ANZIAM J. 56(2014), 66–90
doi:10.1017/S1446181114000212

OPTIMAL TIME-CONSISTENT PORTFOLIO AND CONTRIBUTION SELECTION FOR DEFINED BENEFIT PENSION SCHEMES UNDER MEAN–VARIANCE CRITERION

XIAOQING LIANG[1], LIHUA BAI[1] and JUNYI GUO[✉1]

(Received 30 October, 2013; revised 29 April, 2014)

Abstract

We investigate two mean–variance optimization problems for a single cohort of workers in an accumulation phase of a defined benefit pension scheme. Since the mortality intensity evolves as a general Markov diffusion process, the liability is random. The fund manager aims to cover this uncertain liability via controlling the asset allocation strategy and the contribution rate. In order to have a more realistic model, we study the case when the risk aversion depends dynamically on current wealth. By solving an extended Hamilton–Jacobi–Bellman system, we obtain analytical solutions for the equilibrium strategies and value function which depend on both current wealth and mortality intensity. Moreover, results for the constant risk aversion are presented as special cases of our models.

2010 *Mathematics subject classification*: primary 93E20; secondary 91B30, 91G10, 91B70.

Keywords and phrases: time-consistent strategy, stochastic mortality intensity process, mean–variance criterion, Hamilton–Jacobi–Bellman equation.

1. Introduction

The optimal management of dynamic pension schemes is becoming increasingly important since pension funds currently play an influential role in financial markets for their high capitalization, and participants are starting to pay more attention to the security of promised benefits.

There are two major types of pension funds; defined contribution (DC) pension plans and defined benefit (DB) pension plans. In a DC scheme contributions are fixed, and benefits depend on the returns of the fund portfolio, so the participants bear the financial risk. In a DB scheme benefits are fixed in advance by the sponsor, and

[1]School of Mathematical Sciences, Nankai University, Tianjin 30007, SC, PR China;
e-mail: liangxqnk@mail.nankai.edu.cn, lihuabai@126.com, jyguo@nankai.edu.cn.

contributions are set in order to maintain the fund in balance. Future benefits due to the participants are, thus, a liability for the sponsor who bears the financial risk.

Our aim in this paper is to analyse a DB pension scheme which is a common model in the employment system. DB pension funds have been extensively studied in the literature. They have been generally modelled as linear quadratic (LQ) optimal control problems, and the manager's objectives are usually related to the minimization of the risk of solvency and contribution. See, for instance, the work of Cairns [3], Josa-Fombellida and Rincón-Zapatero [9–11], Ngwira and Gerrard [15] and Delong et al. [4].

The mean–variance (MV) approach proposed by Markowitz [14] is well known as the foundation of modern finance theory and a vast number of papers have been published on this topic. Most of them deal with the single-period case, since the MV criterion in the multiperiod framework lacks the iterated expectation property, which results in inconsistency in MV problems in the sense that the Bellman optimality principle does not hold, and consequently the traditional dynamic programming approach cannot be directly applied.

In the literature, there are two basic ways of handling time-inconsistent problems. One is to find the pre-committed strategies, where "optimal" is interpreted as "optimal from the point of view of the initial time". In the context of MV portfolio choice, the traditional objective can be described as minimizing

$$J_0(0, x, \pi) = \frac{\gamma}{2} \mathbb{V}ar_{0,x}[X^\pi(T)] - \mathbb{E}_{0,x}[X^\pi(T)]$$

over all admissible policies, where γ is a pre-specified risk aversion coefficient. The term "pre-commitment" involves the target given implicitly by considering the variance as the quadratic deviation from the target $\mathbb{E}_{0,x}[X^\pi(T)]$, that is, the manager pre-commits to the target determined at time 0 but does not update the target at subsequent dates. Li and Ng [12] and Zhou and Li [20] developed an embedding technique to transform the original time-inconsistent problem into a tractable stochastic LQ problem in discrete and continuous time settings, respectively. In the pension fund literature, similar problems are investigated by Delong et al. [4] and Josa-Fombellida and Rincón-Zapatero [10]. Note that all these works derived only pre-committed strategies.

Another possibility for dealing with time-inconsistent problems is to study them within a game-theoretic framework and seek the corresponding time-consistent strategy (subgame perfect Nash equilibrium point), which means that the optimal strategy derived at time t will agree with that derived at time $t + \Delta t$. Strotz [17] first formally treated a deterministic Ramsay problem which was time inconsistent using the game-theoretic approach. Björk and Murgoci [1] considered a fairly general class of time-inconsistent objective functions in a Markovian setting, derived an extended Hamilton–Jacobi–Bellman (HJB) equation and provided a verification theorem. Björk et al. [2] further studied the time-consistent strategy for a continuous-time MV portfolio problem where the risk aversion depends dynamically on current wealth.

