J. Appl. Prob. **49,** 627–638 (2012)
Printed in England

COALESCENCE IN CRITICAL AND SUBCRITICAL GALTON–WATSON BRANCHING PROCESSES

K. B. ATHREYA,* *Iowa State University*

Abstract

In a Galton–Watson branching process that is not extinct by the nth generation and has at least two individuals, pick two individuals at random by simple random sampling without replacement. Trace their lines of descent back in time till they meet. Call that generation X_n a *pairwise coalescence time*. Similarly, let Y_n denote the *coalescence time* for the whole population of the nth generation conditioned on the event that it is not extinct. In this paper the distributions of X_n and Y_n, and their limit behaviors as $n \to \infty$ are discussed for both the critical and subcritical cases.

Keywords: Branching process; coalescence; critical; subcritical

2010 Mathematics Subject Classification: Primary 60J80
Secondary 60F10

1. Introduction

Let $\{p_j\}_{j\geq 0}$ be a probability distribution on the nonnegative integers $\mathbb{N}^+ \equiv \{0, 1, 2, \dots\}$. Let $\{\xi_{n,i} : i \geq 1,\ n \geq 0\}$ be a doubly infinite family of independent random variables with distribution $\{p_j\}_{j\geq 0}$. Let $\{Z_n\}_{n\geq 0}$ be a sequence of random variables defined by the stochastic recurrence relation $1 \leq Z_0 < \infty$ nonrandom with values in $\mathbb{N}^+$ and, for $n = 0, 1, 2, \dots$,

$$Z_{n+1} = \begin{cases} \displaystyle\sum_{i=1}^{Z_n} \xi_{n,i} & \text{if } Z_n > 0, \\ 0 & \text{if } Z_n = 0. \end{cases}$$

Then the sequence $\{Z_n\}_{n\geq 0}$ is called the population size sequence of a *Galton–Watson branching process* with *offspring distribution* $\{p_j\}_{j\geq 0}$ and initiated size Z_0. (See [2].)

Here Z_n is the size of the nth generation and, for any n and i, $\xi_{n,i}$ denotes the number of offspring of the ith individual in the nth generation. If $\mathbb{T}$ denotes the full family tree generated, every individual in $\mathbb{T}$ can be identified by a finite string $(i_0, i_1, \dots, i_n)$, meaning that this individual is in the nth generation and is the i_nth offspring of individual $(i_0, i_1, \dots, i_{n-1})$ in the $(n-1)$th generation. For any $0 \leq k < n$, individual $(i_0, i_1, \dots, i_k)$ is a *common ancestor* of all individuals $(i_0, i_1, \dots, i_k, i_{k+1}, \dots, i_n)$, where $i_{k+1}, i_{k+2}, \dots, i_n$ are any positive integers.

The coalescence time of two individuals of any generation is the generation number of their *last common ancestor*. Clearly, every ancestor of this common ancestor is also a common ancestor.

Now, for each $n \geq 1$, consider the event $A_n \equiv \{Z_n \geq 2\}$. For a family tree that is in A_n, choose two individuals at random from those in the nth generation by simple random sampling

Received 17 September 2010; revision received 18 January 2012.
* Postal address: Departments of Mathematics and Statistics, Iowa State University, Ames, Iowa, 50011, USA.
Email address: kba@iastate.edu

without replacement. Now trace their lines of descent back in time till they meet at their *last common ancestor*. Call the generation of the last common ancestor of those randomly chosen two individuals a *pairwise coalescence time* X_n.

Next, for each $n \geq 1$, consider the event $B_n \equiv \{Z_n \geq 1\}$. For a family tree $\mathbb{T}$ in B_n, trace the lines of descent of all the individuals in the nth generation till they meet. Call that generation number Y_n *the total coalescence time*.

In this paper we determine the limit behavior of the distribution of X_n conditioned on A_n and that of Y_n conditioned on B_n as $n \to \infty$ in the nonsupercritical cases, i.e. when $\{p_j\}_{j\geq 0}$ satisfies the condition $0 < m \equiv \sum_{j=1}^{\infty} jp_j \leq 1$.

The supercritical case, i.e. $m > 1$, including the explosion case, $m = \infty$, is treated in [1]. See Remark 2.2 in the next section.

2. Main results

Let $\{Z_n, A_n, B_n, X_n, Y_n, \{p_j\}_{j\geq 0}, m, \mathbb{T}\}$ be as in the introduction.

Theorem 2.1. (Critical case.) *Let* $m \equiv \sum_{j=1}^{\infty} jp_j = 1$, $p_1 < 1$, *and* $\sigma^2 \equiv \sum_{j=1}^{\infty} j^2 p_j - 1 < \infty$. *Then the following statements hold.*

(i) *For* $0 < u < 1$,

$$\lim_{n\to\infty} \mathrm{P}(X_n < nu \mid Z_n \geq 2) \equiv H(u)$$

exists and equals $1 - \mathrm{E}\,\phi(N_u)$, *where* N_u *is a positive integer-valued random variable with a geometric distribution* $\mathrm{P}(N_u = k) = (1-u)u^{k-1}$, $k \geq 1$, *and, for* $j \geq 1$,

$$\phi(j) = \mathrm{E}\biggl(\frac{\sum_{i=1}^{j} \eta_i^2}{(\sum_{i=1}^{j} \eta_i)^2}\biggr),$$

where $\{\eta_i\}_{i\geq 1}$ *are independent and identically distributed (i.i.d.) exponential random variables with mean* 1.

