.2}$$

has a solution $z \in AC[0, h]$ for some $h > 0$ if $f(t, x)$ satisfies the Carathéodory conditions in a neighbourhood of $(0, z_0)$ and $g(x)$ is a left-continuous non-decreasing function in a neighbourhood of z_0. Also solutions to differential inequalities associated with (3.2) are bounded by a solution to (3.2). This remark is also established for equations of this type in [**23**].

Existence and uniqueness theorems as well as explicit bounds for solutions to functional differential equations of the type

$$x'(t) = f \circ x \circ h \circ x(t)$$

given by Dunkel [**6**] follow from Theorem 3.1.

It is of interest to note that Theorem 3.1 and Example 2.1 imply the classical result of Kamke [**14**] which states that a solution to the system of differential inequalities

$$x_i' \leqq g_i(t, x_1, \ldots, x_n), \qquad x_i(0) \leqq x_{i0}, \quad i = 1, \ldots, n$$

is bounded above by a solution to

$$x_i' = g_i(t, x_1, \ldots, x_n), x_i(0) = x_{i0}, \quad i = 1, \ldots, n,$$

on an interval $[0, h)$ provided $g_i(t, x_1, \ldots, x_n)$ satisfies a monotonicity condition of type K (cf. [**4**, p. 27]). It is not difficult to see from Example 2.1 that the operator Hx defined by

$$(Hx)_i(t) = (x_i'(t) - f_i(t, x_i(t)), x_i(0)), \qquad i = 1, \ldots, n,$$

has the property (I), so that, from Theorem 3.1, operators of the form

$$(Hx)_i(t) = (x_i'(t) - f_i(t, x_i(t), x(\cdot)), x_i(0)), \qquad i = 1, \ldots, n,$$

also have the property (I) under the appropriate monotonicity assumptions on f. The Kamke result will follow from there by choosing $f_i(t, x_i, y) = g_i(t, y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$.

Theorems 3.2, 3.3 and 3.4, which follow, on boundary value problems for functional differential equations, are immediate consequences of Examples 2.2, 2.3 and 2.4 by the use of Lemma 2.2.

THEOREM 3.2. *Let* $a < 0$, $1 < b$, $X = C(a, b) \cap [\mathrm{loc}\ AC^1(0, 1)]$, $Y = C(a, 0] \cap C[1, b) \cap L^1[0, 1]$. *Suppose that* $f(t, x, y(\cdot))$ *is continuous on* $[0, 1] \times \mathbf{R}$ *for each* $y \in X$, *nonincreasing in* y, *and that* y_n, $y \in X$, $y_n \uparrow y$ *implies* $f(t, x, y_n(\cdot)) \to f(t, x, y(\cdot))$ *uniformly on compact subsets of* $[0, 1] \times \mathbf{R}$. *If*

$$(Mx)(t) = \begin{cases} f(t, x(t), x(\cdot)) - x'', & t \in (0, 1) \\ x(t), & t \in (a, 0] \cup [1, b), \end{cases}$$

then M *has property* (I) *on* X.

COROLLARY 3.2.1. *Suppose* $g(t, x_1, \ldots, x_n) \in C([0, 1] \times \mathbf{R}^n)$,

$$\{2, \ldots, n\} = N_0 \cup N_1 \quad \textit{with} \quad N_0 \cap N_1 = \phi \quad \textit{and} \quad c \in [0, 1].$$