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

In addition, Hu et al. [7] investigated an equilibrium for a general time-inconsistent stochastic LQ control problem within the class of open-loop controls. Zeng and Li [18] explored time-consistent strategies for optimal investment and reinsurance problems under an MV criterion in a Black–Scholes market. Zeng et al. [19] extended the model to incorporate jumps, and Li and Li [13] focused on the case with state-dependent risk aversion.

It seems very difficult to require investors not to deviate from the optimal strategy chosen at the initial time during the entire investment horizon. A reasonable investor sitting at time t would consider starting from $t + \Delta t$, and would follow the policy that would be optimal sitting at time $t + \Delta t$. Therefore, in our paper, we take the time-inconsistency seriously by updating the target to $\mathbb{E}_{t,x,\lambda}[X^{\pi}(T) - Da(\lambda(T))]$ and considering a state-dependent risk aversion. More explicitly, we consider an objective function of the form

$$J(t, x, \lambda, \pi) = \tfrac{1}{2}\mathbb{V}ar_{t,x,\lambda}[X^{\pi}(T) - Da(\lambda(T))] - (\mu_1 x + \mu_2)\mathbb{E}_{t,x,\lambda}[X^{\pi}(T) - Da(\lambda(T))],$$

where $Da(\lambda(T))$ is the liability of the fund and $\mu_1 x + \mu_2$ represents the state-dependent risk aversion.

To our knowledge, little work has been done in the pension fund literature on time-consistent strategies for portfolio optimization problems, except for the work of He and Liang [5] who studied the time-consistent investment strategy for a DC pension plan. In this paper, we investigate the optimal time-consistent investment strategy and supplementary contribution rate for a DB pension scheme under the MV criterion. The evolution of the mortality intensity follows a general Markov diffusion process. We solve two problems. The supplementary contribution rate in the first problem is set by the spread method of the fund amortization, so that the control process depends only on the investment strategy. In the second problem, the objective is to determine the contribution rate and the investment strategy, minimizing both the contribution and the solvency risk. Delong et al. [4] considered a similar model. However, they managed to obtain the precommitted strategies for the optimization problems.

In this paper, we also study a state-dependent risk aversion, where the risk aversion dynamically depends on current wealth. For a constant risk aversion, the optimal amount invested in the risky asset is independent of current wealth, which is economically unreasonable. We formulate our problem in a game-theoretic framework and provide analytical solutions where the equilibrium investment strategy and supplementary contribution rate are dependent on both the current wealth and the mortality intensity. Moreover, results for constant risk aversion are derived as comparisons with the general cases.

The outline of this paper is as follows. We introduce the model and basic assumptions in Section 2. In Section 3, we formulate the MV optimization problem within a game-theoretic framework and provide a verification theorem. An explicit time-consistent strategy and an equilibrium value function are derived. In Section 4, a generalized MV problem with the contribution risk is also considered and analytical results are obtained. Finally, we give numerical examples to illustrate our results in Section 5.

2. Model and assumptions

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T'}$ for a finite T' which denotes the maximum future lifetime of the fund's participants. The filtration $\mathbb{F}$ consists of two subfiltrations: we set $\mathbb{F} = \mathbb{F}^F \vee \mathbb{F}^M$, where $\mathbb{F}^F$ contains information about the financial market and $\mathbb{F}^M$ contains information about the mortality intensity. Suppose that the subfiltrations $\mathbb{F}^F$ and $\mathbb{F}^M$ are independent of each other. Let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$.

2.1. The financial market In the financial market, there are one risk-free asset and one risky asset. The price of the risk-free asset $R(t)$ is modelled by $dR(t) = rR(t)\,dt$, and the dynamics of the price of the risky asset follows a jump-diffusion process

$$dS(t) = S(t-)\Big[\mu\,dt + \sigma\,dW(t) + \int_{z>-1} z\widetilde{M}(dt, dz)\Big],$$

where r, σ, μ are positive constants and $\mu > r$. Here, $\{W(t)\}_{0\le t\le T'}$ is a standard $\mathbb{F}^F$-adapted Brownian motion and $\widetilde{M}(dt, dz) = M(dt, dz) - \nu(dz)\,dt$ is an $\mathbb{F}^F$-compensated Poisson martingale measure, where M is a Poisson random measure, independent of W, with the intensity measure ν satisfying $\int_{\mathbb{R}}(z^2 \wedge 1)\nu(dz) < \infty$ and $\int_{z\ge1} z^4\nu(dz) < \infty$, ensuring that

$$\sup_{t\in[0,T']} \mathbb{E}\Big[\Big|\int_0^t \int_{z>-1} z\widetilde{M}(dt, dz)\Big|^4\Big] < \infty.$$