Furthermore, $H(\cdot)$ *is an absolutely continuous cumulative distribution function on* $[0, 1]$ *with* $H(0+) = 0$ *and* $H(1-) = 1$.

(ii) *For* $0 < u < 1$,

$$\lim_{n\to\infty} \mathrm{P}(Y_n > nu \mid Z_n \geq 1) = 1 - u.$$

Remark 2.1. Theorem 2.1(ii) is a known result proved by Zubkov [6].

Theorem 2.2. (Subcritical case.) *Let* $0 < m \equiv \sum_{j=1}^{\infty} jp_j < 1$. *Then the following statements hold.*

(i) *For* $k \geq 1$,

$$\lim_{n\to\infty} \mathrm{P}(n - X_n > k \mid Z_n \geq 2) = \frac{\mathrm{E}\,\phi_k(Y)}{\mathrm{E}\,\psi_k(Y)} \equiv \pi_k, \quad \textit{say},$$

where

$$\phi_k(j) = \mathrm{E}\biggl(\frac{\sum_{i_1\neq i_2=1}^{j} Z_{k,i_1} Z_{k,i_2}}{(\sum_{i=1}^{j} Z_{k,i})(\sum_{i=1}^{j} Z_{k,i} - 1)} \mathbf{1}\biggl(\sum_{i=1}^{j} Z_{k,i} \geq 1\biggr)\biggr),$$

https://doi.org/10.1239/jap/1346955322 Published online by Cambridge University Press

$$\psi_k(j) = \mathrm{P}\Big(\sum_{i=1}^{j} Z_{k,i} \geq 2\Big),$$

$\{Z_{r,i} : r \geq 0\}$, $i = 1, 2, \ldots$, are i.i.d. copies of a Galton–Watson branching process $\{Z_r : r \geq 0\}$ with $Z_0 = 1$ and the given offspring distribution $\{p_j\}_{j\geq 0}$, and Y is a random variable with distribution $\{b_j\}_{j\geq 1}$, where $b_j \equiv \lim_{n\to\infty} \mathrm{P}(Z_n = j \mid Z_n > 0, Z_0 = 1)$, which exists (as asserted in Theorem 3.3 below).

Furthermore, if $\sum_{j=1}^{\infty} j \log j p_j < \infty$ then $\lim_{k\uparrow\infty} \pi_k = 0$ and, hence, $n - X_n$ conditioned on $Z_n \geq 2$ converges to a proper distribution on $\{1, 2, \ldots\}$.

(ii) *For $k \geq 1$, $\lim_{n\to\infty} \mathrm{P}(n - Y_n > k \mid Z_n \geq 1) \equiv \tilde{\pi}_k$ exists and equals*

$$\mathrm{E}\Big(\frac{1 - q_k^{Y}}{m^k}\Big) - \mathrm{E}\Big(\frac{Y q_k^{Y-1}(1 - q_k)}{m^k}\Big),$$

where Y is a random variable with distribution

$$\mathrm{P}(Y = j) = b_j = \lim_{n\to\infty} \mathrm{P}(Z_n = j \mid Z_n > 0,\ Z_0 = 1)$$

and $q_k = \mathrm{P}(Z_k = 0 \mid Z_0 = 1)$.

Furthermore, if $\sum_{j=1}^{\infty} j \log j p_j < \infty$ then

$$\lim_{k\to\infty} \tilde{\pi}_k = 0.$$

That is, $n - Y_n$ conditioned on $\{Z_n > 0\}$ converges in distribution as $n \to \infty$ to a proper distribution on $\{1, 2, \ldots\}$.

Remark 2.2. Theorem 2.2 provides a sharp contrast to Theorem 2.1. In the subcritical case, the coalescence times X_n and Y_n are close to the present time n, whereas in the critical case they are both of the order of magnitude n and, hence, could be a long time before the present but also a long time from the initial ancestor. In the supercritical case with $1 < m \equiv \sum_{j=1}^{\infty} j p_j < \infty$, it was shown in [1] that X_n converges in distribution, i.e. coalescence takes place close to the ancestor, with the same being true for Y_n. Thus, we have a trichotomy based on the value of the offspring mean m. For $1 < m < \infty$, the coalescence is close to the ancestor, i.e. beginning of the tree. For $m = 1$, it is of the order n, neither close to the ancestor nor close to the present. For $0 < m < 1$, it is close to the present. Another unexpected result is the following. It turns out that, when $m = \infty$ and $\{p_j\}_{j\geq 0}$ is in the domain of attraction of a stable law of order α, $0 < \alpha < 1$, the coalescence time X_n is close to the present and, in fact, $n - X_n$ converges in distribution to a proper distribution (as in the subcritical case). See [1] for details on this case.

Remark 2.3. The referee pointed out that Theorem 2.1 could also be proved using the excursion representation of the limiting continuum random trees (see [5]). According to the referee,

> in the limiting continuum tree conditioned to reach height 1 (which is known to have an exponentially distributed width at height 1) there is a geometric number N_u of subtrees whose MRCAs lived before time $u < 1$ and whose widths at time 1 are constant multiples of the η_is in Theorem 2.1. Further, the event that two individuals sampled randomly at time 1 have a 'coalescence time' earlier than u is just the event that these individuals do not belong to the same subtree.

Remark 2.4. The referee also pointed out that the $L \log L$ condition in Theorem 2.2 can be dropped for the case $Z_0 = 1$ using the results of Geiger [3], and raised the question of whether Theorem 2.1 can be proved for a random Z_0 with just $\mathrm{E}\, Z_0 < \infty$ but without the $L \log L$ condition. The present author hopes to investigate these issues in the near future.