Suppose further that $(-1)^{\nu(k)} g(t, x_1, \ldots, x_{n+1})$ *is nonincreasing in* x_k, $k = 2, \ldots, n$, *and* $(t - c)(-1)^{\nu(n)} g(t, x_1, \ldots, x_{n+1})$ *is nonincreasing in* x_{n+1} *where* $\nu(k)$ *denotes the number of terms in* N_1 *which do not exceed* k. *If* $\alpha, \beta \in X = C^{(n)}[0, 1] \cap [\mathrm{loc}\ AC^{(n+1)}(0, 1)]$ *are such that*

$$\begin{aligned} &g(t, \alpha^{(n)}(t), \ldots, \alpha(t)) - \alpha^{(n+2)}(t) \leqq 0 \leqq g(t, \beta^{(n)}(t), \ldots, \beta(t)) \\ &\qquad\qquad\qquad\qquad - \beta^{(n+2)}(t), \quad t \in (0, 1), \\ &a^{(n)}(t) \leqq \beta^{(n)}(t), \quad t \in [0, 1], \end{aligned}$$

and $\mu_0, \ldots, \mu_{n+1}$ *satisfy*

$$\begin{aligned} &\alpha^{(n)}(i) \leqq \mu_i \leqq \beta^{(n)}(i); \quad i = 0, 1, \\ &(-1)^{\nu(k)} \alpha^{(n-k+1)}(i) \leqq (-1)^{\nu(k)} \mu_k \leqq (-1)^{\nu(k)} \beta^{(n-k+1)}(i); \\ &\qquad\qquad\qquad\qquad k \in N_i, i = 0, 1, \quad \textit{and} \\ &(-1)^{\nu(n)} \alpha(c) \leqq (-1)^{\nu(n)} \mu_{n+1} \leqq (-1)^{\nu(n)} \beta(c), \quad \textit{if } c = 0, \\ &(-1)^{\nu(n)+1} \alpha(c) \leqq (-1)^{\nu(n)+1} \mu_{n+1} \leqq (-1)^{\nu(n)+1} \beta(c), \quad \textit{if } c = 1, \\ &\alpha(c) = \mu_{n+1} = \beta(c), \quad \textit{if } c \in (0, 1) \end{aligned}$$

then there exists a solution $x \in X$ *to the boundary value problem*

$$\begin{aligned} x^{(n+2)} &= g(t, x^{(n)}, \ldots, x), \\ x^{(n)}(i) &= \mu_i; \quad i = 0, 1, \\ x^{(n-k+1)}(i) &= \mu_k; \quad k \in N_i, i = 0, 1, \\ x(c) &= \mu_{n+1}. \end{aligned}$$

Furthermore,

$$\alpha^{(n)}(t) \leqq x^{(n)}(t) \leqq \beta^{(n)}(t)$$

$$(-1)^{\nu(k)}\alpha^{(n-k+1)}(t) \leqq (-1)^{\nu(k)}x^{(n-k+1)}(t) \leqq (-1)^{\nu(k)}\beta^{(n-k+1)}(t); \quad k \in N_i,\ i = 0, 1,$$

$$(t-c)(-1)^{\nu(n)}\alpha(t) \leqq (t-c)(-1)^{\nu(n)}x(t) \leqq (t-c)(-1)^{\nu(n)}\beta(t).$$

THEOREM 3.3. *Let* $T > 0$, $X = \{x \in C^2(-\infty, \infty) : x(t+T) = x(t)\}$, $Y = \{x \in C(-\infty, \infty) : x(t+T) = x(t)\}$, *and* $f(t, x, y(\cdot))$ *be continuous on* $\mathbf{R}^2$ *and* T*-periodic in* t, *and be nonincreasing in* y. *If* $y_n, y \in X$, $y_n \uparrow y$ *implies* $f(t, x, y_n(\cdot)) \to f(t, x, y(\cdot))$ *uniformly on compact subsets of* $\mathbf{R}^2$, *then* $H : X \to Y$ *has property* (I) *where*

$$(Hx)(t) = f(t, x(t), x(\cdot)) - x''.$$

In Theorem 3.4, t_k, r_k, m, μ_{jk}, L and P are as in Example 2.4; $\omega(t, x)$ satisfies the Carathéodory conditions locally in $[0, 1] \times \mathbf{R}$ and $P(t)\omega(t, x)$ is nondecreasing in x for each t.

