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Fano threefolds in positive characteristic

N. I. SHEPHERD-BARRON
D.P.M.M.S., Cambridge University, 16 Mill Lane, Cambridge CB2 1SB, U.K.

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Abstract. We show that the birational classification in positive characteristic of smooth Fano threefolds X with Picard number 1 is the same as in characteristic zero. In particular, there are no exotic such Fanos; as a consequence of the classification, X is shown to be liftable without ramification to characteristic zero and to contain a line. The main techniques employed are those of Ekedahl and of Mori and Takeuchi.

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Introduction

In characteristic zero, Fano 3-folds X can be classified along the following lines, worked out by Mori and Takeuchi [M], [T]; their arguments represent a simplification of those due to Iskovskikh and Shokurov [I 1,2], [Sh 1,2].

(1) Kodaira vanishing shows that $H^i(\mathcal{O}_X) = 0$ and $H^i(\mathcal{O}(-K_X)) = 0$ for $i > 0$.

(2) An elementary argument involving (essentially) varieties of minimal degree shows that the index r of X is at most 4, and that if $r = 4$ (resp. $r = 3$) then $X \cong \mathbb{P}^3$ (resp. $X \cong Q_2 \hookrightarrow \mathbb{P}^4$).

(3) For $\rho(X) = 1$ and $r > 1$, the complete list was made by Iskovskikh; for $r = 2$ he assumed that if H is the positive generator of $\mathrm{NS}(X)$, then $|\ H\ |$ contains a smooth member. This assumption was subsequently shown by Fujita to be unnecessary (in all characteristics). Also, when $r \geqslant 2$ Megyesi (unpublished) has classified these varieties and shown that they are liftable to characteristic zero.

(4) For $\rho(X) = 1$ and $r = 1$ ('primitive and of the first species'), the first point is to show that $|-K_X|$ has a member H with at worst RDPs; H is then a K3 surface (cf. [Sh 1]). Then Saint-Donat's theorems about linear systems on K3's [SD] can be used to show that if $\deg X \geqslant 8$, then X is an intersection of quadrics.

(5) Then consideration of respectively triple projection from a general point and double projection from a conic (which, after Mori, can be best understood as respectively a double projection followed by a flop and ordinary projection followed by a flop) leads to the inequality $\deg X \leqslant 24$ (equivalent to $g \leqslant 13$, where $g = \frac{1}{2}c_1^3(X) + 1$ is, as usual, the genus of a curve section). Moreover, the analysis of the triple projection (involving a comparison of the various kinds of

extremal rays with the results of computing certain intersection numbers) shows that X is covered by conics, and then double projection from a general conic (which is an ordinary projection followd by a flop) excludes the possibilities $g = 11$ and $g = 13$. Moreover, the analysis of this conic projection shows that X contains a line. Taken together, the two kinds of projection give a birational classification (that is, a detailed description of various birational equivalences between Fano's of different degrees) [T]. Since this approach does not require the existence of lines to be proved a priori, it gives a substantial simplification of the classification (Shokurov's proof [Sh 2], [R 1] that lines exist is complicated). Iskovskikh [I3, I4] and Cutkosky [C] have also given other derivations of this birational classification using extremal rays and flops; however, they start by assuming the existence of lines (in [I3] and [C]) or lines and conics (in [I4]), rather than prove it using this part of Mori theory.

(6) The analysis of the cone $NE(X)$ and his description of extremal rays permits a classification if $\rho(X) \geqslant 2$ [MM]. This depends upon the fact, established in (5), that if $r = \rho = 1$, then X contains a surface swept out by lines, so that for any smooth curve C on X, $\mathrm{Bl}_C X$ is not Fano.

Step (5) is the main part of the classification and depends upon a number of other results, for example

(i) Grauert-Riemenschneider vanishing;
(ii) a description of flops;
(iii) the facts that X is not covered by lines, and that through a generic point P of X there is a finite (non-zero) number of conics and only finitely many rational quartic curves that are singular at P.

The aim of this paper is to establish (1)–(5) in characteristic $p > 0$; Kollár has already shown that extremal rays on smooth threefolds have the same description in char. p as in char. 0 [K3], except that conic bundles and pencils of del Pezzo surfaces may have wild behaviour. The additional arguments can be summarized as follows.

(1) By combining ideas of Ekedahl [E] with the elaboration developed by Kollár [K1] of bend-and-break techniques we prove a large piece of Kodaira vanishing (in particular, enough to show that $-K$ has at least as many sections as are predicted by Riemann–Roch; a priori, $|-K|$ may be empty).

(2) Following a suggestion by Mori, we overcome the failure of Bertini's theorem by considering generic members of linear systems rather than geometric generic members; this forces the consideration of linear systems on normal K3-like surfaces that are not geometrically normal, but there are no difficulties. This enables us to dispose of those Fano 3-folds for which $\mathrm{Bs}\,|-K|$ is not empty and to deal with the singularities arising from the projections just as in characteristic zero. The arguments are modifications of Mori's [M] in characteristic zero.

(3) If X is cut out by quadrics, then in odd characteristic a monodromy argument shows that it is not covered by lines. This is needed because double projection from

a point P on a line breaks down. When $\rho = 1$ this is extended to characteristic 2. The Hoffman–Singleton graph proves relevant at this point; it does not seem to have appeared in the context of algebraic geometry before.

Now assume that $r = \rho = 1$.

(4) Double projection from a general point P collapses only finitely many curves. The arguments of Mori and Takeuchi then carry over without change to show that $g \leqslant 13$ and that there is a conic through P.

(5) Projection from a generic conic also collapses only finitely many curves.

(6) Conic bundles and del Pezzo fibrations behave sufficiently well (although there is a subtlety involving del Pezzo fibrations in characteristic 2).

(7) If $g \geqslant 7$, then we get a birational description as in characteristic zero.

(8) We use this description to prove that $H^1(\Omega^1_X)$ is generated by the Chern classes of divisors. We then deduce that $H^2(\Theta_X) = 0$, so that X can be lifted, without ramification, to characteristic zero.

(9) Once the variety can be lifted to characteristic zero the existence of lines is immediate. This result is crucial in Mori and Mukai's classification of Fano 3-folds of Picard number at least 2.

On the other hand, Mukai has given a complete *biregular* classification of embedded primitive Fano 3-folds of the first species. Given a Fano 3-fold with $11 \neq g \leqslant 12$, he constructs a vector bundle to embed the variety in some homogeneous space G/P, where P is a maximal parabolic. I have not checked whether or to what extent this carries over to all characteristics. Perhaps it is worth pointing out in this context that in characteristic p there are no exotic homogeneous spaces of Picard number 1. The reason is that if P is a maximal parabolic and $P_0 = P_{\mathrm{red}}$, then $\mathrm{Lie}(P)$ contains $\mathrm{Lie}(P_0)$, while $\mathrm{Lie}(P_0)$ is a maximal Lie sub–algebra of $\mathrm{Lie}(G)$. Hence the morphism $G/P_0 \to G/P$ is a power of the geometric Frobenius, so that G/P_0 and G/P are conjugate varieties.

Mukai has also given a simple proof of the degree bound $11 \neq g \leqslant 12$, independent of any classification (birational or biregular), using the moduli of curves and K3 surfaces. However, this depends on proving that for a smooth hyperplane section S, the versal deformation space of the pair (X, S) maps onto the space of polarized deformations of S, which in turn depends on knowing that $H^1(\Omega^1_X)$ is spanned by algebraic classes. This is not clear a priori in positive characteristic (except as a consequence of the Tate conjecture).

Finally, the classification when $\rho \geqslant 2$ is left open, although some of the results used by Mori and Mukai in characteristic zero are established here (such as the existence of lines when $r = \rho = 1$ and the analysis of conic bundles and del Pezzo fibrations).

1. Kodaira vanishing

THEOREM 1.1 *If X is a Fano variety of arbitrary dimension, then the irregularity $q(X)$ is zero.*

Proof. Let $\alpha\colon X \to A$ be the Albanese mapping, and let D be an ample divisor on A. By the cone theorem, there are finitely many rational curves $C_1, \dots, C_n$ on X such that for any curve Γ on X, there are rational numbers $m_1, \dots, m_n$ with $\Gamma \equiv \sum m_i C_i$. Since α collapses every rational curve, we have $\Gamma \,.\, \alpha^* D = \sum m_i C_i \,.\, \alpha^* D = 0$. Then α collapses Γ, so that α is constant. □

COROLLARY 1.2 *If X is a Fano 3-fold, then $\chi(\mathcal{O}_X) \geqslant 1$.*

Proof. $H^1(\mathcal{O}_X)$ is the tangent space to $\mathrm{Pic}^0\ X$ and $H^2(\mathcal{O}_X)$ is the obstruction space. By Theorem 1.1 $\dim\ \mathrm{Pic}^0\ X = 0$, and the Corollary follows. □

THEOREM 1.3 *If X is a Fano n-fold, but possibly with local complete intersection (lci) singularities, then through any smooth point x of X there is a rational curve L with $L \,.\, (-K_X) \leqslant n+1$.*

Proof. Kollár has shown [K1] that if C is a curve and $f: C \to X$ is a morphism such that $f(C)$ does not lie in the singular locus of X, then

$$\dim_{[f]}\ \mathrm{Mor}(C, X) \geqslant \deg f^*(-K_X) + n \,.\, \chi(\mathcal{O}_C).$$

So we can bend-and-break just as usual, using the fact that $-K_X$ is ample. □

THEOREM 1.4 *Suppose that X is a normal lci Fano 3-fold and that $D \in$ Pic X is ample. Then $H^1(\mathcal{O}(-D)) = 0$ if either $p \geqslant 5$ or X is smooth and $D \,.\, c_2 \geqslant 0$.*

Proof. Assume that $H^1(X, \mathcal{O}(-D)) \neq 0$. By Serre vanishing, we may assume that $H^1(X, \mathcal{O}(-pD)) = 0$. Take a non-zero class $\sigma \in H^1(\mathcal{O}(-D))$. Then $\sigma^p = 0$, so that by [E] there is a morphism $\rho : Y \to X$ which is a torsor under some α_p-group scheme over X; here, Y is reduced and irreducible with lci singularities, $\deg \rho = p$ and $\omega_Y \cong \rho^* \mathcal{O}(K_X - (p-1)D)$.

(One can also construct Y as follows. The class σ corresponds to a non-split extension $\mathcal{E}$ of $\mathcal{O}(-D)$ by $\mathcal{O}_X$. The vanishing of σ^p means that $F^*\mathcal{E} = \tilde{\mathcal{E}}$, say, splits. Set $\mathbb{P} = \mathbb{P}(\mathcal{E})$, which has a section X_0 corresponding to the description of $\mathcal{E}$ as an extension. Put $U = \mathbb{P} - X_0$; this is just the torsor corresponding to σ under the line bundle $\mathcal{O}(-D)$, regarded as a $\mathbb{G}_a$-group scheme, and as such is an affine line bundle. Put $\tilde{\mathbb{P}} = \mathbb{P}(\tilde{\mathcal{E}})$; then $\tilde{\mathbb{P}}$ contains a section X_1 that maps to a copy of Y in $\mathbb{P}$ which is disjoint from X_0. The advantage of the torsorial description, however, is that it makes it clear, even if X is singular, that Y is a Cartier divisor on $\mathbb{P}$ so that all information about Y that we need (mainly a description of ω_Y and $\rho_* \mathcal{O}_Y$) can be derived from the adjunction formula.)

(N.B. If X were non-normal, then this non-trivial torsor Y might be non-reduced, and the argument that follows would not apply. In fact, Reid [R] has constructed non-normal del Pezzo surfaces in char. p on which Kodaira vanishing fails, and if $p = 2$ or 3 then some of his examples even have hypersurface singularities.)