2.2. The pension model Denote by $\lambda(t)$ the level of the mortality intensity of the fund participants, t years after they enter the plan. This evolves as a general Markov diffusion process given by

$$d\lambda(t) = \theta(t, \lambda(t))\,dt + \eta(t, \lambda(t))\,d\overline{W}(t), \tag{2.1}$$

where $\overline{W}(t)$ is an $\mathbb{F}^M$-adapted BM independent of W. Assume that the coefficients fulfil the following regularity conditions to ensure the existence of a strictly positive solution to (2.1).

- (A1) The functions $\theta : [0, T'] \times (0,\infty) \to (0,\infty)$ and $\eta : [0, T'] \times (0,\infty) \to (0,\infty)$ are continuous, locally Lipschitz continuous in λ and uniformly continuous in t.
- (A2) There exists a sequence $(D_n)_{n\in\mathbb{N}}$ of bounded domains with closure $\overline{D}_n \subseteq (0,\infty)$, and $\bigcup_{n\ge1} D_n = (0,\infty)$, such that $\theta(t, \lambda)$ and $\eta(t, \lambda)$ are uniformly Lipschitz continuous on $[0, T'] \times \overline{D}_n$.
- (A3) For all $(t, \lambda) \in [0, T'] \times (0,\infty)$, $P(\{s \in [t, T'] \mid \lambda(s) \in (0,\infty) \text{ and } \lambda(t) = \lambda\}) = 1$.

In this paper, we analyse a DB pension fund during the accumulation phase $[0, T]$, where $T < T'$ denotes the retirement time of all participants. Then the liability of the plan, also known as the expected discounted present value of future pension benefits to the cohort of the participants conditioned on the given level of the mortality intensity at time T, is equal to

$$Da(\lambda) = D\mathbb{E}_{T,\lambda}\Big[\int_T^{T'} e^{-r(s-T)}e^{-\int_T^s \lambda(u)\,du}\,ds\Big], \tag{2.2}$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

where D is the promised pension benefit, defined in advance, and the residual probability of payments after time T' is set to zero.

Let $X(t)$ be the value of the accumulated fund at time t, $AL(t)$ the actuarial liability, and $C(t)$ the contribution rate which funds the liability and consists of two elements: the normal cost rate $\mathrm{NC}(t)$ and the supplementary cost rate $\mathrm{SC}(t)$ amortizing the unfunded actuarial liability, which is defined as $UAL(t) = AL(t) - X(t-)$. Then $X(t)$ satisfies the stochastic differential equation (SDE)

$$dX^{\pi}(t) = \pi(t-)\Big(\mu\,dt + \sigma\,dW(t) + \int_{z>-1} z\widetilde{M}(dt,dz)\Big) + (X^{\pi}(t) - \pi(t))r\,dt + C(t)\,dt,$$

where $\pi(t)$ is the amount invested in the risky asset.

Denote by $M(t)$ a distribution function on $[0,T]$, which represents the percentage of the actuarial value of the future benefits (2.2) accumulated during the first t years. We assume that $M : [0,T] \to [0,1]$ is absolutely continuous with respect to Lebesgue measure, and its density function $m(t)$ is Lipschitz continuous on $[0,T]$. The actuarial liability and the normal cost can be defined as in the paper by Delong et al. [4]:

$$AL(t,\lambda) = e^{-\rho(T-t)}M(t)\mathbb{E}_{t,\lambda}[Da(\lambda(T))], \quad 0 \le t \le T, \tag{2.3}$$

$$\mathrm{NC}(t,\lambda) = e^{-\rho(T-t)}m(t)\mathbb{E}_{t,\lambda}[Da(\lambda(T))], \quad 0 \le t \le T, \tag{2.4}$$

where ρ is the fund's valuation rate. Moreover, we investigate two MV optimization problems. In the first one, the supplementary contribution rate is set as

$$\mathrm{SC}(t,\lambda) = \kappa(AL(t,\lambda) - X^{\pi}(t-)), \tag{2.5}$$

where κ is some predefined constant. The value of the fund process $(X^{\pi}(t), 0 \le t \le T)$ depends only on the investment strategy π. In the second one, we add the supplementary contribution rate into our optimization problem and minimize the MV objective along with the expected value of squares of future supplementary costs.