3. Some preliminary results

In this section we collect some well-known results, and present some new results on Galton–Watson branching processes in the critical and subcritical cases.

Theorem 3.1. (Critical case.) *Let $m = 1$, $p_1 < 1$, and $0 < \sigma^2 \equiv \sum_{j=1}^{\infty} j^2 p_j - 1 < \infty$. Let $Z_0 = k < \infty$. Then, as $n \to \infty$,*

(i) $n\,\mathrm{P}(Z_n > 0) \to 2/\sigma^2$;

(ii) $\mathrm{P}(Z_n/n > u \mid Z_n > 0) \to \mathrm{e}^{-2u/\sigma^2}$, $0 < u < \infty$.

For a proof, see [2, p. 20].

Let $\{Z_{j,i}^{(k)} : j \ge 0\}$ be the Galton–Watson branching process initiated by the ith individual in the kth generation, $1 \le i \le Z_k$.

Theorem 3.2. *Under the hypothesis and notation of Theorem 3.1, consider the point process*

$$V_n \equiv \left\{ \frac{Z_{n-k,i}^{(k)}}{n-k} : 1 \le i \le Z_k,\ Z_{n-k,i}^{(k)} > 0 \right\}$$

on the event $B_n \equiv \{Z_n > 0\}$. Then, as $n \to \infty$, $k \to \infty$, and $k/n \to u$, $0 < u < 1$, conditioned on the event B_n, the point process sequence V_n converges to $V \equiv \{\eta_i : 1 \le i \le N_u\}$, where $\{\eta_i\}_{i \ge 1}$ are i.i.d. exponential random variables with mean $\sigma^2/2$ and N_u is independent of $\{\eta_i\}_{i \ge 1}$ with distribution $\mathrm{P}(N_u = k) = (1-u)u^{k-1}$, $k \ge 1$.

Proof. Let $f : \mathbb{R}^+ \equiv [0, \infty) \to \mathbb{R}^+$ be a bounded continuous function. Let, for $n \ge 1$ and $1 \le k \le n$,

$$Y_{n,k} = \exp\biggl(-s \sum_{i=1}^{Z_k} f\biggl(\frac{Z_{n-k,i}^{(k)}}{n-k} \biggr) \mathbf{1}(Z_{n-k,i}^{(k)} > 0) \biggr).$$

Then

$$\begin{aligned}
&\mathrm{E}(Y_{n,k}\, \mathbf{1}(Z_n > 0) \mid Z_j : j \le k) \\
&\quad = \mathrm{E}(Y_{n,k}\, \mathbf{1}(Z_k > 0)\, \mathbf{1}(Z_n > 0) \mid Z_k) \qquad \text{(by the Markov property)} \\
&\quad = \mathrm{E}(Y_{n,k}\, \mathbf{1}(Z_k > 0) \mid Z_k) - \mathrm{E}(Y_{n,k}\, \mathbf{1}(Z_k > 0)\, \mathbf{1}(Z_n = 0) \mid Z_k) \\
&\quad = (g_{n-k}(s))^{Z_k}\, \mathbf{1}(Z_k > 0) - q_{n-k}^{Z_k}\, \mathbf{1}(Z_k > 0),
\end{aligned}$$

where

$$g_j(s) = \mathrm{E}\biggl(\exp\biggl(-s f\biggl(\frac{Z_j}{j} \biggr) \mathbf{1}(Z_j > 0) \biggr) \biggm| Z_0 = 1 \biggr)$$

and

$$q_j = \mathrm{P}(Z_j = 0 \mid Z_0 = 1), \qquad j \ge 1.$$

Now,

$$g_j(s) = q_j + (1-q_j)\,\mathrm{E}\bigg(\exp\bigg(-sf\bigg(\frac{Z_j}{j}\bigg)\bigg)\;\bigg|\; Z_j > 0,\ Z_0 = 1\bigg).$$

From Theorem 3.1(i), under the hypothesis of Theorem 2.1, as $j \to \infty$,

$$j(1-q_j) \to \frac{2}{\sigma^2}$$

and

$$\frac{Z_j}{j}\;\bigg|\; Z_j > 0 \xrightarrow{\mathrm{D}} \text{an exponential distribution with mean } \frac{2}{\sigma^2}.$$

Clearly,

$$g_j(s) = (1 + (1-q_j)(\tilde{g}_j(s) - 1)),$$

where

$$\tilde{g}_j(s) = \mathrm{E}\bigg(\exp\bigg(-sf\bigg(\frac{Z_j}{j}\bigg)\bigg)\;\bigg|\; Z_j > 0,\ Z_0 = 1\bigg).$$

Since f is bounded and continuous, it follows from Theorem 3.1 that, as $j \to \infty$,

$$\tilde{g}_j(s) \to \frac{2}{\sigma^2}\int_0^\infty \mathrm{e}^{-sf(x)}\mathrm{e}^{-2x/\sigma^2}\,\mathrm{d}x \equiv \tilde{g}(s), \quad \text{say,}$$

implying that $(g_j(s))^j \to \mathrm{e}^{2(\tilde{g}(s)-1)/\sigma^2}$.