Note that from the description above of ω_Y, Y is Fano. Choose a smooth point y on Y. By Theorem 3 there is a rational curve C through y with $C\,.\,(-K_Y) \leqslant 4$. Hence $\rho^*((p-1)D - K_X)\,.\dot{C} \leqslant 4$, so that $p \leqslant 3$. Note also that this bound means that we can assume that D is a maximal counterexample to Kodaira vanishing, in that $H^1(X, \mathcal{O}(-c_1 - rD)) = 0$ and $H^1(X, \mathcal{O}(-sD)) = 0$ for all $r \geqslant 1$ and $s \geqslant 2$.

Assume now that X is smooth and $D\,.\,c_2 \geqslant 0$. Also, we know that $\rho_*\mathcal{O}_Y$ has an increasing filtration whose graded pieces are $\mathcal{O}_X, \mathcal{O}_X(D), \ldots, \mathcal{O}_X((p-1)D)$. (This is proved by Ekedahl [E], and is derived from the description of Y as a subvariety of the affine bundle U above.) So from Riemann-Roch and the inequalities $\chi(\mathcal{O}_X) \geqslant 1$, $D\,.\,c_2 \geq 0$, we get $\chi(\mathcal{O}_Y) > \chi(\mathcal{O}_X)$. Then $h^2(\mathcal{O}_Y) > h^2(\mathcal{O}_X)$, so that $h^2(\mathcal{O}_X(rD)) > 0$ for some $r \geqslant 1$. Serre duality now gives $h^1(\mathcal{O}_X(-(c_1 + rD))) > 0$, contrary to the maximality of D. □

COROLLARY 1.5 *If X is a smooth Fano threefold, then*

(1) $H^i(\mathcal{O}_X) = 0$ *for* $i > 0$ *and* $\chi(\mathcal{O}_X) = 1$.
(2) Pic X *has no torsion.*
(3) $\pi_1^{\text{alg}}(X) = 1$.
(4) $h^0(\mathcal{O}(-K_X)) \geqslant c_1(X)^3/2 + 3$.

Proof. (1) By Corollary 1.2, Theorem 1.4 and the Riemann–Roch theorem, it follows that $H^1(\mathcal{O}(nK)) = 0$ for all $n \geqslant 1$. Then by Serre duality and Corollary 1.2, we see that $0 = h^1(\mathcal{O}(K)) = h^2(\mathcal{O}) \geqslant h^1(\mathcal{O})$, which proves (1).

Suppose that $D \in$ Pic X with $nD = 0$ and $D \neq 0$. By (1) and R–R, $\chi(\mathcal{O}(D)) = 1$, so that $h^2(\mathcal{O}(D)) = 1$. Via Serre duality, this gives a contradiction to Theorem 1.4, and so proves (2).

Suppose that $Y \to X$ is a finite étale cover of degree n. Then Y is Fano, so that by (1) $n = 1$. This proves (3).

(4) is now a consequence of Riemann–Roch. □

2. Linear systems on K3-like surfaces

In this section we consider connected projective surfaces F defined over the function field K of some k-variety. F will be normal, but possibly not geometrically normal.

DEFINITION. F is K3-like if in addition $\omega_F \cong \mathcal{O}_F$ and $H^1(\mathcal{O}_F) = 0$.

Recall that the notion of RDP makes sense in the context of arbitrary normal excellent 2-dimensional schemes.

PROPOSITION 2.1 *Suppose that F is K3-like and has only RDPs, and that H is a nef and big Cartier divisor on F. Then $H^1(\mathcal{O}(-H)) = 0$.*

PROPOSITION 2.2 *Suppose that F is K3-like with only RDPs, that H is ample and that* $\mathrm{Bs}\,|H|$ *is not empty and is defined over k.*

(1) *If $H^2 \geqslant 4$, the base locus is isomorphic to $\mathbb{P}^1_K$.*
(2) *If $H^2 = 2$, then the base locus is a single RDP, F embeds into $\mathbb{P}(1,1,1,2,3)$ as a $(2,6)$ complete intersection and $H = \mathcal{O}(1)$.*

Proof. Over an algebraically closed field these results are well known. One proof of them depends upon the fact that if E is a rank two vector bundle on the minimal resolution $f : \widetilde{F} \to F$ with $c_1(E)^2 - 4c_2(E) \geqslant -2$, then the Riemann–Roch theorem shows that $\dim \mathrm{Hom}(E,E) \geqslant 3$, so that, by the Cayley–Hamilton theorem, E has a non-zero nilpotent endomorphism. This proof carries over to the present context, where $\widetilde{F}$ is a regular scheme rather than a smooth surface. For example, if P is a K-rational RDP in the base locus, let Z denote the corresponding fundamental cycle on $\widetilde{F}$. There is then a non-split extension E of $\mathcal{O}(f^*H - Z)$ by $\mathcal{O}$, and Reider's arguments show that case (2) holds. (Recall that the defining property of Z is that $Z^2 = -2$ and that $Z\,.\,A \leqslant 0$ for any curve A contracted by f.) □

We shall apply these results when K is the function field of the anti–canonical system $|-K_X|$ of a Fano 3-fold X, F is the generic member of $|-K_X|$ and $|H|$ is the complete linear system cut out on F by $|-K_X|$ (via Corollary 1.5).

The next three sections deal with the separate cases where $|-K|$ has base points or defines a morphism that is not birational or defines a morphism that is birational.

3. The base locus of $|-K_X|$

In this section $\mathrm{Bs}\,|-K_X|$ is assumed to be non–empty. The aim is to describe X under this hypothesis; the arguments are taken over almost unchanged from Mori's notes covering the case of characteristic zero, which in turn simplify Shokurov's arguments [Sh 1] by using the geometry of extremal rays. We give most of the details since these notes are unpublished.

LEMMA 3.1 $|-K_X|$ *is not composite with a pencil.*

Proof. Omitted; the proof is as in Mori's notes or [Sh 1]. □

THEOREM 3.2 (1) *The generic member of $|-K_X|$ is K3-like.*

(2) $H^1(\mathcal{O}(-K_X)) = 0$ *and* $h^0(\mathcal{O}_X(-K_X)) = \frac{1}{2}(-K_X)^3 + 3$.

(3) *The geometric generic member of $|-K|$ is reduced and irreducible.*

Proof. Put $H = -K_X$ and $|H| = D_0 + |D|$, where D_0 is the fixed part. According to Abhyankar, there is a sequence of blow-ups $X_{i+1} \to X_i$ with smooth centres C_i and exceptional divisor E_i whose composite $\sigma : \widetilde{X} \to X$ resolves the base locus. That is, if $\pi_{i+1} : \widetilde{X} \to X_{i+1}$ is the composite and $F_i = \pi_{i+1}^* E_i$, then we can write

$$\sigma^* D = \widetilde{D} + \sum n_i F_i,$$

where $n_i \geqslant 1$, $|\widetilde{D}|$ has no base points and $\sum n_i F_i$ is the fixed part of $|\sigma^* D|$.

Now consider the generic member $\widetilde{F}$ of $|\widetilde{D}|$. By Lemma 3.1, this is a regular K-scheme that is geometrically reduced and irreducible, where K is the function field of $|H|$. Let F be its image in $X \otimes K$ and $\widetilde{F} \to F' \to F$ the Stein factorization. Since $K_{\widetilde{X}} \sim \sigma^* K_X + \sum a_i F_i$, where $a_i = 1$ if F_i maps to a curve in X, we see that $K_{\widetilde{F}} \sim (-\sigma^* D_0 - \sum (n_i - a_i) F_i)|_{\widetilde{F}}$. Hence

$$K_{F'} \sim \left(-\sigma^* D_0 - \sum (n_i - 1) F_i\right)|_{F'},$$

where the sum is over those i for which F_i maps to a curve in X. In particular, $K_{F'}$ is anti–effective or zero.

Define $G = \sigma^* H|_{\widetilde{F}}$ and $L = K_{\widetilde{F}} + G$. So by Proposition 2.1 $H^i(\mathcal{O}_{\widetilde{F}}(-G)) = 0$ for $i \geqslant 1$. Then

$$h^0(\mathcal{O}_{\widetilde{F}}(L)) = \chi(\mathcal{O}_{\widetilde{F}}(L)) = \tfrac{1}{2} L \,.\, (L - K_{\widetilde{F}}) + \chi(\mathcal{O}_{\widetilde{F}}).$$

We also know that $h^0(\mathcal{O}_{\widetilde{F}}(\widetilde{F})) = h^0(\mathcal{O}_{\widetilde{X}}(\widetilde{F})) - 1$, since $H^1(\mathcal{O}_{\widetilde{X}}) = 0$, so that

$$h^0(\mathcal{O}_{\widetilde{F}}(L)) \geqslant h^0(\mathcal{O}_{\widetilde{X}}(\widetilde{F})) - 1 = h^0(\mathcal{O}_X(H)) - 1 \geqslant \tfrac{1}{2} H^3 + 2,$$

where the last inequality comes from Corollary 1.5. Hence

$$\tfrac{1}{2} H^3 + 2 \leqslant \frac{1}{2} \left(\widetilde{D} + \sum a_i F_i\right) . \, \widetilde{D} \,.\, \sigma^* H + \chi(\mathcal{O}_{\widetilde{F}}).$$

Since $\widetilde{D} = \sigma^* H - \sum n_i F_i - \sigma^* D_0$, we see that

$$\begin{aligned} \tfrac{1}{2} H^3 + 2 \ \leqslant\ & \frac{1}{2} \left(\sigma^* H + \sum (a_i - n_i) F_i - \sigma^* D_0\right) \\ & . \left(\sigma^* H - \sum n_i F_i - \sigma^* D_0\right) . \, \sigma^* H + \chi(\mathcal{O}_{\widetilde{F}}). \end{aligned}$$

Now for all divisors A, B on X we have $F_i \,.\, \sigma^* A \,.\, \sigma^* B = 0$, and also $F_i \,.\, F_j \,.\, \sigma^* A = 0$ unless both $i = j$ and F_i maps to a curve in X, in which case $a_i = 1$. Hence

$$\tfrac{1}{2} H^3 + 2 \leqslant \tfrac{1}{2} H \,.\, (H - D_0)^2 + \sum n_i (n_i - 1) F_i^2 \,.\, \sigma^* H + \chi(\mathcal{O}_{\widetilde{F}}),$$

where the sum is over those i such that F_i maps to a curve C_i in X. Then $F_i^2 \,.\, \sigma^* H = C_i \,.\, H$ and the Hodge index theorem on a general member of $|nH|$ for $n \gg 0$ shows that $H \,.\, (H - D_0)^2 \leqslant H^3$, so that

$$\chi(\mathcal{O}_{\widetilde{F}}) \geqslant 2.$$

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Since $K_{F'}$ is anti-effective or zero, it follows readily that F' is K3-like and that $F' = F$. This proves the first part of the theorem. Also, it follows that all the inequalities above are in fact equalities, and in particular that $D_0 = 0$ and $h^0(\mathcal{O}_X(H)) - 1 = \frac{1}{2}H^3 + 2$. The second part of the theorem now follows, and the third is an immediate consequence of the first. $\square$

LEMMA 3.3 *If X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ and is Fano, then* $\mathrm{Bs}\,|-K_X|$ *is empty.*

Proof. Write $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a) \oplus \mathcal{O}(b))$ with $0 \geqslant a \geqslant b$ and let $C = \mathbb{P}(\mathcal{O}(b))$ be the most negative section. Let F be the class of a fibre and D the tautological class. Then $K_X \sim -3D + (a+b-2)F$, so that $0 < (-K_X)\,.\,C = 3b - a - b + 2$. Thus $a = b = -1$ or $a = b = 0$. In the first case $-K \sim 3D + 4F$ and $|D + F|$ has no base locus, and in the second $-K \sim 3D + 2F$ and $|D|$ has no base locus. In each case $\mathrm{Bs}\,|-K|$ is empty. $\square$

THEOREM 3.4 *Assume that* $\mathrm{Bs}\,|-K_X|$ *is non–empty. Then either*

(1) *$X \cong S \times \mathbb{P}^1$, where S is a del Pezzo surface of degree 1 or*
(2) *$X \cong \mathrm{Bl}_C Y$, where $Y \hookrightarrow \mathbb{P}(1,1,1,2,3)$ as a hypersurface of degree 6 and $C = Y \cap (1) \cap (1)$ is a smooth elliptic curve.*

Proof. By Theorem 3.2 a generic member F of $|H| = |-K_X|$ is K3-like, and $|H|$ cuts out a complete linear system $|D|$ on it. Clearly $\mathrm{Bs}\,|D| = \mathrm{Bs}\,|H| \otimes K$ as subschemes of $X \otimes K$, so that by Proposition 2.2 either F embeds in $\mathbb{P}(1,1,1,2,3)$ as a $(2,6)$ complete intersection or $\mathrm{Bs}\,|H|$ is, as a scheme, a copy Γ of $\mathbb{P}^1$.