THEOREM 3.4. *Suppose there exist* $\alpha, \beta \in AC^{(n-1)}[0, 1]$ *such that*

$$L\alpha - \omega(t, \alpha) \leqq 0 \leqq L\beta - \omega(t, \beta); P(t)\alpha(t) \leqq P(t)\beta(t);$$

$$\alpha^{(k)}(t_j) = \beta^{(k)}(t_j) = \mu_{jk};\ k = 0, \ldots, r_j - 2 \quad \text{for} \quad j = 0, m;$$

$$k = 0, \ldots, r_j - 1 \quad \text{for} \quad j = 1, \ldots, m-1 \quad (m > 1).$$

If $m \geqq 1$, *the equation* $Lx = \omega(t, x)$ *has a solution* x *satisfying*

$$P(t)\alpha(t) \leqq P(t)x(t) \leqq P(t)\beta(t);$$

$$x^{(k)}(t_j) = \mu_{jk};\ k = 0, \ldots, r_j - 2, j = 0, m;\ k = 0, \ldots, r_j - 1, \quad j = 1, \ldots, m-1;$$

$$x^{(r_0-1)}(t_0) = \nu_0,\ x^{(r_m-1)}(t_m) = \nu_m$$

for any ν_0, ν_m *such that*

$$P^{(r_0)}(t_0)\alpha^{(r_0-1)}(t_0) \leqq P^{(r_0)}(t_0)\nu_0 \leqq P^{(r_0)}(t_0)\beta^{(r_0-1)}(t_0),$$

$$\alpha^{(r_m-1)}(t_m) \geqq \nu_m \geqq \beta^{(r_m-1)}(t_m).$$

If $m = 0$, *the initial value problem* $Lx = \omega(t, x)$,

$$x^{(k)}(t_0) = \begin{cases} \mu_{0k}, & k = 0, \ldots, n-2, \\ \nu, & k = n-1, \end{cases}$$

where $\alpha^{(n-1)}(t_0) \leqq \nu \leqq \beta^{(n-1)}(t_0)$ *if* $t_0 = 0$ *and* $\alpha^{(n-1)}(t_0) \geqq \nu \geqq \beta^{(n-1)}(t_0)$ *if* $t_0 = 1$, *has a solution* x *such that*

$$(t-t_0)^n\alpha(t) \leqq (t-t_0)^n x(t) \leqq (t-t_0)^n\beta(t).$$

Theorems 3.2, 3.3, 3.4 follow from Lemma 2.2 by consideration of

$$K(x, y)(t) = \begin{cases} f(t, x(t), y(\cdot)) - x''(t), & t \in (0, 1) \\ x(t), & t \in (a, 0] \cup [1, b), \end{cases}$$

$$K(x, y)(t) = f(t, x(t), y(\cdot)) - x''(t),$$

$$K_m(x, y)(t) = \begin{cases} (Lx(t) - \omega(t, y(t)), (-1)^{t_0} x^{(n-1)}(t_0)), & \text{if } m = 0, \\ (Lx(t) - \omega(t, y(t)), P^{(r_0)}(t_0) x^{(r_0-1)}(t_0), - \\ \qquad -P^{(r_m)}(t_m) x^{(r_m-1)}(t_m)), & \text{if } m \geqq 1, \end{cases}$$

and Examples 2.2, 2.3 and 2.4, respectively. The convergence of the sequence $\{z_n\}$ of Lemma 2.2 follows from writing the equations $K(z_{n+1}, z_n) = \xi$ in appropriate integral equation form in each case.

Corollary 3.2.1 follows from Theorem 3.2 by considering $(f(t, x(t), x(\cdot)) - x''(t), x(0), x(1))$ where

$$f(t, x, y(\cdot)) = g(t, x, y_2(t), \ldots, y_{n+1}(t)), y_1(t) = y(t), y_k(t) = \mu_k + \int_i^t y_{k-1},$$

$$k \in N_i, i = 0, 1 \text{ and } y_{n+1}(t) = \mu_{n+1} + \int_c^t y_n.$$

It should be noted that in the case $n = 1$, the sets N_0 and N_1 are empty.