Now, let $n \to \infty$, $k \to \infty$, and $k/n \to u$, $0 < u < 1$. Then

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{E}((g_{n-k}(s))^{Z_k}\,\mathbf{1}(Z_k > 0) \mid Z_k > 0) \\
&\quad = \lim_{n\to\infty} \mathrm{E}(((g_{n-k}(s))^{n-k})^{(Z_k/k)k/(n-k)} \mid Z_k > 0) \\
&\quad = \frac{2}{\sigma^2}\int_0^\infty \exp\bigg(\frac{2}{\sigma^2}(\tilde{g}(s)-1)\bigg(\frac{u}{1-u}\bigg)x\bigg)\mathrm{e}^{-2x/\sigma^2}\,\mathrm{d}x \\
&\quad = \frac{2/\sigma^2}{(2/\sigma^2)(1-(\tilde{g}(s)-1)u/(1-u))} \\
&\quad = \frac{1-u}{1-u\tilde{g}(s)}.
\end{aligned}$$

Similarly,

$$\mathrm{E}(q_{n-k}^{Z_k} \mid Z_k > 0) \to 1-u.$$

Thus,

$$\begin{aligned}
\mathrm{E}(Y_{n,k} \mid Z_n > 0) &= \mathrm{E}(\mathrm{E}(Y_{n,k}\,\mathbf{1}(Z_n > 0) \mid Z_j\colon j \le k) \mid Z_n > 0) \\
&\to \frac{1-u}{u}\bigg(\frac{1}{1-u\tilde{g}(s)} - 1\bigg) \\
&= \frac{(1-u)u\tilde{g}(s)}{u(1-u\tilde{g}(s))} \\
&= (1-u)\sum_{j=0}^{\infty} u^j(\tilde{g}(s))^{j+1} \\
&= \sum_{j=1}^{\infty}(1-u)u^{j-1}(\tilde{g}(s))^j.
\end{aligned}$$

https://doi.org/10.1239/jap/1346955322 Published online by Cambridge University Press

This equals $\mathrm{E}(\exp(-s\sum_{i=1}^{N_u} f(\eta_i)))$, where $\{\eta_i\}_{i\geq 1}$ and N_u are as in the statement of Theorem 3.1. Now, by the continuity theorem for point processes (see [4]), Theorem 3.2 follows.

Theorem 3.3. (Subcritical case.) *Let* $0 < m \equiv \sum_{j=1}^{\infty} jp_j < 1$ *and* $Z_0 = 1$. *Then, for* $j \geq 1$, *the following statements hold.*

(i) $\lim_{n\to\infty} \mathrm{P}(Z_n = j \mid Z_n > 0) \equiv b_j$ *exists and* $\sum_{j=1}^{\infty} b_j = 1$. *Furthermore,* $B(s) \equiv \sum_{j=1}^{\infty} b_j s^j$, $0 \leq s \leq 1$, *is the unique solution of the functional equation*

$$B(f(s)) = mB(s) + (1-s), \qquad 0 \leq s \leq 1,$$

in the class of probability generating functions vanishing at 0.

(ii) $\sum_{j=1}^{\infty} jb_j < \infty$ *if and only if* $\sum_{j=1}^{\infty} j \log jp_j < \infty$.

(iii) *If* $\sum_{j=1}^{\infty} j \log jp_j < \infty$ *then*

$$\lim_{n\to\infty} \frac{\mathrm{P}(Z_n > 0)}{m^n} = \frac{1}{\sum_{j=1}^{\infty} jb_j} > 0.$$

For a proof, see [2, pp. 16, 18, 40].

The next result extends the above to random initial Z_0.

Theorem 3.4. *Let* $0 < m < 1$ *and* Z_0 *be a random variable such that* $\mathrm{P}(Z_0 \geq 1) = 1$ *and* $\mathrm{E}\, Z_0 < \infty$. *Then,*

(i) *for* $j \geq 1$,

$$\lim_{n\to\infty} \mathrm{P}(Z_n = j \mid Z_n > 0) = b_j,$$

where $\{b_j\}_{j\geq 1}$ *is as in Theorem 3.3(i);*

(ii) *if* $\sum_{j=1}^{\infty} j \log jp_j < \infty$, $\lim_{n\to\infty} \mathrm{P}(Z_n > 0)/m^n$ *exists in* $(0, \infty)$ *and equals* $\mathrm{E}\, Z_0 / \sum_{j=1}^{\infty} jb_j$.

Proof. For $0 \leq s \leq 1$,

$$\mathrm{E}(s^{Z_n} \mid Z_0) = (f_n(s))^{Z_0},$$

where $f_n(s) = \mathrm{E}(s^{Z_n} \mid Z_0 = 1)$. So,

$$\begin{aligned}
\mathrm{E}(s^{Z_n} \mid Z_n > 0) &= \frac{\mathrm{E}(s^{Z_n} \mathbf{1}(Z_n > 0))}{\mathrm{P}(Z_n > 0)} \\
&= \frac{\mathrm{E}(s^{Z_n}) - \mathrm{E}(s^{Z_n} \mathbf{1}(Z_n = 0))}{\mathrm{P}(Z_n > 0)} \\
&= \frac{\mathrm{E}(f_n(s))^{Z_0} - \mathrm{E}(f_n(0))^{Z_0}}{1 - \mathrm{E}(f_n(0))^{Z_0}} \\
&= 1 - \frac{\mathrm{E}(1 - (f_n(s))^{Z_0})}{\mathrm{E}(1 - (f_n(0))^{Z_0})}.
\end{aligned}$$

By the monotone convergence theorem,

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{1 - (f_n(s))^{Z_0}}{1 - f_n(s)}\right) = \mathrm{E}\, Z_0 \quad \text{for } 0 \leq s < 1.$$

Also, by Theorem 3.3(i),

$$\lim_{n\to\infty}\frac{f_n(s)-f_n(0)}{1-f_n(0)}=B(s)\quad\text{for } 0\le s<1,$$

and, hence,

$$\lim_{n\to\infty}\frac{1-f_n(s)}{1-f_n(0)}=1-B(s)\quad\text{for } 0\le s<1.$$

Since, by hypothesis, $0<\mathrm{E}\,Z_0<\infty$,

$$\lim_{n\to\infty}\mathrm{E}(s^{Z_n}\mid Z_n>0)=\lim_{n\to\infty}\biggl(1-(1-B(s))\frac{\mathrm{E}\,Z_0}{\mathrm{E}\,Z_0}\biggr)=B(s),$$

proving (i).