In the first case X then embeds in $\mathbb{P} = \mathbb{P}(1,1,1,1,2,3)$ as a $(2,6)$ complete intersection. Since $|-K_X| = |\mathcal{O}_X(1)|$ has base points, X must pass through the singular locus of $\mathbb{P}$, which contradicts the smoothness of X. So the first case is impossible.

So consider the second case, and put $f\colon V = \mathrm{Bl}_\Gamma X \to X$ with exceptional divisor E. Let $\phi\colon V \to \widetilde{Y}$ be the Stein factorization of the morphism defined by $|-K_V|$. Since on F there is a pencil of genus one curves C with $C\,.\,H = 1$, the strict transforms of these curves are collapsed by ϕ, so that $\widetilde{Y}$ is a surface (by Lemma 3.1) and the induced morphism $E \to \widetilde{Y}$ is birational.

Next, I claim that $g = 4$. For this, some intersection numbers are required. We have $(H - E)^3 = 0$, so that $H^3 + 2 = 4\deg\Gamma$. Since by definition $H^3 = 2g - 2$, it follows that $g = 2\deg\Gamma$. Also, $H|_F \sim \Gamma + gA$, where A is a genus one curve, so that $H\,.\,\Gamma = g - 2$. This proves the claim.

Note that $(H - E)^2\,.\,E = H^2\,.\,E - 2H\,.\,E^2 + E^3 = 3\deg\Gamma - 2 = 4$, so that $\widetilde{Y} = Y$ is a surface of degree 4 in $\mathbb{P}^5$.

Suppose now that the map $E \to Y$ is not an isomorphism. Then it contracts a negative section Σ on E and Y is a quartic cone. Since $-K_V \sim \phi^*\mathcal{O}(1)$, we get $-K_V \sim F_1 + \cdots + F_4$, where F_i is the inverse image of a line. Then H has index 4, while $H\,.\,\Gamma = 2$. Hence $E \to Y$ is an isomorphism.

Let $Y \to \mathbb{P}^1$ be the map corresponding to $E \to \Gamma$; then the composite $V \to \mathbb{P}^1$ factors through X, say via a morphism $\psi\colon X \to \mathbb{P}^1$ of which Γ is a section. Since X is Fano, there is an extremal ray R on X whose associated contraction contr_R does *not* factor through ψ. Then contr_R cannot contract a divisor to a point, and so is of type E_1 or D or C.

Let Σ be a negative section on Y and m a fibre of the map $Y \to \mathbb{P}^1$. Put $Z = f_*\phi^*\Sigma$ and $D = f_*\phi^*m$. Let ℓ be an extremal curve of minimal degree spanning R. Then $D\,.\,\ell > 0$, since contr_R does not factor through ϕ, and $1 \leqslant H\,.\,\ell \leqslant 3$, e.g. from the classification of extremal rays. If $H\,.\,\ell = 3$, then contr_R makes X a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, which contradicts Lemma 3.3. So $H\,.\,\ell \leqslant 2$. There are two cases to consider:

(a) $E \cong \mathbb{F}_2$. Then $H \sim Z + 3D$. If $Z\,.\,\ell < 0$, then R is of type E_1 and Z is the divisor contracted by contr_R; then $Z\,.\,\ell = -1$ and $K_X\,.\,\ell = -1$, which is absurd. So $Z\,.\,\ell \geqslant 0$, so that $-K_X\,.\,\ell \geqslant 3$, which is impossible. So this case cannot happen.

(b) $E \cong \mathbb{F}_0$. Then $H \sim Z + 2D$; since both Z and D move in their linear equivalence classes, we have $Z\,.\,\ell \geqslant 0$ and $D\,.\,\ell \geqslant 0$. Hence $Z\,.\,\ell = 0$ and $D\,.\,\ell = 1$, and contr_R makes X a $\mathbb{P}^1$-bundle over a smooth surface S, say. By considering both $X \to S$ and $\psi\colon X \to \mathbb{P}^1$, it is easy to see that $X \cong S \times \mathbb{P}^1$; since X is Fano and $\mathrm{Bs}\,|-K_X|$ is not empty, it follows that S is del Pezzo of degree 1. □

Convention. Henceforth X will denote a Fano 3-fold (of Picard number $\rho(X) = \rho$) on which $|-K|$ has no base points, and so defines a morphism $\phi\colon X \to Y$ where $Y \hookrightarrow \mathbb{P}^{g+1}$. We shall also assume that X is of index 1; Megyesi has shown that if not, then X is classified exactly as in characteristic zero.

4. The morphism defined by $|-K|$

PROPOSITION 4.1 *If ϕ is not birational, then* $\deg\phi = 2$ *and Y is a 3-fold of minimal degree $g-1$ in $\mathbb{P}^{g+1}$. This occurs if and only if one of the following conditions holds:*

(i) $g = 2$, $Y = \mathbb{P}^3$ *and X is the double cover of $\mathbb{P}^3$ branched in a sextic.*

(ii) $g = 3$, *Y is a quadric and X is the double cover of Y branched in a quartic section.*

(iii) $g \geqslant 4$ *and Y is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Moreover,* $\rho \geqslant 2$ *and there is a morphism $X \to \mathbb{P}^1$ whose fibres are del Pezzo surfaces of degree 2.*

Proof. Exactly as in char. zero. □

PROPOSITION 4.2 *If ϕ is birational, then it is an isomorphism onto its image and the image is projectively normal.*

Proof. This is proved exactly as in characteristic zero, using Noether's theorem on canonical curves. □

PROPOSITION 4.3 *If $|-K|$ is very ample, then there are four possibilities:*

(i) $g = 3$ *and X is a quartic 3-fold in $\mathbb{P}^4$;*
(ii) $g = 4$ *and X is a quadro-cubic complete intersection in $\mathbb{P}^5$;*
(iii) $g \geqslant 5$, *the intersection Y of the quadrics containing X is a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$ and the induced map $X \to \mathbb{P}^1$ exhibits a pencil of cubic surfaces on X;*
(iv) $g \geqslant 5$ *and X is an intersection of quadrics.*

Proof. Just as in characteristic zero. □

COROLLARY 4.4 *If $\rho(X) = 1$ and $g \geqslant 5$, then $|-K|$ is very ample and the anti–canonical model of X is cut out by quadrics.* □

5. Coverings by lines

In this section X is Fano of index one and is anti-canonically embedded in $\mathbb{P}^{g+1}$ as an intersection of quadrics.

We shall show that if $p \neq 2$, then X is not covered by lines. This is crucial for making multiple projection from a general point. In characteristic zero this is easy, but in general there can be too many lines. For example, the Fermat quartic 3-fold X in characteristic 3 has such a covering; if P is a general point on X, then the tangent space to X at P meets X with multiplicity 3, and then projecting this intersection from P shows that that are lines on X through P. This behaviour turns out to be closely related to the fact that the embedding of X in $\mathbb{P}^4$ is not Lefschetz. Recall that a general pencil of hyperplane sections in a Lefschetz embedding is Lefschetz, in the sense that the singular members have just one singular point, a node, and the general member is smooth. A point P is Lefschetz if there is a hyperplane section that is smooth away from P and has a node at P. If the characteristic is odd (or zero) or the dimension is even, then the embedding is Lefschetz and some member of every Lefschetz pencil is singular if and only if a general point (or if some point) is Lefschetz [SGA 7]. Since we are dealing with 3-folds, characteristic 2 be more delicate and is discussed in Section 7.

PROPOSITION 5.1 *Fix $P \in X$, and let $\sigma: \widetilde{X} = \mathrm{Bl}_P X \to X$ be the blow-up with exceptional divisor $E \cong \mathbb{P}^2$. Let $L \subset |\mathcal{O}_E(2)|$ be the system of conics cut out by $|-K_{\widetilde{X}}| = |-K_X - 2P|$. Then the base points of L correspond to the lines on X through P, and there are four possibilities:*

(i) *L has a smooth member (in which case P is a Lefschetz point);*
(ii) *L has a fixed line, or is empty (when there is a plane in X through P);*
(iii) *L has a unique base point (in which case there is a unique line in X through P);*

(iv) *$p = 2$ and L is the set of all double lines in E.*

Proof. This is an elementary and well known consequence of X being cut out by quadrics. □

THEOREM 5.2 (1) *If $p \neq 2$, then the embedding is Lefschetz and in a general pencil some member is singular.*

(2) *If $p = 2$, then either a general point is Lefschetz or X is a wild conic bundle over a surface or L is the set of double lines in E.*

Proof. Assume that the result is false; then either (ii) or (iii) of 5.1 holds.

If (ii) holds, then as P moves on X we get a 1-dimensional family $\{M_t\}_{t\in T}$ of planes on X. Let ℓ be a line in one of these planes M; then $(-K_X) \,.\, \ell = 1$, so that $(N_{M/X} \,.\, \ell)_M = -2$, by adjunction. But this contradicts the fact that M moves.

If (iii) holds, let T be an irreducible projective surface in the Hilbert scheme of lines on X, $\rho\colon L \to T$ the universal family and $\pi\colon L \to X$ the canonical projection. If T is chosen appropriately, then π is dominant and generically one-to-one (so either birational or purely inseparable).

Suppose that π is not finite; then there is an irreducible curve $\Gamma \subset L$ such that $\pi(\Gamma)$ is a point Q, say. Then $\rho(\Gamma)$ cannot be a point, and so it is a curve Δ, say. Put $\Sigma = \rho^{-1}(\Delta)$. Then $\pi(\Sigma)$ is a cone with vertex Q. Since X is smooth at Q, $\pi(\Sigma)$ is the cone $\hat{C}$ over an irreducible plane curve C. Since X is an intersection of quadrics, C is a conic. Let $\ell \subset \hat{C}$ be a generator. Since $(-K_X \,.\, \ell) = 1$, we get $\hat{C} \,.\, \ell = -1$, by adjunction. This contradicts the fact that ℓ moves in a 2-dimensional family on X.

Hence π is finite and purely inseparable. After iterating the Frobenius, we get morphisms $X \to L^{(n)} \to T^{(n)}$ for some $n \gg 0$. Then the morphism $X \to T^{(n)}$ exhibits the lines parametrized by T as spanning an extremal ray, say R. But then the classification of extremal rays shows that contr_R makes X a wild conic bundle. This can only happen if $p = 2$. □

THEOREM 5.3 *Assume that $p \neq 2$. Then X is not covered by lines.*

Proof. Take a Lefschetz pencil $\{S_t\}_{t\in T}$ on X and let S be a geometric generic member. Each S_t is birationally K3, and so cannot carry a pencil of smooth rational curves, so that if X carries a 2-dimensional family of lines, then every S_t has a non-zero but finite number of lines. Fix a prime $\ell \neq p$, and let x be the class in $H^2(S, \mathbb{Q}_\ell(1))$ of a line on H.

If x is not monodromy invariant, then by [SGA 7], there is a vanishing cycle $\delta \in H^2(S, \mathbb{Q}_\ell(1))$ such that $x \,.\, \delta \neq 0$. Let σ denote the reflection in δ, so that $\sigma(x) \neq x$. Since $\sigma(x)$ is the image of x under some Galois conjugation, $\sigma(x)$ is a line on H. Hence $x \,.\, \sigma(x) = 0$ or 1. This gives $x \,.\, (x + (\delta.x)\delta) = 0$ or 1, so that $(x \,.\, \delta)^2 = 2$ or 3. But $x \,.\, \delta \in \mathbb{Q}_\ell$, which by quadratic reciprocity is impossible if $\ell = 5$ or 19. Thus every line in S is monodromy invariant.