Theorem 3.2, when modified by Nagumo-type conditions, includes results of Schmitt [**25**; **26**] and of Grimm and Schmitt [**10**] for boundary value problems with deviating arguments and of Schmitt [**28**] for systems of differential equations where the same type of argument as we have used in treating the Kamke results for systems is applicable. No new fixed point theorem has been required here other than that required in the proof of the basic result of Jackson and Schrader (Example 2.2). In a later paper we will present an elementary proof for results such as Example 2.2 which together with Lemma 2.2 will facilitate a completely elementary discussion of these problems. If one uses as one's basic result a theorem such as that of Erbe [**7**] for differential equations $x'' = f(t, x', x)$ with nonlinear boundary data on x and x', then even greater generality can be achieved.

A simple modification of Corollary 3.2.1 to cover equations of the form $x^{(n+2)} = g(t, x^{(n+1)}, x^{(n)}, \ldots, x)$ includes Theorems 7 and 8 of Klaasen [**16**] for boundary value problems for $x''' = g(t, x'', x', x)$. Theorem 9 of [**16**] follows from Theorem 3.4 with $Lx = x'''$. Klaasen's Theorem 10 can be obtained also by the present techniques provided the general existence theory of second order boundary value problems is utilized in place of Example 2.2.

In the case $n = 2$, $m = 1$ Theorem 3.4 has essentially been proved by Grimmer and Waltman [**9**] by a polar coordinate technique; their result also follows easily from the Sturm comparison theorem.

Only a special case of Theorem 1 of Schmitt [**24**], namely Example 2.3, was used to prove Theorem 3.3. If the full generality of Schmitt's Theorem 1 is used, then it can be shown by Lemma 2.1 that differential equations of the form $x'' = f(t, x, x', x(\cdot))$ have periodic solutions provided f is periodic in t and satisfies a Nagumo condition in x' with respect to an upper and a lower solution (cf. [**24**]). This result then generalizes Theorems 1 and 2 of [**27**] for linear second order differential equations with deviating argument.

4. Applications to integral equations. The classical integral inequalities all furnish examples of operators with the intermediate value property (I). The operators involved are usually of the form

$$x(t) - \int_I f(t, \tau, x(\tau))d\tau$$

(or may be reduced to this form by a change of variables) where $f(t, \tau, x)$ is nondecreasing in x. Thus these results follow from Example 2.5. The following theorem is not a consequence of Example 2.5 and does not involve the monotonicity assumption on f. This theorem generalizes most of the classical inequalities including that of Wajewski when it is written in the form (2.6).

Let $t_0, \ldots, t_m, r_0, \ldots, r_m$ $(m \geqq 0)$, L and $P(t)$ be as in Example 2.4 except that it is not assumed that $Lx = 0$ is disconjugate on $[0, 1]$.

THEOREM 4.1. *Suppose that*

(i) $k(t, \tau)$ *is the Green's function for the operator* L *with homogeneous boundary conditions*

$$x^{(k)}(t_j) = 0, k = 0, \ldots, r_j - 1; \quad j = 0, \ldots, m, \tag{4.1}$$

(ii) $a \in L^1[0, 1]$, $P(t)a(t) \geqq 0$, $\omega(t, x)$ *satisfies the Carathéodory conditions and* $P(t)x \leqq P(t)y$ *implies* $\omega(t, x) \leqq \omega(t, y)$,

(iii) $[L - a(t)]x = 0$ *is disconjugate on* $[0, 1]$ *and* $Lx = 0$ *has only the trivial solution satisfying* (4.1).