Next,

$$\frac{\mathrm{P}(Z_n>0)}{m^n}=\frac{\mathrm{E}(1-(f_n(0))^{Z_0})}{m^n}=\frac{1-f_n(0)}{m^n}\,\mathrm{E}\biggl(\sum_{j=0}^{Z_0-1}(f_n(0))^j\biggr).$$

By Theorem 3.3(iii), under the hypothesis that $\sum_{j=1}^{\infty} j\log jp_j<\infty$, the first term on the right-hand side above converges to $1/\sum_{j=1}^{\infty} jb_j$, which is positive and finite. By the monotone convergence theorem, the second term on the right-hand side above converges to $\mathrm{E}\,Z_0$. This proves (ii).

Corollary 3.1. *Let $0<m<1$ and Z_0 be a finite nonrandom integer greater than or equal to 1. Then, for $j\ge 1$,*

$$\lim_{n\to\infty}\mathrm{P}(Z_n=j\mid Z_n>0)=b_j,$$

where $\{b_j\}_{j\ge1}$ is as in Theorem 3.3(i).

4. Proofs of the main results

4.1. Proof of Theorem 2.1

(i) The event $\{X_n<k\}$ for $1\le k\le n$ occurs if and only if the two randomly chosen individuals from the nth generation come from the $(n-k)$th generation of the trees initiated by two distinct individuals of the kth generation. Also, the total number of choices of the two individuals from the nth generation is $Z_n(Z_n-1)$. Thus, for $n\ge1$ and $k\ge1$,

$$\mathrm{P}(X_n<k\mid Z_n\ge2)=\mathrm{E}\biggl(\frac{\sum_{i_1\ne i_2=1}^{Z_k}Z^{(k)}_{n-k,i_1}Z^{(k)}_{n-k,i_2}}{Z_n(Z_n-1)}\biggm| Z_n\ge2\biggr).$$

Since

$$\sum_{i_1\ne i_2=1}^{Z_k}Z^{(k)}_{n-k,i_1}Z^{(k)}_{n-k,i_2}=\biggl(\sum_{i=1}^{Z_k}Z^{(k)}_{n-k,i}\biggr)^2-\sum_{i=1}^{Z_k}(Z^{(k)}_{n-k,i})^2,$$

https://doi.org/10.1239/jap/1346955322 Published online by Cambridge University Press

it suffices to show that, as $k \to \infty$, $n \to \infty$, and $k/n \to u$, $0 < u < 1$, the quantity

$$\mathrm{E}\biggl(\frac{\sum_{i=1}^{Z_k}(Z_{n-k,i}^{(k)})^2}{(\sum_{i=1}^{Z_k} Z_{n-k,i}^{(k)})^2} \biggm| Z_n \ge 2\biggr)$$

converges to $1 - H(u)$ defined in (i).

Next, by Theorem 3.1(ii), Z_n/n conditioned on $Z_n > 0$ converges in distribution to an exponential distribution with mean $2/\sigma^2$; thus, it follows that

$$\mathrm{P}(Z_n \ge 2 \mid Z_n > 0) \to 1 \quad \text{as } n \to \infty.$$

Also, the quantity

$$\frac{\sum_{i=1}^{Z_k}(Z_{n-k,i}^{(k)})^2}{(\sum_{i=1}^{Z_k} Z_{n-k,i}^{(k)})^2} = \frac{\sum_{i\in J_k}(Z_{n-k,i}^{(k)}/(n-k))^2}{(\sum_{i\in J_k} Z_{n-k,i}^{(k)}/(n-k))^2},$$

where $J_k \equiv \{i : Z_{n-k,i}^{(k)} > 0\}$, $0 \le k \le n$, is a continuous functional in the appropriate topology of point processes on $\mathbb{R}^+$ of the point process V_n conditioned on the event $\{Z_n > 0\}$. Thus, by the weak convergence result in Theorem 3.2, as $n \to \infty$, $k \to \infty$, and $k/n \to u$, $0 < u < 1$,

$$\mathrm{E}\biggl(\frac{\sum_{i\in J_k}(Z_{n-k,i}^{(k)})^2}{(\sum_{i\in J_k} Z_{n-k,i}^{(k)})^2} \biggm| Z_n \ge 2\biggr) \to \mathrm{E}\biggl(\frac{\sum_{i=1}^{N_u}\eta_i^2}{(\sum_{i=1}^{N_u}\eta_i)^2}\biggr),$$

where $\{\eta_i\}_{i\ge 1}$ and N_u are as in Theorem 3.2. This proves the convergence part of Theorem 2.1(i).