So suppose that $l \subset S$ is a Galois-invariant line. We shall show that l is defined over $K = k(\eta)$, where η is the generic point of T. For this, we can replace K be any separable extension of itself. In particular, we can assume that S_η has a K-point. Put $\bar{\mathcal{L}} = \mathcal{O}_S(l)$ and $G = \mathrm{Gal}(\bar{K}/K)$. So $\bar{\mathcal{L}} \in (\mathrm{Pic}\, S)^G$. Now the locally finite group scheme $\mathrm{Pic}\, S_\eta$ is étale over K, so that $\bar{\mathcal{L}}$ is defined over the separable closure K^{sep} of K. Now $H^2(G,(K^{\mathrm{sep}*})) = 0$, by Tsen's Theorem, so that the Hochschild–Serre spectral sequence gives an isomorphism $\mathrm{Pic}\, S_\eta \to (\mathrm{Pic}\, S_\eta \otimes K^{\mathrm{sep}})^G$. Hence there is a line bundle $\mathcal{L}$ on S_η such that $\bar{\mathcal{L}} \cong \mathcal{L} \otimes \bar{K}$. A nonzero section of $\mathcal{L}$ then defines a K-rational divisor D such that $l = D \otimes \bar{K}$. Then the Zariski closure of D in the threefold maps to a subvariety L of X such that L meets a general hyperplane in a line. But then L is a plane, so that X is covered by planes. Then $\mathcal{N}_{L/X} \cong \mathcal{O}(n)$, with $n \geqslant 0$, and $-K_X\,|_L \cong \mathcal{O}(1)$. However, this contradicts the adjunction formula. □

Remark. If $p = 2$, then there are Fano 3-folds that are wild conic bundles (i.e., conic bundles all of whose fibres are double lines), and these are covered by lines.

We shall sharpen these results when $\rho(X) = 1$, but to do this we shall need a version of Grauert–Riemenschneider vanishing.

6. Grauert–Riemenschneider vanishing

In this section we shall prove an ad hoc version of G.–R. vanishing which is enough, for example, to make double projection from a point to work.

PROPOSITION 6.1 *Suppose that V is a smooth threefold, that $|D| = |-K_V|$ has no base points and that the Stein factorization $\phi\colon V \to Y$ of the morphism defined by $|D|$ is birational. Assume also that $H^i(V,\mathcal{O}_V) = 0$ for $i > 0$.*

Then Y has Gorenstein singularities, $\omega_V = \phi^*\omega_Y$ and $R^i\phi_*\mathcal{O}_V = 0$ for $i > 0$.

Proof. We have $D \sim \phi^*H$ for some ample H. Since ϕ is birational, it is clear that $\omega_V = \phi^*\omega_Y$.

Let S be the generic member of $|D|$ defined over the function field K of $|D|$. Since $|D|$ has no base points, S is a regular scheme, and is clearly K3-like. By abuse of notation, we shall not distinguish between V and $V \otimes K$.

Consider the exact sequence

$$0 \to \mathcal{O}_V((n-1)D) \to \mathcal{O}_V(nD) \to \mathcal{O}_S(nD) \to 0.$$

Now $H^1(S,\mathcal{O}_S(nD)) = 0$ for all $n \in \mathbb{Z}$, by Proposition 2.1 (the vanishing theorem for K3 surfaces with RDPs), so that $H^1(V,\mathcal{O}_V((n-1)D)) \twoheadrightarrow H^1(V,\mathcal{O}_V(nD))$ for all $n \in \mathbb{Z}$. Since $H^1(\mathcal{O}_V) = 0$, it follows that $H^1(V,\mathcal{O}_V(nD)) = 0$ for all $n \geqslant 0$. Also, $H^0(S,\mathcal{O}_S(nD)) = 0$ for all $n < 0$, so that $H^1(V,\mathcal{O}((n-1)D)) \hookrightarrow H^1(V,\mathcal{O}(nD))$ for all $n < 0$. Hence $H^1(V,\mathcal{O}(nD)) = 0$ for all n, and then by Serre duality $H^2(V,\mathcal{O}(nD)) = 0$ for all n.

There is a Leray spectral sequence

$$E_2^{pq} = H^p(Y,(R^q\phi_*\mathcal{O}_V) \otimes \mathcal{O}_Y(nH)) \Rightarrow H^{p+q}(V,\mathcal{O}_V(nD)).$$

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Take $n \gg 0$; then $E_2^{pq} = 0$ for all $p > 0$, by Serre vanishing. It follows that $H^0(Y, (R^q\phi_*\mathcal{O}_V) \otimes \mathcal{O}_Y(nH)) \cong H^q(V, \mathcal{O}_V(nD))$ for all q, so that $R^q\phi_*\mathcal{O}_V = 0$ for $q > 0$.

Since $\omega_V = \phi^*\omega_Y$, it follows that $R^q\omega_V = 0$ for $q > 0$ and $\phi_*\omega_V = \omega_Y$, and now Kempf's proof [KKMS] that in char. zero rational singularities are Cohen-Macaulay goes through to give the result. □

Remark We shall use this result when V is the blow-up of an embedded Fano 3-fold along either a line or a conic or a point lying on no line.

7. The case $\rho(X) = 1$

In Sections 7–10 we assume that $\rho(X) = 1$, in addition to the hypotheses of Section 5. This enables us to get the answers to the questions of projective geometry that must be solved to give the birational classification.

THEOREM 7.1 *If $p \neq 5$, then X is not covered by lines.*

Proof. Assume that X is covered by lines. By Theorem 5.2(2) a general point of X is Lefschetz, so that every member S of a general pencil of hyperplane sections of X has at most finitely many singularities, all of which are nodes. So S is birationally K3, and so cannot carry a pencil of lines. Hence every hyperplane section of X contains a line. Let S be a general such section, and l a line in S (so that in particular l is a general line on X). Since $\rho = 1$, X is not a wild conic bundle, and so the proof of Proposition 5.2 shows that a general point on l lies on another line in X, so that there is an irreducible surface $\Sigma \subset X$ swept out by lines through l.

Put $|H| = |-K_X|$ and let $\sigma : V = \mathrm{Bl}_l\, X \to X$ be the blow-up. Note that by construction, a general member S of $|\ H - l\ |$ is smooth, so that if $\widetilde{S}$ is its strict transform on V, then $\widetilde{S} \to S$ is an isomorphism.

A general member of $|H - l| = |-K_V|$ meets Σ set-theoretically in a union of lines (as is easy to see), including l; besides l, these lines are disjoint, since X is cut out by quadrics. Also, $|H - l|$ defines a birational morphism $\psi : V \to W \hookrightarrow \mathbb{P}^{g-1}$, with Stein factorization $\phi : V \to Y$, say. By cutting down to a curve section and using Noether's theorem in the usual well known way, we show that $W = Y$. Say $C = \phi(\Sigma)$; then a general hyperplane section of W cuts C transversely and has nodes there, since by Proposition 6.1 $R^i\phi_*\mathcal{O}_V = 0$ for $i > 0$ and for a geometric generic point $x \in C$ the fibre $\phi^{-1}(x)$ is, as a set, a copy of $\mathbb{P}^1$. Then ϕ is a minimal resolution near x. Note also that the transform $\widetilde{\Sigma}$ of Σ is, over a neighbourhood of x, the whole exceptional locus of ϕ.

Then by Artin's result, that the ideal sheaf defining the fundamental cycle of a rational surface singularity is generated by the maximal ideal of the singularity, $\mathcal{I}_C \,.\, \mathcal{O}_V = \mathcal{I}_{\widetilde{\Sigma}}$ over a neighbourhood of x. Thus if S' is a general section of Y, with $\phi^{-1}(S') = \widetilde{S}$, then $\widetilde{S} \cap \widetilde{\Sigma}$ is a reduced sum of lines. So if $S = \sigma(\widetilde{S})$, then

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$S \,.\, \Sigma = ml + l_1 + \cdots + l_n$, where $m \geqslant 1$ and $l_1, \ldots, l_n$ are lines on S distinct from l that meet l and are disjoint from each other. Note that since C is embedded in $\mathbb{P}^{g-1}$, the Galois group of $k(\mathbb{P}^{(g-1)\vee})$ permutes the points of $C \cap S'$ transitively, so that $l_1, \ldots, l_n$ are Galois conjugate.

Since Pic $X = \mathbb{Z}[-K]$, we have $\Sigma \sim dH$ for some $d \in \mathbb{N}$. Computing intersection numbers on the K3 surface S gives $d = -2m + n$ and $d = m - 2$. Hence

$$d^2 \,.\, \deg X = -2m^2 - 2n + 2m \,.\, n = 4d^2 + 6d.$$

Since $\deg X \geqslant 8$ and is even, we get $d = 1$ and $\deg X = 10$ (i.e. $g = 6$). Hence if L is any other line on S that meets l, we have $1 = L \,.\, H \mid_S = L \,.\, \Sigma \mid_S = L \,.\, (3l + \sum l_i) \geqslant 3$, so that $\{l_1, \ldots, l_n\}$ is the complete set of lines in S that meet l.

Note that any monodromy invariant configuration on S supports a very ample divisor, since $\rho(X) = 1$, and so is connected. So l is conjugate to some (and hence all) of the l_i. Hence if $\Gamma^\vee$ is the dual graph of the configuration Γ formed by the lines on S, then the Galois group acts transitively on the vertices of $\Gamma^\vee$. So from the values of m and n, it follows that every vertex meets just 7 others. Moreover, since $3l + \sum l_i$ is a hyperplane section, every line except $l, l_1, \ldots, l_7$ meets just one of the l_i. So $\Gamma^\vee$ has no squares or triangles and is of diameter 2. It follows easily that $\Gamma^\vee$ has 50 vertices. Moreover, $\Gamma^\vee$ is regular of valency 7 (for this and other definitions, see BCN], p. 434). Since any 2 adjacent (resp. non-adjacent) vertices have 0 (resp. 1) common neighbours, $\Gamma^\vee$ is strongly regular with parameters $(\nu, k, \lambda, \mu) = (50, 7, 0, 1)$. Then by [loc. cit., Theorem 1.3.1, p. 8] the eigenvalues of the adjacency matrix M (which is defined with zeroes down the diagonal, so that the intersection matrix A is given by $A = M - 2I$) has eigenvalues $7, 2, -3$ with respective multiplicities $1, 28, 21$. Hence A has eigenvalues $5, 0, -$ with multiplicities $1, 28, 21$. Hence rank $A = 22$, so that S is supersingular, and all eigenvalues of A are ± 5. Since on any supersingular surface the intersection pairing on the Néron-Severi group is unimodular away from p, and there is no even unimodular $\mathbb{Z}$-lattice of rank 22 and signature $(1, 21)$, we have $p = 5$, contrary to assumption. □

Remark. (i) The properties of $\Gamma^\vee$ described above characterize it as the*Hoffman-Singleton graph* HS [BCN, p. 391]. The obvious question is whether there is a K3 surface S (necessarily supersingular in characteristic 5) containing an HS configuration of lines. However, this cannot occur, since [loc. cit.] HS contains 5 disjoint 5-cycles, and the existence of this on a K3 surface S is impossible, since for example $c_2(S) = 24$. Moreover, the lattice L generated by the vertices of HS, where each vertex has self-intersection -2, turns out to have discriminant -5, while on any supersingular K3 surface in char. p the Néron–Severi group has discriminant $-p^{2\sigma}$, where σ is the Artin invariant. Hence NS(S) cannot be embedded in L.

However, the double cover S of $\mathbb{P}^2$ branched in the Fermat sextic C contains a copy of Aut(HS), namely a split extension of $G = \mathrm{PSU}_3(\mathbb{F}_{25})$ by $\langle\tau\rangle$, where τ is the involution of G given by $\mathrm{Frob}_{\mathbb{F}_{25}}$ [loc. cit.].