Then, $M : X \rightarrow Y$ *has property* (I) *where*

$$(Mx)(t) = x(t) - \int_0^1 k(t, \tau)[a(\tau)x(\tau) + \omega(\tau, x(\tau))]d\tau,$$

$X = \{x \in [0, 1] : k(t, \tau)[a(\tau)x(\tau) + \omega(\tau, x(\tau))] \in L^1[0, 1]$ *for each* $t \in [0, 1]\}$, $Y = L^1[0, 1]$, *and* X, Y *are partially ordered by* $x \succ 0$ *if* $P(t)x(t) \geqq 0$.

Proof. Let

$$(Tx)(t) = \int_0^1 k(t, \tau)[a(\tau)x(\tau) + \omega(\tau, x(\tau))]d\tau,$$

so that if $\alpha, \beta \in X$, $\alpha \prec \beta$, and $M\alpha \prec \xi \prec M\beta$, $\xi \in Y$, then

$$\alpha - \xi - T\alpha \prec 0 \prec \beta - \xi - T\beta, \tag{4.2}$$

which implies

$$\begin{aligned} LT\alpha &\leqq a\xi + aT\alpha + \omega(t, \alpha) \\ LT\beta &\geqq a\xi + aT\beta + \omega(t, \beta) \end{aligned} \tag{4.3}$$

and

$$(4.4) \quad (L - a)T\alpha - a\xi - \omega(t, \xi + T\alpha) \leqq 0 \leqq (L - a)T\beta - \alpha\xi - \omega(t, \xi + T\beta).$$

Since $\omega(t, \alpha) \leqq \omega(t, \beta)$, (4.3) implies $(L - a)(T\beta - T\alpha) \geqq 0$, and since, in addition, $T\alpha$ and $T\beta$ satisfy (4.1), Lemma 2.1 implies that $T\beta - T\alpha > 0$. Hence, Theorem 3.4 implies the existence of a solution x to $(L - a)x - a\xi - \omega(t, \xi + x) = 0$ satisfying (4.1) and such that

$$(4.5) \qquad T\alpha \prec x \prec T\beta.$$

Clearly $z = \xi + x$ satisfies $Mz = \xi$, and (4.2) and (4.5) imply $\alpha \prec z \prec \beta$.
Theorem 4.1 and Lemma 2.2 imply that operators of the form

$$(Mx)(t) = x(t) - \int_0^1 f(t, \tau, x(\tau))d\tau - \int_0^1 k(t, \tau)[a(\tau)x(\tau) + \omega(\tau, x(\tau))]d\tau$$

have property (I) provided k, a, ω satisfy the assumptions of Theorem 4.1 and $P(t)f(t, \tau, x)$ is nondecreasing in x.

The following Corollary illustrates the application of Theorem 4.1 to the solution of integral inequalities which do not necessarily have a monotone kernel. The special case $\omega(\alpha) \equiv 0$ is a result of Azbelev and Tsalyuk [**1**].

COROLLARY 4.1.1. *If* $\mu \in \mathbf{R}$, $\alpha(t)$, $\omega(\alpha(t)) \in \operatorname{loc} L^1[0, h)$, $\mu + \omega(\mu) > 0$ *and* ω *is continuous and nondecreasing, then*

$$\alpha(t) \leqslant \mu + \int_0^t \sin(t - \tau)[\alpha(\tau) + \omega(\alpha(\tau))]d\tau, \quad t \in [0, h)$$

implies

$$\alpha(t) \leqq \mu + \Omega^{-1}(t), \qquad t \in [0, h) \cap [0, \Omega(\infty)),$$

where

$$\Omega(z) = \int_0^z \left[2\mu u + 2\int_\mu^{\mu+u} \omega(x)dx\right]^{-\frac{1}{2}} du.$$

Proof. It follows from Theorem 4.1 that $\alpha(t) \leqq \mu + z(t)$ where $z(t)$ is the maximal solution of $z'' = \mu + \omega(z + \mu)$, $z(0) = z'(0) = 0$ which can be determined explicitly as $z(t) = \Omega^{-1}(t)$.

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