Next, since $\mathrm{E}\,\eta_1^2 < \infty$ and $\mathrm{E}\,\eta_1 > 0$, by the strong law,

$$\frac{\sum_{i=1}^{j}\eta_i^2}{(\sum_{i=1}^{j}\eta_i)^2} \to 0 \quad \text{with probability 1 as } j \to \infty.$$

Now, by the bounded convergence theorem, $\phi(j) \to 0$ as $j \to \infty$. Also, as $u \uparrow 1$, $N_u \to \infty$ in distribution and, thus, $H(u) \uparrow 1$ as $u \uparrow 1$. Since $H(u) = 1 - \sum_{j=1}^{\infty}\phi(j)(1-u)u^{j-1}$, $H(\cdot)$ is an absolutely continuous cumulative distribution function on $[0, 1]$ with $H(0+) = 0$ and $H(1-) = 1$. Thus, the proof of Theorem 2.1(i) is complete.

(ii) The event $\{Y_n \ge k\}$ for $1 \le k \le n$ conditioned on $\{Z_n > 0\}$ occurs if and only if all the Z_n individuals of the nth generation come from the $(n-k)$th generation of a branching process initiated by exactly one individual of the kth generation, i.e. $Z_{n-k,i}^{(k)} = 0$ for all but one i, $1 \le i \le Z_k$, and $Z_k > 0$. This yields

$$\begin{aligned}
&\mathrm{P}(Y_n \ge k \mid Z_n \ge 1)\\
&\quad= \frac{\mathrm{E}(Z_{n-k,i}^{(k)} = 0 \text{ for all but one } i,\ 1 \le i \le Z_k,\ \text{and } Z_k > 0)}{\mathrm{P}(Z_n > 0)}\\
&\quad= \mathrm{E}(Z_k q_{n-k}^{Z_k-1}(1-q_{n-k}) \mid Z_k > 0)\frac{\mathrm{P}(Z_k > 0)}{\mathrm{P}(Z_n > 0)} \quad (\text{where } q_n = \mathrm{P}(Z_n > 0 \mid Z_0 = 1))\\
&\quad= \mathrm{E}\biggl(\frac{Z_k}{k} q_{n-k}^{Z_k-1}\frac{k}{n-k}(n-k)(1-q_{n-k}) \biggm| Z_k > 0\biggr)\frac{\mathrm{P}(Z_k > 0)}{\mathrm{P}(Z_n > 0)}.
\end{aligned}$$

Now, as $n \to \infty$, $k \to \infty$ and $k/n \to u$, $0 < u < 1$, by Theorem 3.1,

$$(n-k)(1-q_{n-k}) \to \frac{2}{\sigma^2}, \qquad q_{n-k}^{n-k} = \bigg(1 - \frac{(n-k)(1-q_{n-k})}{n-k}\bigg)^{n-k} \to \mathrm{e}^{-2/\sigma^2},$$

and

$$\frac{Z_k - 1}{n-k} = \frac{Z_k - 1}{k}\,\frac{k}{n-k},$$

conditioned on $Z_k > 0$, converges to $u/(1-u)$ times an exponential random variable with mean $\sigma^2/2$. Also,

$$\frac{\mathrm{P}(Z_k > 0)}{\mathrm{P}(Z_n > 0)} = \frac{k\,\mathrm{P}(Z_k > 0)}{n\,\mathrm{P}(Z_n > 0)}\,\frac{n}{k} \to \frac{1}{u}.$$

Thus, as $n \to \infty$,

$$\mathrm{P}(Y_n \geq k \mid Z_n \geq 1) \to \frac{1}{u}\,\mathrm{E}\bigg(\frac{\sigma^2}{2}\eta \exp\bigg(-\frac{2}{\sigma^2}\,\frac{u}{1-u}\,\frac{\sigma^2}{2}\eta\bigg)\frac{2}{\sigma^2}\,\frac{u}{1-u}\bigg),$$

where η is an exponential random variable with mean 1.

The above limit equals

$$\frac{1}{1-u}\,\mathrm{E}\bigg(\eta \exp\bigg(-\bigg(\frac{u}{1-u}\bigg)\eta\bigg)\bigg) = \frac{1}{1-u}(1-u)^2 = 1-u,$$

since, for any $\theta > 0$,

$$\mathrm{E}(\eta \mathrm{e}^{-\theta\eta}) = \int_0^\infty x\mathrm{e}^{-\theta x}\mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{(1+\theta)^2}.$$

This proves Theorem 2.1(ii).

4.2. Proof of Theorem 2.2

(i) For $0 \leq k < n < \infty$,

$$\mathrm{P}(n - X_n > k \mid Z_n \geq 2) = \frac{\mathrm{P}(X_n < n-k,\, Z_n \geq 2)}{\mathrm{P}(Z_n \geq 2)} = \frac{a_n}{c_n}, \quad \text{say.}$$

Now,

$$\begin{aligned}
a_n &= \mathrm{E}\Bigg(\frac{\sum_{i_1\neq i_2=1}^{Z_{n-k}} Z_{k,i_1}^{(n-k)} Z_{k,i_2}^{(n-k)}}{(\sum_{i=1}^{Z_{n-k}} Z_{k,i}^{(n-k)})(\sum_{i=1}^{Z_{n-k}} Z_{k,i}^{(n-k)} - 1)}\,\mathbf{1}\bigg(\sum_{i=1}^{Z_{n-k}} Z_{k,i}^{(n-k)} \geq 2\bigg)\Bigg) \\
&= \mathrm{E}\Bigg(\frac{\sum_{i_1\neq i_2=1}^{Z_{n-k}} Z_{k,i_1}^{(n-k)} Z_{k,i_2}^{(n-k)}}{(\sum_{i=1}^{Z_{n-k}} Z_{k,i}^{(n-k)})(\sum_{i=1}^{Z_{n-k}} Z_{k,i}^{(n-k)} - 1)}\,\mathbf{1}\bigg(\sum_{i=1}^{Z_{n-k}} Z_{k,i}^{(n-k)} \geq 2\bigg) \;\Bigg|\; Z_{n-k} > 0\Bigg)\,\mathrm{P}(Z_{n-k>0}) \\
&= \mathrm{E}(\phi_k(Z_{n-k}) \mid Z_{n-k} > 0)\,\mathrm{P}(Z_{n-k} > 0),
\end{aligned}$$