(ii) As already noted, the Fermat quartic 3-fold in characteristic 3 is a Fano with $\rho = r = 1$ which is covered by lines. Another example is the double cover in char. 5 of $\mathbb{P}^3$ branched in the Fermat sextic. I do not know whether there are any other examples, although of course the genus could be at most 5.

NOTATION. Henceforth X will be a Fano 3-fold with $\rho = 1$ that is embedded in $\mathbb{P}^{g+1}$ by $|H| = |-K_X|$ and cut out by quadrics. P will denote a general point on X, so that in particular P lies on no line, and $\sigma: \widetilde{X} = \mathrm{Bl}_P X \to X$ will be the blow-up at P, with exceptional divisor E. We denote the Stein factorization of the morphism $\psi: \widetilde{X} \to Y$ defined by the linear system $|\widetilde{H}| = |-K_{\widetilde{X}}|$ (that is, the double projection from P) by $\phi: \widetilde{X} \to X_1$. Sometimes we shall abuse the notation by also regarding ϕ as a rational map on X.

COROLLARY 7.2 (1) $|\widetilde{H}|$ *has no base points and* ϕ *collapses just the strict transforms of the curves of degree* d *with multiplicity* $2d$ *at* P.

(2) *Assume that* $g \geqslant 6$. *Then either* Y *is of minimal degree in* $\mathbb{P}^{g-3}$ *or* $X_1 \to Y$ *is an isomorphism.*

(3) *If* $g \geqslant 6$, *then a general member of* $|-K_{\widetilde{X}}|$ *is K3, maybe with nodes.*

Proof. For (1) there is nothing to do and (2) is an immediate consequence of Noether's theorem on canonical curves. For (3) it is enough to exclude 5.1(iv). Suppose that this does happen. Then Y is normal and $\deg\phi \leqslant 2$, while ϕ induces a morphism from E that is of degree 4. This is impossible. $\square$

LEMMA 7.3 *Suppose that* $g \geqslant 6$. *Then the natural map*

$$\rho: H^0(\widetilde{X}, \mathcal{O}(-K_{\widetilde{X}})) \to H^0(E, \mathcal{O}_E(-K_{\widetilde{X}}))$$

has maximal rank. In particular, E *is embedded as a Veronese surface if* $g \geqslant 8$.

Proof. Suppose that ρ is neither injective nor surjective. Then $|\widetilde{H} - E|$ is not empty; let D denote a general member of it. We have $D = G + rE$ with $r \geqslant 0$ and G reduced and irreducible.

Let F denote a general member of $|\widetilde{H}|$; this is a K3 surface, maybe nodal, by Corollary 7.2(3). There is an exact commutative diagram

$$\begin{tikzcd}
H^0(\mathcal{O}_{\widetilde{X}}(\widetilde{H})) \arrow[r] \arrow[d] & H^0(\mathcal{O}_E(\widetilde{H})) \arrow[r] \arrow[d] & H^1(\mathcal{O}_{\widetilde{X}}(D)) \arrow[r] \arrow[d] & 0 \\
H^0(\mathcal{O}_F(\widetilde{H})) \arrow[r] & H^0(\mathcal{O}_{F\cap E}(\widetilde{H})) \arrow[r] & H^1(\mathcal{O}_F(\widetilde{H})) \arrow[r] & 0
\end{tikzcd}$$

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(cf. [R 1, p. 29]). It follows that ρ is surjective if $H^1(\mathcal{O}_F(-D|_F)) = 0$. For this, it will suffice to show that $D|_F$ is numerically connected.

There are three cases to consider.

(a) Assume that G is not contracted by ϕ.
Then $F|_G$ moves in a big linear system with no base points, so that for general F we have $F \,.\, G = p^m A$, where A is reduced and irreducible. Since $\deg \phi \leqslant 2$, it follows that $p^m = 1$. Define $\Gamma = F \,.\, E$; then $D|_F = A + r\Gamma$.
Now suppose that $D|_F = C_1 + C_2$, with $C_i > 0$. We can then assume that $C_1 = A + a\Gamma$ and $C_2 = (r-a)\Gamma$. Since $(D|_F) \,.\, \Gamma = 2$, we see that $C_1 \,.\, C_2 = (r-a)(A \,.\, \Gamma - 2a) = (r-a)(2r+2-2a) > 0$, so that $D|_F$ is numerically connected.

(b) Assume that G is contracted by ϕ and that $g \neq 7$. Then there is an equality $(H-2E)^2 \,.\, (H-(3+r)E) = 0$, so that $H^3 = 4(3+r)$. Now $H^3 \neq 12$, so that $r \geqslant 1$. Then $H^3 \geqslant 16$, so that $\dim \ker \rho \geqslant 2$. Then G moves in a pencil, while ϕ is generically finite, so that we can choose G not to be contracted.

(c) Assume that G is contracted and that $g = 7$. Since G is contracted, it cannot move, and so $\dim |\widetilde{H} - E| = 0$. Then $|\pi^* H - 2E|$ cuts out a base point free linear system of dimension 3 on the Veronese surface E. Since $(\pi^* H - 2E) \,.\, E \,.\, G = 6$ and the image of G is irreducible, we get a contradiction to the fact that the singular locus of the image of E under ϕ (which is the image of the Veronese surface in $\mathbb{P}^5$ under projection from a disjoint line) is the union of three lines, so reducible. □

COROLLARY 7.4 (1) *If* $g \geqslant 8$, *then the only curves contracted by* ϕ *are the conics through* P.

(2) *If* $g = 7$, *then in addition* ϕ *contracts the quartic curves (maybe reducible) that are singular at* P.

(3) *If* $g = 6$, *then also* ϕ *contracts the unique sextic rational curve that is triple at* P.

Proof. This is a consequence of the projective geometry of the Veronese surface. For example, if $g = 7$, then the Veronese surface E is mapped by a 4-dimensional linear system, and such a system either is very ample or maps a conic 2-to-1 onto a line. □

PROPOSITION 7.5 *If* $g \geqslant 6$, *then* ϕ *contracts at most finitely many curves.*

Proof. If not, then ϕ collapses a divisor D that meets E. Since no curve in E is contractible, the image of D is a curve, say W. Strictly Henselize at the generic point of W; then we get an RDP, so that $\mathcal{I}_W \,.\, \mathcal{O}_{\widetilde{X}} = \mathcal{O}(-Z)$, where $Z = \sum n_i D_i$ is the fundamental cycle, by Artin's theorem. But D is irreducible, so that the D_i are Galois conjugate. Then the n_i are equal. Since a fundamental cycle is not multiple, $Z = D$.

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Now return to the global context and let C denote the fibre of ϕ over a geometric generic point of W. Then the properties of fundamental cycles show that $D\,.\,C = -2$.

Write $D \sim x(-K_{\widetilde{X}}) - yE$, where x, y are integers. Pushing down to X shows that $x > 0$, while the immobility of D shows that $y > 0$. Recall that $E\,.\,C > 0$ and $(-K_{\widetilde{X}})\,.\,C = 0$, while the contraction shows that $(-K_{\widetilde{X}})^2\,.\,D = 0$. Since $(-K_{\widetilde{X}})^3 = 2g - 10$ and $(-K_{\widetilde{X}})^2\,.\,E = 4$, we get $0 = (-K_{\widetilde{X}})^2\,.\,D = x(-K_{\widetilde{X}})^3 - 4y$, so that $(g-5)x - 2y = 0$.

Thus $y(E\,.\,C) = 2$. Now consider the various possibilities separately.

(1) $E\,.\,C = 2$. Then the Veronese surface E is mapped to a singular quartic surface E_1. Then by Lemma 7.3, either $g = 7$ and E is singular along a line or $g = 6$ and E_1 is singular along three concurrent lines. Since D is irreducible, it is mapped to one of these lines. Then a general member of $|-K_{\widetilde{X}}|$ cuts D in a scheme that is the fundamental cycle of an RDP, so that $(-K_{\widetilde{X}})\,.\,D^2 = -2$. However,

$$(-K_{\widetilde{X}})\,.\,D^2 = x^2(-K_{\widetilde{X}})^3 - 2xy(-K_{\widetilde{X}})^2\,.\,E + y^2(-K_{\widetilde{X}})\,.\,E^2,$$

so that

$$-2 = x^2(2g-10) - 8xy - 2y^2.$$

Substituting $y = x(g-5)/2$ gives

$$4 = x^2(g-5)(g-1),$$

which is absurd.

(2) $E\,.\,C = 1$. Since E meets every component of C, C is reduced and irreducible. As in (1), a general member of $|-K_{\widetilde{X}}|$ cuts D in a number of copies of C; since $D\,.\,E$ is a plane curve of degree y, it follows that $(-K_{\widetilde{X}})\,.\,D \sim yC$. Then

$$-4y = (-K_{\widetilde{X}})\,.\,D^2 = x^2(2g-10) - 8xy - 2y^2.$$

Substituting $y = 2$ and $(g-5)x = 2y$ gives an immediate contradiction. $\square$

COROLLARY 7.6 *If $g \geqslant 6$, then there are at most finitely many conics through a general point P of X and at most finitely many quartic curves singular at P (including reducible curves).* $\square$

8. Projection from a conic

In this section too we assume that $-K_X$ generates Pic X and that X is an intersection of quadrics. We shall assume also that X is covered by conics, so that if C is a general conic on X and P is a general point on C, then it is a general point on X. Let $\sigma\colon X^* = \mathrm{Bl}_C\, X \to X$ be the blow-up along C with exceptional divisor F. Put $H^* = -K_{X^*}$; then $|H^*|$ has no base points. We let $\phi\colon X^* \to X_1$ be the Stein factorization of the morphism defined by $|H^*|$.

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LEMMA 8.1 (1) *The normal bundle $N_{C/X}$ is $\mathcal{O}(a) \oplus \mathcal{O}(-a)$, where $a = 0, 1$ or 2.*

(2) *ϕ collapses only the conics meeting C twice, the lines meeting C and, if $a = 2$, the negative section of F.*

Proof. (1) Since $K_X \,.\, C = -2$, it follows that $N_{C/X} = \mathcal{O}(a) \oplus \mathcal{O}(-a)$. Then $F \cong \mathbb{F}_{2a}$ and $|H^*|$ cuts out a linear system on F with no base points. Also, $H^*|_F \sim D + 2\phi$, where D is the tautological class and ϕ is a fibre. Suppose that Γ is a negative section (unique if $a \neq 0$) in F; then $D \,.\, \Gamma = -a$, so that $-a + 2 \geqslant 0$.

(2) Suppose that $B \subset X$ is an irreducible curve collapsed byϕ and that $B \neq C$. Then $B \cup C$ is contained in a copy of $\mathbb{P}^3$. Since X is cut out by quadrics, it follows that B is as described. It remains to consider curves in F collapsed by ϕ; it is clear that the only one is Γ, and then only if $a = 2$. □

PROPOSITION 8.2 *If $g \geqslant 7$, then ϕ collapses only finitely many curves.*

Proof. Assume that ϕ collapses a divisor D. We have $D \sim x(-K_{X^*}) - yF$ where x, y are positive integers. Since $(-K_{X^*})^2 \,.\, D = 0$, we get $2y = (g-4)x$.

Suppose that $\phi(D)$ is a point. Then $D^2 \,.\, (-K_{X^*}) = 0$, so that

$$x^2(2g-8) - 8xy - 2y^2 = 0.$$

Substituting $2y = (g-4)x$ gives a contradiction.

So $\phi(D)$ is a curve. Since the general 1–dimensional fibre of ϕ is the strict transform Γ of a conic meeting C twice, it follows that $F \,.\, \Gamma = 2$ and an argument involving fundamental cycles shows that $D.\Gamma = -2$. Then $y = 1$, so that $(g-4)x = 1$, contradicting $g \geqslant 7$. □

Remark. In fact the same result holds if $g = 6$. For this it is necessary to prove that the homomorphism $H^0(\mathcal{O}_{X^*}(-K_{X^*})) \to H^0(\mathcal{O}_F(-K_{X^*}))$ has maximal rank.