3. Mean–variance optimization problem

In this section, we consider the optimal investment strategy for a DB pension scheme where the supplementary contribution rate is predetermined as in (2.5). The manager aims to manage the fund in order to cover the liability $Da(\lambda(T))$ at the retirement time T. In this case, the dynamics of the fund process $X^{\pi}(t)$ can be given by

$$\begin{aligned} dX^{\pi}(t) &= [\pi(t)(\mu - r) + (r-\kappa)X^{\pi}(t) + \mathrm{NC}(t,\lambda) + \kappa AL(t,\lambda)]\,dt \\ &\quad + \sigma\pi(t)\,dW(t) + \pi(t-)\int_{z>-1} z\widetilde{M}(dt,dz). \end{aligned} \tag{3.1}$$

Moreover, an investment strategy $\pi(t)$ is *admissible* if it satisfies the following conditions:

(i) For all $t \in [0,T], \pi(t)$ is a predictable mapping with respect to $\mathcal{F}_t$.
(ii) $\int_0^T \pi^2(t)\,dt < \infty$ almost everywhere.

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

(iii) The stochastic differential equation (3.1) has a unique solution X^π on $[0, T]$.

The set of all admissible strategies is denoted by $\mathcal{A}$.

In the literature, MV problems are usually transformed into LQ control problems, which can be solved by standard methods. Contrary to the existing literature in this field, our paper aims to obtain the optimal time-consistent policies instead of the pre-committed policies.

Our main goal is to formalize the MV optimization problem without pre-commitment, and consider a state-dependent risk aversion. Thus, the objective function is of the form

$$J(t, x, \lambda, \pi) = \frac{1}{2}\mathbb{V}ar_{t,x,\lambda}[X^\pi(T) - Da(\lambda(T))] - (\mu_1 x + \mu_2)\mathbb{E}_{t,x,\lambda}[X^\pi(T) - Da(\lambda(T))], \tag{3.2}$$

where μ_1, μ_2 are constants and the term $\mu_1 x + \mu_2$ represents the state-dependent risk aversion. This problem can be viewed as a dynamic MV problem, we analyse it in a game-theoretic framework which is developed by Björk and Murgoci [1].

Definition 3.1. An admissible control $\hat{\pi}$ is an equilibrium control law if for any given $\pi \in R, h > 0$ and $(t, x, \lambda) \in [0, T] \times R \times R^+$,

$$\limsup_{h \to 0} \frac{J(t, x, \lambda, \hat{\pi}) - J(t, x, \lambda, \pi_h)}{h} \leq 0,$$

where the control law

$$\pi_h(s, y, \lambda) = \begin{cases} \pi & \text{for } t \leq s < t + h, \quad y \in R, \lambda \in R^+, \\ \hat{\pi}(s, y, \lambda) & \text{for } t + h \leq s \leq T, \quad y \in R, \lambda \in R^+. \end{cases}$$

The corresponding equilibrium value function V is defined by

$$V(t, x, \lambda) = J(t, x, \lambda, \hat{\pi}).$$

We will now provide a verification theorem for the MV problem (3.2) without pre-commitment. To generalize this theorem, we consider a general optimization problem of the form

$$J(t, x, \lambda, \pi) = f(t, x, \lambda, y^\pi(t, x, \lambda), z^\pi(t, x, \lambda)),$$

where f is a function in $C^{1,2,2,2,2}([0, T] \times R^4)$ (that is, f is continuous on $[0; T] \times R^4$, continuously differentiable with respect to the first variable, and has continuous derivatives up to order 2 with other variables), and

$$y^\pi(t, x, \lambda) = \mathbb{E}_{t,x,\lambda}[X^\pi(T) - Da(\lambda(T))],$$
$$z^\pi(t, x, \lambda) = \mathbb{E}_{t,x,\lambda}[(X^\pi(T) - Da(\lambda(T)))^2].$$

In particular, if we set

$$f(t, x, \lambda, y, z) = \frac{1}{2}(z - y^2) - (\mu_1 x + \mu_2)y, \tag{3.3}$$

then the problem reduces to our MV optimization problem without pre-commitment.