where, for $j \geq 1$,

$$\phi_k(j) = \mathrm{E}\Bigg(\frac{\sum_{i_1\neq i_2=1}^{j} Z_{k,i_1} Z_{k,i_2}}{(\sum_{i=1}^{j} Z_{k,i})(\sum_{i=1}^{j} Z_{k,i} - 1)}\,\mathbf{1}\bigg(\sum_{i=1}^{j} Z_{k,i} \geq 2\bigg)\Bigg)$$

https://doi.org/10.1239/jap/1346955322 Published online by Cambridge University Press

and $\{Z_{r,i} : r \geq 0\}$, $i = 1, 2, \ldots, j$, are i.i.d. copies of a Galton–Watson branching process with $Z_0 = 1$ and offspring distribution $\{p_i\}_{i\geq 0}$ satisfying the hypothesis of Theorem 2.2.

Similarly,

$$c_n = \mathrm{E}(\psi_k(Z_{n-k}) \mid Z_{n-k} > 0)\,\mathrm{P}(Z_{n-k} > 0),$$

where, for $j \geq 1$, $\psi_k(j) = \mathrm{E}(\mathbf{1}(\sum_{i=1}^{j} Z_{k,i} \geq 2))$.

By Theorem 3.4, for any Z_0 with $\mathrm{E}\,Z_0 < \infty$, and $k < \infty$,

$$Z_{n-k} \mid Z_{n-k} > 0 \xrightarrow{\mathrm{D}} \{b_j\}_{j\geq 1} \quad \text{as } n \to \infty,$$

where $\{b_j\}_{j\geq 1}$ is as in Theorem 3.3(i). Thus, for each fixed $k \geq 1$,

$$\mathrm{P}(n - X_n > k \mid Z_n \geq 2) \to \frac{\mathrm{E}\,\phi_k(Y)}{\mathrm{E}\,\psi_k(Y)} \equiv \pi_k,$$

where Y is a random variable with distribution $\mathrm{P}(Y = j) = b_j$, $j \geq 1$.

It remains to show that if $\sum_{j=1}^{\infty} j \log j p_j < \infty$ then $\lim_{k\uparrow\infty} \pi_k = 0$.

For $1 \leq j < \infty$,

$$\begin{aligned}\phi_k(j) &\leq \mathrm{P}(\text{there exist } i_1, i_2,\ i_1 \neq i_2,\ 1 \leq i_1, i_2 \leq j,\ Z_{k,i_1} > 0,\ Z_{k,i_2} > 0)\\ &\leq 1 - (f_k(0))^j - j(f_k(0))^{j-1}(1 - f_k(0))\end{aligned}$$

and

$$\psi_k(j) = \mathrm{P}\bigg(\sum_{i=1}^{j} Z_{k,i} \geq 2\bigg) = \mathrm{P}\bigg(\sum_{i=1}^{j} Z_{k,i} \geq 2 \,\bigg|\, \sum_{i=1}^{j} Z_{k,i} \geq 1\bigg)\,\mathrm{P}\bigg(\sum_{i=1}^{j} Z_{k,i} \geq 1\bigg).$$

So,

$$\mathrm{E}\,\phi_k(Y) = \mathrm{E}(1 - (f_k(0))^Y - Y(f_k(0))^{Y-1}(1 - f_k(0)))$$

and

$$\mathrm{E}\,\psi_k(Y) = \mathrm{E}(1 - (f_k(0))^Y)\,\mathrm{P}\bigg(\sum_{i=1}^{Y} Z_{k,i} \geq 2 \,\bigg|\, \sum_{i=1}^{Y} Z_{k,i} \geq 1\bigg).$$

Since $\sum_{j=1}^{\infty} j \log j p_j < \infty$ implies that $\mathrm{E}\,Y < \infty$, by Theorem 3.4(i),

$$\lim_k \mathrm{P}\bigg(\sum_{i=1}^{Y} Z_{k,i} \geq 2 \,\bigg|\, \sum_{i=1}^{Y} Z_{k,i} \geq 1\bigg)$$

exists and equals $1 - b_1$, $0 < b_1 < 1$.

Next,

$$\begin{aligned}\frac{\mathrm{E}\,\phi_k(Y)}{\mathrm{E}\,\psi_k(Y)} &\leq \frac{\mathrm{E}(1 - (f_k(0))^Y)}{\mathrm{E}(1 - (f_k(0))^Y)}\bigg(1 - \frac{(1 - f_k(0))\,\mathrm{E}(Y(f_k(0))^{Y-1})}{\mathrm{E}(1 - (f_k(0))^Y)}\bigg)\\ &\quad\times \frac{1}{\mathrm{P}(\sum_{i=1}^{Y} Z_{k,i} \geq 2 \mid \sum_{i=1}^{Y} Z_{k,i} \geq 1)}.\end{aligned}$$