9. Rational Gorenstein singularities and flops

In characteristic zero it is known that given a rational Gorenstein 3-fold singularity (Y, P), either it is a double point of a hypersurface or a generic section through P is elliptic, in which case the canonical divisor class K_Y pulls back to the canonical class of $\mathrm{Bl}_P Y$. As a consequence, if Y has a small resolution, then Y is a double point. Here we prove analogous results in characteristic p.

PROPOSITION 9.1 (1) *Suppose that Y is a normal projective Gorenstein 3-fold having a resolution $f : \widetilde{Y} \to Y$ such that $f_*\omega_{\widetilde{Y}} = \omega_Y$. Fix a very ample linear system $|H|$ and a point P on Y. Let K denote the function field of the system $|H - P|$. Let S denote the generic member of $|H - P|$ and $\tilde{S}$ its strict transform on Y. Then S is a normal projective K-scheme on which P is either regular or an RDP or elliptic, in the sense that $\mathfrak{m}_{S,P}\omega_S \subset (f_*\omega_{\tilde{S}})_P$.*

(2) *If f is small, then all singular points of Y are double points.*

Proof. By Abhyankar's theorem already quoted, we can blow up $\widetilde{Y}$ to get a smooth model Y^* on which the ideal sheaf $\mathfrak{m}_P \,.\, \mathcal{O}_{Y^*}$ is Cartier. Then we can assume that $Y^* = \widetilde{Y}$, so that $\widetilde{S}$ moves in a linear system with no base points, and so is a regular projective K-scheme. Now the proofs of (1) and (2) are just as given by Reid; for the sake of convenience, a sketch follows.

We have $\widetilde{S} \sim f^*S - E$, where E is the effective Cartier divisor with ideal sheaf $\mathfrak{m}_P\mathcal{O}_{\widetilde{Y}}$, and $K_{\widetilde{Y}} \cong f^*K_Y + Z$, where Z is effective in a neighbourhood of $f^{-1}(P)$. Let $g : \widetilde{S} \to S$ be the induced morphism. Then $K_{\widetilde{S}} \sim g^*K_S + (Z - E)|_{\widetilde{S}}$, by the adjunction formula, so that in a neighbourhood of $f^{-1}(P)$ there is an inclusion $\mathcal{O}_{\widetilde{S}}(g^*K_S - E|_{\widetilde{S}}) \hookrightarrow \mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}})$. Then taking g_* gives (1).

For (2), we appeal to (1) and the fact that if (S, P) is an elliptic Gorenstein surface singularity of multiplicity at least 3, then $\mathrm{Bl}_P\, S$ is the canonical model of the minimal resolution, so that if $h \colon Y_1 \to Y$ is the blow-up at P, then Y_1 is normal and Gorenstein, and $K_{Y_1} \sim h^*K_Y$, so that the components of the exceptional locus are divisors with zero discrepancy. However, if there is a small resolution, then any exceptional divisor has strictly positive discrepancy. □

In our applications, Y will be the anti-canonical model of $\widetilde{X}$ (that is, $\phi \colon \widetilde{X} \to Y$ will be the Stein factorization of the morphism defined by the base–point–free linear system $|-K_{\widetilde{X}}|$), where $\pi : \widetilde{X} \to X$ is the blow-up along either a general point or, assuming that X is covered by conics, a general conic. We know now that ϕ is small.

PROPOSITION 9.2 *For all singular points Q of Y there is an involution ι of $(Y, Q)^h$ that induces -1 on its local class group.*

Proof. We use Kollár's argument, slightly extended to include characteristic 2. First, henselize at Q. There is a finite morphism $f \colon (Y, Q)^h \to (\mathbb{A}^3, 0)^h$ of degree 2. If f is inseparable, then $p = 2$ and for any divisor class $D \in \mathrm{Pic}\; U$ (where U is a punctured neighbourhood of Q in Y^h), we have $2D = f^*f_*D$, so that $2D$ is principal. Thus the local class group is 2-elementary, and we can take $\iota =$ identity. If f is separable, then it is Galois; let ι be the covering involution. Then $D + \iota^*D = f^*f_*D$, which is principal, so that $\iota^*D \sim -D$. □

COROLLARY 9.3 (1) *$\widetilde{X}$ has a smooth flop $\widetilde{X}^+$.*

(2) *If E is the exceptional divisor on $\widetilde{X}$ and Γ is a flopping curve, then $E.\Gamma > 0$.*

(3) *If E is the exceptional divisor on $\widetilde{X}$ and E^+ its strict transform on $\widetilde{X}^+$, then $(E^+)^3 \leqslant E^3$, with equality if and only if $\widetilde{X}$ is isomorphic to $\widetilde{X}^+$, which in turn happens if and only if $\widetilde{X}$ is Fano.*

Proof. (1) For this, we just use Kollár's argument [K2].

(2) If $\widetilde{X}$ is a point blow-up, then the flopping curves do not lie in E, but do meet it. If $\widetilde{X}$ is a conic blow-up, then $E + (-K_{\widetilde{X}}) = \pi^*(-K_X)$. By Lemma 8.1, $\pi^*(-K_X) \,.\, \Gamma > 0$ for any flopping curve Γ, and the result follows.

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(3) We follow an argument of Mori's. By (2), the class $m(E+n(-K_{\widetilde{X}}))$ is very ample relative to $f: \widetilde{X} \to Y$ for suitable m and n. Then there exist $B_1, B_2, B_3 \in |mn(-K_{\widetilde{X}})|$ and $A_1, A_2, A_3 \in |m(E+n(-K_{\widetilde{X}}))|$ such that the B_i are disjoint from the flopping locus Γ and the A_i are disjoint from each other near Γ. Then $B_i \,.\, F \,.\, D = B_i^+ . F^+ . D^+$ for all divisors F, D on $\widetilde{X}$, so that $m^3(E^3 - (E^+)^3) = A_1 \,.\, A_2 \,.\, A_3 - A_1^+ . A_2^+ . A_3^+$. Since $E^3 - (E^+)^3$ is supported on the flopping locus, it follows that

$$\begin{aligned} m^3(E^3 - (E^+)^3) &= (A_1 \,.\, A_2 \,.\, A_3)_\Gamma - (A_1^+ . A_2^+ . A_3^+)_{\Gamma^+} \\ &= (A_1^+ . A_2^+ . A_3^+)_{\Gamma^+}. \end{aligned}$$

Since $A_1^+ . A_2^+$ is a non-zero cycle supported on Γ^+, unless A_1 is disjoint from Γ (in which case Γ is empty), and since $D \,.\, \Delta = -D^+ \,.\, \Delta^+$ for all divisors D on $\widetilde{X}$ and components Δ of Γ, we are done. □

10. Wild fibrations

A conic bundle $f : X \to B$ is *wild* if every fibre is a double line. A del Pezzo fibration is *wild* if the geometric generic fibre is wild, in the sense that $H^1(\mathcal{O}) \neq 0$.

If $f : X \to B$ is a wild conic bundle, then there is no discriminant curve and so no analogue of the formula $-4K_B \sim f_*(-K_X)^2 + \Delta$. In fact it is unclear a priori whether B need be rational if X is Fano, or even whether it might be of general type. The next result disposes of this intriguing possibility.

PROPOSITION 10.1 *If X is Fano and $f : X \to B$ is a wild conic bundle, then $B \cong \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Since $H^i(\mathcal{O}_X) = 0$ for $i > 0$, the Leray spectral sequence shows that $H^i(\mathcal{O}_B) = 0$ for $i > 0$. Since Pic X is torsion–free so is Pic B, and then the classification of surfaces shows that either B is rational or $\mathrm{Kod}(B) \geqslant 1$. Assume that B is irrational, and fix an ample divisor class D on B. Then $D.K_B > 0$. Take an irreducible element S of $|-K|$ (it is clear from Section 3 and 4 that such an S exists). If V is the normalization of the Stein factorization S_1 of S then $K_V \leqslant 0$ and $V \to B$ is a purely inseparable double cover. Let $V \to B \to V^{(1)}$ be the corresponding factorization of the geometric Frobenius; we can regard $\beta : B \to V^{(1)}$ as the quotient by a foliation $\mathcal{O}(A) \hookrightarrow T_B$. Then $\beta^* c_1(V^{(1)}) \sim 2A + (c_1(B) - A) \sim A - K_B$, so that $D \,.\, (A - K_B) \geqslant 0$. Hence the foliation $\mathcal{O}(A) \hookrightarrow T_B$ is independent of the choice of S, so that V is independent of S. In fact, if S is *any* member of $|-K|$, then it has a unique component S' dominating B, and this argument shows that the normalization of S' is independent of S. This is clearly impossible, and hence B is rational.

Suppose now that there is a smooth rational curve E on B with $E^2 = -d \leqslant -1$. Put $W = f^{-1}(E)$, with induced morphism $h : W \to E$, and let ϕ denote a scheme-theoretic fibre of h. If W is not reduced, then since $H \,.\, \phi = 2$ it follows that $W = 2U$ and U is a $\mathbb{P}^1$-bundle over E, say with fibre $f = \phi_{\mathrm{red}}$. Since $\mathcal{N}_{U/X} \,.\, f = 0$, the adjunction formula gives $K_U \,.\, f = -1$, which is impossible. Hence W is reduced.

Let $\nu : \widetilde{W} \to W$ be the normalization; then there is a factorization $\widetilde{W} \to E^{(-1)} \to E$ of $h\nu$ through the geometric Frobenius; since $\nu^* H$ is ample and cuts out $\mathcal{O}(1)$ on the geometric generic fibre f of $\widetilde{W} \to E^{(-1)}$, it follows that $\widetilde{W}$ is a $\mathbb{P}^1$-bundle over $E^{(-1)}$. Let $C \subset \widetilde{W}$ be the curve defined by the conductor ideal; then

$$\omega_{\widetilde{W}} \cong \nu^* \omega_W \otimes \mathcal{O}(-C) \cong \nu^* \mathcal{O}(-H) \otimes \nu^* h^* \mathcal{O}(-d),$$

so that $-K_{\widetilde{W}} \sim \nu^* H + C + 2df$. Since $f \,.\, \nu^* H = 1$, it follows that $f \,.\, C = 1$, so that $C = D + \sum r_i f_i$, where D is a section and the f_i are fibres. Then $0 = D^2 + D \,.\, K_{\widetilde{W}} + D \,.\, \nu^* H + 2d + \sum r_i$. Since $\nu^* H$ is ample, we see that $2 > 2d + \sum r_i$, which is impossible if $d > 0$. $\square$

PROPOSITION 10.2 *Suppose that X is a normal Gorenstein 3-fold and that $f : X \to U$ is a projective morphism to a smooth quasi–projective curve whose geometric generic fibre X_s is a del Pezzo surface. Then one of the following statements is true.*

(1) *X_s is tame (that is, $\chi(\mathcal{O}_{X_s}) = 1$).*
(2) *$p = 2$ and $(\omega_{X_s})^2 = 1$ or 2.*

Proof. Note first that X_s is a variety, by Bertini's theorem.

We shall assume that U is affine and that ω_U is trivial, and we shall shrink U tacitly and arbitrarily whenever this may be convenient.

Assume that X_s is wild. Put $D = -K_X$. Then $H^1(\mathcal{O}_X(-D)) \neq 0$. Pick a non-zero class $\tau \in H^1(\mathcal{O}_X(-D))$, and pick the least integer $n \geqslant 1$ such that the image of τ in $H^1(\mathcal{O}_X(-p^n D))$ under the n'th Frobenius is zero. Then there is a finite and purely inseparable morphism $\rho \colon Y \to X$, where Y is a variety (since X is normal) with Gorenstein singularities, since ρ is locally an α_{p^n}-bundle in the flat topology. Moreover, $\omega_Y \cong \rho^* \mathcal{O}_X(K_X - (p^n - 1)D) \cong \rho^* \mathcal{O}_X(-p^n D)$ and $\rho_* \mathcal{O}_Y$ has a filtration whose graded pieces are $\mathcal{O}_X, \mathcal{O}_X(D), \ldots, \mathcal{O}_X((p^n - 1)D)$ [E]. (Note that the construction of Y given by Ekedahl is not unique, given that X fails to be complete. However, the ambiguity in the construction is entirely accounted for by the existence of non-constant global functions on X; since all such functions pull back from U, we can avoid all difficulties by shrinking U.)

Let $\nu : \widetilde{Y} \to Y$ be the normalization, so that $K_{\widetilde{Y}} \sim \nu^* K_Y - \Delta$, where Δ is the codimension 1 part of the subscheme defined by the conductor ideal. Put $\tilde{\rho} = \rho \circ \nu : \widetilde{Y} \to X$, and let $\widetilde{Y} \xrightarrow{\tilde{f}} U_r \xrightarrow{F_r} U$ be the Stein factorization. Note that $F_r : U_r \to U$ is the r'th geometric Frobenius, where $0 \leqslant r \leqslant n$. Let t be a geometric generic point of U_r and $s = F_r(t)$. Since $\widetilde{Y}$ is normal, the fibre $\widetilde{Y}_t = V$, say, is a variety, by Bertini's theorem, that is Cohen–Macaulay and has

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hypersurface singularities in codimension 1. Put $L = (\tilde{\rho}^* D)|_V$, and let $\pi : \widetilde{V} \to V$ be the minimal resolution. Then $-K_{\widetilde{V}} = p^n \pi^* L + \widetilde{\Delta} + Z$, where $\widetilde{\Delta}, Z \geqslant 0$, Z is contracted by π and $\widetilde{\Delta}$ is the strict transform of the curve defined by the conductor ideal. Hence $\Delta|_V = 0$ if $\widetilde{\Delta} = 0$.

CLAIM *If* $\widetilde{\Delta} \neq 0$, *then* $p = 2$ *and* $(\omega_{X_s})^2 = 1$.

it Proof of claim. If $\widetilde{V} \cong \mathbb{P}^2$, then this is obvious. So assume that $\widetilde{V} \neq \mathbb{P}^2$; then there is a ruling $\widetilde{V} \to C$, where C is a curve of genus g, say. Let F be a fibre. Then

$$2 = (-K_{\widetilde{V}}) \,.\, F = p^n (\pi^* L) \,.\, F + (\widetilde{\Delta} + Z) \,.\, F,$$

so that $p^n = 2$ and $\widetilde{\Delta} + Z$ is vertical. Now $\widetilde{V}$ has a section σ with $\sigma^2 \leqslant g$; suppose first that σ is not contracted by π, so that $\sigma \,.\, \pi^* L > 0$ and $\sigma \,.\, (\widetilde{\Delta} + Z) \geqslant 0$. Then $2 - g \geqslant (-K_{\widetilde{V}}) \,.\, \sigma \geqslant p^n \,.\, 1 + 0$, so that $g = 0$ and $\sigma^2 = 0$. However, if $\widetilde{V} \neq \mathbb{F}_0$, then there exists σ with $\sigma^2 < 0$; hence $\widetilde{V} = \mathbb{F}_0$. Then $\widetilde{\Delta} + Z = 0$.

If σ is contracted, then it forms part of the exceptional locus of an RDP, since it is not contained in $\widetilde{\Delta} + Z$; hence $\sigma^2 = -2$. Then σ is disjoint from $\widetilde{\Delta} + Z$, so that there is a vertical curve C not contained in $\widetilde{\Delta} + Z$ with $C \,.\, (\widetilde{\Delta} + Z) > 0$. If $C \,.\, \pi^* L = 0$, then $C^2 \leqslant -2$, since the resolution is minimal. Then $0 \geqslant -K_{\widetilde{V}} \,.\, C = (\widetilde{\Delta} + Z) \,.\, C$, which is absurd. If $C \,.\, \pi^* L > 0$, then $1 \geqslant -K_{\widetilde{V}} \,.\, C > (\widetilde{\Delta} + Z) \,.\, C$, which is also absurd. Hence the claim is established.

Hence we may assume that Y is normal. Note that then Y_t is a variety and is a Cartier divisor on Y, and so has Gorenstein singularities. Let $\sigma : Y_t \to X_s$ be the induced morphism; $\deg \sigma = p^{n-r}$. The filtration above shows that

$$p^r \chi(\mathcal{O}_{Y_t}) = \sum_{i=0}^{p^n - 1} \chi(\mathcal{O}_{X_s}(iD)) = D^2 (p^n + 1) p^n (p^n - 1)/6 + p^r \chi(\mathcal{O}_{X_s}).$$

By the adjunction formula, $\omega_{Y_t} \cong \rho^* \mathcal{O}_X(-p^n D)|_{Y_t} \cong \sigma^* \mathcal{O}_{X_s}(-p^n D)$, so that Y_t is a del Pezzo surface whose index is divisible by p^n. Then from Reid's classification [R 2, 1.1] $p^n \leqslant 3$.

Suppose that $p^n = 3$. Then $(\omega_{Y_t})^2 = \deg \sigma \,.\, p^{2n} (\omega_{X_s})^2 = p^{3n-r} (D|_{X_s})^2$. Since Y_t is del Pezzo and ω_{Y_t} is divisible by 3, Reid's classification shows that $Y_t \cong \mathbb{P}^2$, so that $r = n = 1$ and $(D|_{X_s})^2 = 1$. Also,

$$\chi(\mathcal{O}_{Y_t}) = (D|_{X_s})^2 \,.\, 4 \,.\, 2/6 + \chi(\mathcal{O}_{X_s}),$$

which is absurd.

Suppose that $p^n = 2$. Then Y_t is del Pezzo and ω_{Y_t} is divisible by 2, so that Y_t is normal, from Reid's classification, and is a quadric (maybe a cone). Then $8 = 2^{3-r} (D|_{X_s})^2$, so that $(D|_{X_s})^2 = 2^r$. Also,

$$2^r \chi(\mathcal{O}_{Y_t}) = (D|_{X_s})^2 .3.2.1/6 + 2^r \chi(\mathcal{O}_{X_s}),$$

so that $\chi(\mathcal{O}_{X_s}) = 0$ and $(\omega_{X_s})^2 = (D|_{X_s})^2 = 2^r = 1$ or 2. □

COROLLARY 10.3 *Suppose that* $f: X \to U$ *is as in* 10.2. *Assume that there is a multiple fibre. Then* $p = 2$, *all fibres have multiplicity at most 2 and* $K^2_{X_s} = 2$ *for a general fibre* X_s.

Proof. Suppose that $F = mL$ is a multiple fibre. Then for a geometric generic fibre F_t, m divides $\chi(\mathcal{O}_{F_t})$, so that F_t is wild. Then $p = 2$, by 10.2. Also, $mK^2_X \,.\, L = K^2_X \,.\, F_t = 1$ or 2, by 10.2, so that $m = 2$ and $K^2_X \,.\, F_t = 2$. □

PROPOSITION 10.4 *Suppose that* $f: X \to B$ *is the contraction of an extremal ray on a smooth 3-fold and that* $\dim B = 1$. *Then* $K^2_{X_t} \leqslant 9$ *for any fibre* X_t.

Proof. Suppose that t is a geometric generic point of B, and put $X_t = S$. By 10.2 and 10.3 we can assume that S is tame and that there are no multiple fibres.

If S is normal, then either it has only RDPs or it is the cone over an elliptic curve. In the first case $K^2_S \leqslant 9$, and in the second $K^2_S \leqslant 3$ since S is a divisor on a smooth 3-fold.

Suppose then that S is not normal and that $\widetilde{S} \to S$ is the normalization. Then $\widetilde{S}$, polarized by $-K_S$, is either $\mathbb{P}^2$ or a rational cone or a rational scroll or a Veronese surface, and $\widetilde{S} \to S$ is a projection mapping a conic C to a line L. Since $\widetilde{S}$ has unibranched singularities, the map $C \to L$ must be one–to–one on geometric points. Hence either $p \neq 2$ and C is a double line or $p = 2$, C is smooth and $C \to L$ is the Frobenius. Hence if $p \neq 2$, then either $\widetilde{S}$ is a plane and C a double line or $\widetilde{S}$ is a rational cone and C is a double generator. In the first case $K^2_S = 1$ and in the second S is the cone over a cuspidal rational curve.

If $p = 2$ and C is smooth, then either

(a) $\widetilde{S}$ is a plane and C a conic, or
(b) $\widetilde{S}$ is a Veronese surface and C a conic, or
(c) $\widetilde{S}$ is a quadric cone, or
(d) $\widetilde{S}$ is a rational scroll $\mathbb{F}_{a,k}$, with $k \leqslant 2$. In cases (a)–(c) $K^2_S \leqslant 4$, while in case (d) either $a = 0$ and $k = 1$ (when $\widetilde{S}$ is a quadric, so that $K^2_S = 2$) or there are two classes in Pic $\widetilde{S}$ that are invariant under monodromy. This last possibility, however, contradicts the fact that the contraction of an extremal ray has relative Picard number equal to 1. □

11. The birational classification when $\rho = 1$

In this section X will denote a Fano 3-fold with $\rho = r = 1$ and $g \geqslant 6$.

LEMMA 11.1 *Suppose that* X^+ *is the flop of either* $\mathrm{Bl}_P X$ *or (assuming that* X *is covered by conics)* $\mathrm{Bl}_C X$, *that* $f: X^+ \to B$ *is the contraction of an extremal ray and that* $\dim B = 1$. *Then* f *has no multiple fibres.*

Proof. According to the results of Section 10 we can assume that there is a double fibre, say $F = 2L$, and that $(H^+)^2 \,.\, L = 1$, where $H^+ = -K_{X^+}$. Let E^+ denote the strict transform of the exceptional divisor. Then we can write $L \sim xH^+ - yE^+$ for some integers x, y. Since $(H^+)^3 = 2g - 2s$, where $s = 5$ or 4, and $(H^+)^2 \,.\, E^+ = 4$, we get

$$1 = (H^+)^2.L = x(2g - 2s) - 4y,$$

which is absurd. □

THEOREM 11.2 *X is covered by conics, $g \leqslant 12$ and $g \neq 11$.*

Proof. This exactly follows Takeuchi's argument [T]. What he does is to carry out a detailed analysis involving the calculation of intersection numbers and the consideration of extremal rays on the flop X^+ of a point blow-up. This proves in particular that X^+ has an extremal ray whose contraction is not a morphism to X. This carries over to characteristic p given the results of Section 10 and 11.1 concerning conic bundles and del Pezzo fibrations. The only other difference is that to prove that $g \neq 11$, we must show that the flop X^+ of the blow-up at a general point is not a wild conic bundle.

So assume that $g = 11$; by Takeuchi's argument there is a conic bundle structure $h\colon X^+ \to S$, which we assume wild, where S is a surface with $\rho(S) = 1$. By Proposition 10.1 $S \cong \mathbb{P}^2$. Let L denote the pull–back to X^+ of a line on S and E^+ the strict transform on X^+ of the exceptional divisor on X. Let A denote a fibre of h, with its reduced structure, and put $H^+ = -K_{X^+}$. Then $L = H^+ - E^+$ (Takeuchi) so that $E^+ \,.\, A = 1$. Hence $E^+ \to S$ is purely inseparable. According to Takeuchi X^+ does contain at least one flopping curve ℓ, since the number e defined by the equation $(E^+)^3 = E^3 - e = 1 - e$ is non-zero; by the construction of the flop, $E^+.\ell < 0$, so that E^+ contains ℓ and the normalization of E^+ has Picard number at least 2. Then $E^+ \to S$ is not finite, so that E^+ contains at least one of the curves A. However, $K_{X^+}|_{E^+} \sim 2E^+|_{E^+}$ and so is even; thus $H^+.\, A$ is even, while we know that $H^+.\, A = 1$. □

Remark. In characteristic zero Takeuchi proves the existence of lines on X in the course of considering an extremal ray on the flop of $\mathrm{Bl}_C\, X$, where C is a conic. In characteristic p this breaks down; even though the enumerative arguments show that there is a flopping curve on $\mathrm{Bl}_C\, X$, this curve could be the negative curve on the exceptional divisor if $\mathcal{N}_{C/X}$ is of type $(2, -2)$. If C is generic in characteristic zero then its normal bundle is $(0, 0)$, of course.