By the monotone convergence theorem, as $k \to \infty$,

$$\mathrm{E}(Y(f_k(0))^{Y-1}) \to \mathrm{E}\, Y$$

and

$$\mathrm{E}\bigg(\frac{1-(f_k(0))^Y}{1-f_k(0)}\bigg) \to \mathrm{E}\, Y.$$

By Theorem 3.3(ii), $\sum_{j=1}^{\infty} j \log j p_j < \infty$ implies that $0 < \mathrm{E}\, Y < \infty$ and this in turn implies, by Theorem 3.4(i),

$$\lim_{k\to\infty} \mathrm{P}\bigg(\sum_{i=1}^{Y} Z_{k,i} \geq 2 \;\bigg|\; \sum_{i=1}^{Y} Z_{k,i} \geq 1\bigg) = 1 - b_1, \qquad 0 < b_1 < 1.$$

Thus,

$$\overline{\lim_{k\to\infty}} \frac{\mathrm{E}\,\phi_k(Y)}{\mathrm{E}\,\psi_k(Y)} \leq \frac{1}{1-b_1}\bigg(1 - \frac{\mathrm{E}\, Y}{\mathrm{E}\, Y}\bigg) = 0,$$

i.e. $\lim_{k\to\infty} \pi_k = 0$, completing the proof of Theorem 2.2(ii).

(ii) Clearly,

$$\mathrm{P}(n - Y_n > k \mid Z_n \geq 1) = \mathrm{P}(Y_n < n-k \mid Z_n \geq 1) = \frac{\mathrm{P}((Y_n < n-k) \cap (Z_n \geq 1))}{\mathrm{P}(Z_n \geq 1)}.$$

The numerator is

$$\begin{aligned}
&\mathrm{P}(\text{there exist } i_1, i_2,\ i_1 \neq i_2,\ 1 \leq i_1, i_2 \leq Z_{n-k} \ni Z_{k,i_1}^{(n-k)} > 0,\ Z_{k,i_2}^{(n-k)} > 0) \\
&\quad = \mathrm{E}(1 - (f_k(0))^{Z_{n-k}} - Z_{n-k}(f_k(0))^{Z_{n-k}-1}(1 - f_k(0));\ Z_{n-k} > 0) \\
&\quad = \mathrm{E}(1 - (f_k(0))^{Z_{n-k}} - Z_{n-k}(f_k(0))^{Z_{n-k}-1}(1 - f_k(0)) \mid Z_{n-k} > 0)\,\mathrm{P}(Z_{n-k} > 0).
\end{aligned}$$

For fixed $k \geq 1$, the first term above goes to, by Theorem 3.3(i),

$$\mathrm{E}(1 - (f_k(0))^Y - Y(f_k(0))^{Y-1}(1 - f_k(0))).$$

Also, by Theorem 3.3(iii), for fixed $k \geq 1$,

$$\frac{\mathrm{P}(Z_n > 0)}{\mathrm{P}(Z_{n-k} > 0)} \to m^k \quad \text{as } n \to \infty.$$

Thus, for fixed $k \geq 1$,

$$\lim_{n\to\infty} \mathrm{P}(n - Y_n > k \mid Z_n > 0) = \frac{\mathrm{E}(1 - (f_k(0))^Y - Y(f_k(0))^{Y-1}(1 - f_k(0)))}{m^k} \equiv \tilde{\pi}_k.$$

Now,

$$\tilde{\pi}_k = \frac{\mathrm{E}(1 - (f_k(0))^Y)}{m^k}\bigg(1 - \frac{(\mathrm{E}\, Y(f_k(0))^{Y-1})(1 - f_k(0))}{\mathrm{E}(1 - (f_k(0))^Y)}\bigg).$$

As argued in the proof of (i) above, under the hypothesis that $\sum_{j=1}^{\infty} j \log j p_j < \infty$,

$$\mathrm{E}\bigg(\frac{1-(f_k(0))^Y}{1-f_k(0)}\bigg) \to 1 < \mathrm{E}\,Y < \infty, \qquad \mathrm{E}\,Y(f_k(0))^{Y-1} \to 1 < \mathrm{E}\,Y < \infty.$$

Also,

$$\mathrm{E}\bigg(\frac{1-(f_k(0))^Y}{m^k}\bigg) = \frac{1-f_k(0)}{m^k}\,\mathrm{E}\bigg(\sum_{j=0}^{Y-1}(f_k(0))^j\bigg),$$

and, by Theorem 3.3(iii),

$$\lim_{k\to\infty} \frac{1-f_k(0)}{m^k} = \frac{1}{\mathrm{E}\,Y}.$$

Thus, $\lim_{k\to\infty} \tilde{\pi}_k = 0$. This completes the proof of Theorem 2.2(ii).

Acknowledgement

The author would like to convey his grateful thanks to the anonymous referee for the suggestions given in Remarks 2.3 and 2.4 of using the continuum random tree for Theorem 2.1, the results of Geiger [3] for Theorem 2.2, and also the relevance of the results on Itô's excursion theory due to Le Gall [5].

References

[1] Athreya, K. B. (2012). Coalescence in the recent past in rapidly growing populations. *Stoch. Process. Appl.* **122,** 3757–3766.

[2] Athreya, K. B. and Ney, P. E. (2004). *Branching Processes*. Dover, Mineola, NY.

[3] Geiger, J. (1999). Elementary new proofs of classical limit theorems for Galton–Watson processes. *J. Appl. Prob.* **36,** 301–309.

[4] Kallenberg, O. (1986). *Random Measures*, 4th edn. Academic Press, London.

[5] Le Gall, J.-F. (2010). Itô's excursion theory and random trees. *Stoch. Process. Appl.* **120,** 712–749.

[6] Zubkov, A. M. (1975). Limit distribution of the distance to the nearest common ancestor. *Theory Prob. Appl.* **20,** 602–612.