COROLLARY 11.3 *The birational structure of X is just as given by Takeuchi and Mori. That is, via point blow-up (resp. conic blow-up) and flop, the possibilities for the various values of g are given as follows:*

Point blow-up*:*
$g = 12$*:* X^+ *is the blow-up of* $\mathbb{P}^3$ *along a smooth rational curve* Γ *of degree* 6 *that lies on a cubic surface.*
$g = 10$*:* X^+ *is a pencil of sextic del Pezzo surfaces over* $\mathbb{P}^1$*.*
$g = 9$*:* X^+ *is the blow-up of another Fano of genus* 9 *along a point.*
$g = 8$*:* X^+ *is the blow-up of a cubic threefold along a smooth rational curve of degree* 4*.*
$g = 7$*:* X^+ *is the blow-up of another Fano of genus* 7 *along a point.*

Conic blow-up*:*
$g = 12$*:* X^+ *is the blow-up of* Q^3 *along a sextic rational curve.*
$g = 10$*:* X^+ *is a conic bundle over* $\mathbb{P}^2$*, which is either wild in characteristic* 2 *or has a quartic discriminant locus.*
$g = 9$*:* X^+ *is a pencil of sextic del Pezzo surfaces over* $\mathbb{P}^1$*.*
$g = 8$*:* X^+ *is the blow-up of another Fano of genus* 8 *along a smooth conic.*
$g = 7$*:* X^+ *is the blow-up of* Q^3 *along a curve of degree* 10 *and genus* 7*.*

Proof. Exactly as that of Mori and Takeuchi. □

COROLLARY 11.4 X *is rational if either* $g = 7$ *or* $g \geqslant 9$*.*

Proof. By Corollary 11.3 and the results of Section 10, it is enough to show that if $f : X^+ \to \mathbb{P}^1$ is a pencil of sextic del Pezzo surfaces, all of whose fibres L have $\chi(\mathcal{O}) = 1$ and are reduced and irreducible, then X^+ is rational.

Suppose first that the geometric generic fibre L is not normal. Let $Z \to L$ be the normalization. Since the singularities of L are unibranched and lie on a smooth 3-fold, Reid's classification [R] shows that $Z \to L$ is the projection of a rational scroll $\mathbb{F}$ from a point Q coplanar with a conic D on $\mathbb{F}$, the image ℓ of D on L is a line and $D \to \ell$ is inseparable (so that $p = 2$). Then the scheme-theoretic generic fibre X^+_K defined over $K = k(\mathbb{P}^1)$ contains a K-rational line m (recall that X^+_K is embedded in $\mathbb{P}^6$); projecting X^+_K from m exhibits X^+_K birationally as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1_K$, so that X^+ is rational.

So we can assume that L is normal. Then it has only RDPs.

Suppose that $g = 10$ and X^+ is the flop of a point blow-up $\mathrm{Bl}_P\, X$. Let E^+ denote the strict transform of the exceptional divisor. Any irreducible flopping curve Γ is the transform of a conic through P, so that $\Gamma \,.\, E^+ = -1$. Since $0 = \Gamma \,.\, (-K_{X^+}) = \Gamma \,.\, (E^+ + L)$, it follws that $L \,.\, \Gamma = 1$, so that Γ is a section of f. That is, X^+_K has a K-point; projecting from this shows that X^+_K is K-birational to a quintic del Pezzo surface, which is well known to be rational over K (see [S–B] for a short and easy proof).

Suppose that $g = 9$ and that X^+ is the flop of a conic blow-up $\mathrm{Bl}_C\, X$. If there is an irreducible flopping curve Γ that is the transform of a conic meeting C in one point, then we see as above that Γ is a section of f, so that again X^+ is rational. Otherwise $\mathcal{N}_{C/X} = (2, -2)$ and the only flopping curve Γ is the transform of the negative section on the exceptional divisor E. Then $E^+ \,.\, \Gamma = -2$, so that $L \,.\, \Gamma = 2$.

Assume first that $\Gamma \to \mathbb{P}^1$ is separable. Suppose that X_K^+ is smooth. Then the complement of the lines on X_K^+ is a torsor T_1 under a torus T defined over K, and there is a Galois extension L/K such that $T_1 \otimes_K L$ is trivial. Since K is a C_1 field, $H^1(L/K, T_1(L)) = 0$ [S, p.170], so that T_1 is trivial. Then X_K^+ has a K-point and is K-rational.

If X_K^+ is singular, then either it has a double point Q defined over K or it has two conjugate points Q_1, Q_2 of type A_1. In the first case projection from Q maps X_K^+ to a quartic surface S in $\mathbb{P}^5$. Geometrically, there are three possibilities for S, and in each case S carries a pencil, defined over K, of lines. Hence X_K^+ is K-birational to $\mathbb{P}_K^1 \times Z$, where Z is a conic over K. By Tsen's theorem Z, and so X_K^+, is then K-rational.

In the second case the line m joining Q_1, Q_2 lies in X_K^+ and is defined over K. Projecting from m maps X_K^+ to a cubic surface in $\mathbb{P}^4$, which contains a pencil of lines defined over K. Then X_K^+ is again seen to be rational via Tsen's theorem.

So assume that $\Gamma \to \mathbb{P}^1$ is inseparable. Then $p = 2$. Assume first that Γ passes through a smooth point of L. Put $Y_K = \mathrm{Bl}_{\Gamma_K} X_K^+$. Then Y_K is a quartic del Pezzo surface containing a K-rational line m, the exceptional divisor. Since Y_K is an intersection of two quadrics, projecting Y_K from m shows that Y_K is K-rational, and so X is rational.

If instead Γ passes through an RDP of L, then X_K^+ has a K-rational double point Q. We have already seen that X_K^+ is K-rational in this case. □

12. Liftability and lines

Of course if $H^2(\Theta_X) = 0$ then X is liftable to characteristic zero. However, $h^2(\Theta) = h^1(\Omega_X^1 \otimes \mathcal{O}(-H))$, where $H = -K$; this would vanish if X were liftable even modulo p^2, by Deligne and Illusie's proof of Kodaira–Akizuki–Nakano vanishing. Without some assumption of liftability, however, the well known use of the Cartier operator and the spectral sequences of hypercohomology yields only that $H^2(\Theta)$ is at most 1-dimensional, even if the groups $H^i(\Omega^j \otimes \mathcal{O}(-pH))$ are assumed to vanish for $i + j \leqslant 2$.

PROPOSITION 12.1 *Assume that X^+ is a smooth projective 3-fold containing an anticanonical K3 surface S, that $H^1(\Omega_{X^+}^1)$ is generated by algebraic classes (that is, generated by the first Chern classes of divisors) and that for every non-zero primititive class $D \in$ Pic X^+, the restriction of D to S is also non-zero and primitive.*

Then $H^2(\Theta_{X^+}) = 0$.

Proof. The hypotheses imply that the natural map $H^1(\Omega_{X^+}^1) \to H^1(\Omega_S^1)$ is injective. Since $H^0(\Omega_S^1) = 0$ [R–S], taking cohomology of the exact sequence

$$0 \to \Omega_{X^+}^1(\log S)(-S) \to \Omega_{X^+}^1 \to \Omega_S^1 \to 0$$

shows that $H^1(\Omega^1_{X^+}(\log S)(-S)) = 0$. Then $H^2(\Theta_{X^+}(-\log S)) = 0$, by Serre duality. Then the cohomology of

$$0 \to \Theta_{X^+}(-\log S) \to \Theta_{X^+} \to \Theta_S \to 0,$$

with the Rudakov–Shafarevich theorem, gives the result. □

We shall say that $H^1(\Omega^1_X)$ is algebraic if it is generated by algebraic classes.

LEMMA 12.2 *If X, Y are birational smooth projective 3-folds, then $H^1(\Omega^1_X)$ is algebraic if and only if $H^1(\Omega^1_Y)$ is so. In particular, $H^1(\Omega^1_X)$ is algebraic if X is rational.*

Proof. By Abhyankar's theorem already quoted, there is a sequence $\widetilde{X} \to X$ of blow-ups with smooth centres such that the induced birational map $\widetilde{X} \to Y$ is a morphism. If $H^1(\Omega^1_X)$ is algebraic, then so is $H^1(\Omega^1_{\widetilde{X}})$, and the Leray spectral sequence shows that so is $H^1(\Omega^1_Y)$. □

PROPOSITION 12.3 *Suppose that X is Fano, of index 1 and Picard number 1. Then $H^1(\Omega^1_X)$ is generated by algebraic classes.*

Proof. By Corollaries 11.3 and 11.4 X is either rational or birational to a cubic 3-fold. The result now follows from Lemma 12.2 □

COROLLARY 12.4 *Any Fano 3-fold X of Picard number 1 can be lifted to characteristic zero.*

Proof. If $r \geqslant 2$, then this has been done by Megyesi (unpublished). If $\mathrm{Bs}\,|-K| \neq \emptyset$, then this follows from Theorem 3.4. If $\mathrm{Bs}\,|-K| = \emptyset$ and $|-K|$ is not very ample, then X is a divisor in a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$ if X is trigonal and is a divisor in a line bundle over either $\mathbb{P}^3$ or a quadric 3-fold or a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$; in these cases the liftability is immediate.

Assume then that $|-K|$ is very ample. If $g \leqslant 5$, then X is a complete intersection in projective space. If $g = 6$ then its anticanonical ring is Gorenstein of codimension 3, and so defined by Pfaffians. Then X is a linear section of the Grassmannian $G(2, 5)$. So suppose that $g \geqslant 7$. By Proposition 12.1 it suffices to show that $-K_X$ restricts to a primitive class on a general K3 section S. Suppose that it does not; then $2g - 2 = 2dp^2$ for some integer d. The resulting possibilities are

(1) $g = 10$, $p = 3$ and $d = 1$ and
(2) $g = 9$, $p = 2$ and $d = 2$.

In case (1), consider the flop X^+ of a point blow-up. Let E^+ be the strict transform of the exceptional divisor and S^+ the strict transform of a general element of $|H^+| = |-K_{X^+}|$. Then $H^+|_{S^+} \sim 3G - 2E^+|_{S^+}$ for some divisor class G on S^+. We know that $-K_{X^+} \sim E^+ + L$, where L is a fibre of a del Pezzo fibration, so that $L|_{S^+}$ is divisible by 3. Then $h^0(\mathcal{O}_{X^+}(L)) = 2$, while $h^0(\mathcal{O}_{S^+}(L)) \geqslant 4$. Then the cohomology of an appropriate exact sequence gives $h^1(\mathcal{O}_{X^+}(-S^+ + L)) \geqslant 2$. However, $\mathcal{O}_{X^+}(-S^+ + L) \cong \mathcal{O}_{X^+}(-E^+)$, and taking the cohomology of

$$0 \to \mathcal{O}_{X^+}(-E^+) \to \mathcal{O}_{X^+} \to \mathcal{O}_{E^+} \to 0$$

gives a contradiction, since $H^1(\mathcal{O}_{X^+}) = 0$.

In case (2), consider the flop X^+ of a conic blow-up, and let E^+, S^+, H^+ be as in (1). Then $H^+|_{S^+} \sim 2G - E^+|_{S^+}$ for some divisor class G on S^+, while X^+ has a sextic del Pezzo fibration with fibre L such that $H^+ \sim E^+ + L$. Then $L|_{S^+}$ is divisible by 2, and now the same argument as in (1) can be used. □

COROLLARY 12.5 *X contains a line.*

Proof. Lines exist in characteristic zero, and so arise in characteristic p by specialization. □

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