THEOREM 3.2 (Verification theorem). *Suppose that there exist three real-valued functions $W(t, x, \lambda), G(t, x, \lambda), H(t, x, \lambda) \in C^{1,2,2}([0, T] \times R \times R^+)$ satisfying the following extended HJB system such that, for all $(t, x, \lambda) \in [0, T] \times R \times R^+$,*

$$\inf_{\pi \in R}\{\mathcal{A}^\pi W(t,x,\lambda) - \mathcal{A}^\pi f(t,x,\lambda,G,H) + f_y(t,x,\lambda,G,H)\mathcal{A}^\pi G(t,x,\lambda) + f_z(t,x,\lambda,G,H)\mathcal{A}^\pi H(t,x,\lambda)\} = 0, \tag{3.4}$$

$$W(T,x,\lambda) = f(T,x,\lambda,G,H),$$

$$\mathcal{A}^{\hat{\pi}} G(t,x,\lambda) = 0, \tag{3.5}$$

$$G(T,x,\lambda) = x - Da(\lambda),$$

$$\mathcal{A}^{\hat{\pi}} H(t,x,\lambda) = 0, \tag{3.6}$$

$$H(T,x,\lambda) = (x - Da(\lambda))^2,$$

where

$$\begin{aligned}
&\mathcal{A}^\pi \phi(t,x,\lambda)\\
&\quad = \phi_t + \phi_x\Big[(r-\kappa)x + \Big(\mu - r - \int_{z>-1} z\nu(dz)\Big)\pi + \mathrm{NC}(t,\lambda) + \kappa AL(t,\lambda)\Big] + \phi_\lambda\theta(t,\lambda)\\
&\quad\quad + \frac{1}{2}\phi_{xx}\sigma^2\pi^2 + \frac{1}{2}\phi_{\lambda\lambda}\eta^2(t,\lambda) + \int_{z>-1}[\phi(t,x+\pi z,\lambda) - \phi(t,x,\lambda)]\nu(dz),\\
&\mathcal{A}^\pi f(t,x,\lambda,G,H)\\
&\quad = f_t + f_x\Big[(r-\kappa)x + \Big(\mu - r - \int_{z>-1} z\nu(dz)\Big)\pi + \mathrm{NC}(t,\lambda) + \kappa AL(t,\lambda)\Big] + f_\lambda\theta(t,\lambda)\\
&\quad\quad + f_y\mathcal{A}^\pi G(t,x,\lambda) + f_z\mathcal{A}^\pi H(t,x,\lambda) + \frac{1}{2}U_1(f,G,H)\sigma^2\pi^2\\
&\quad\quad + \frac{1}{2}U_2(f,G,H)\eta^2(t,\lambda)\\
&\quad\quad + \int_{z>-1}[f(t,x+\pi z,\lambda,G(t,x+\pi z,\lambda),H(t,x+\pi z,\lambda)) - f(t,x,\lambda,G,H)]\nu(dz)\\
&\quad\quad - f_y\int_{z>-1}[G(t,x+\pi z,\lambda) - G(t,x,\lambda)]\nu(dz)\\
&\quad\quad - f_z\int_{z>-1}[H(t,x+\pi z,\lambda) - H(t,x,\lambda)]\nu(dz),
\end{aligned}$$

$$\begin{aligned}
U_1(f,G,H) &= f_{xx} + 2f_{xy}G_x + 2f_{xz}H_x + 2f_{yz}H_xG_x + f_{yy}(G_x)^2 + f_{zz}(H_x)^2,\\
U_2(f,G,H) &= f_{\lambda\lambda} + 2f_{\lambda y}G_\lambda + 2f_{\lambda z}H_\lambda + 2f_{yz}H_\lambda G_\lambda + f_{yy}(G_\lambda)^2 + f_{zz}(H_\lambda)^2,
\end{aligned}$$

with $f = f(t,x,\lambda,G,H), G = G(t,x,\lambda)$ and $H = H(t,x,\lambda)$ for simplicity, and

$$\hat{\pi} = \arg\inf_{\pi\in R}\{\mathcal{A}^\pi W(t,x,\lambda) - \mathcal{A}^\pi f(t,x,\lambda,G,H) + f_y\mathcal{A}^\pi G(t,x,\lambda) + f_z\mathcal{A}^\pi H(t,x,\lambda)\}.$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

Then

$$\begin{aligned} V(t,x,\lambda) &= W(t,x,\lambda) \\ G(t,x,\lambda) &= \mathbb{E}_{t,x,\lambda}[X^{\hat{\pi}}(T) - Da(\lambda(T))], \\ H(t,x,\lambda) &= \mathbb{E}_{t,x,\lambda}[(X^{\hat{\pi}}(T) - Da(\lambda(T)))^2] \end{aligned}$$

and the equilibrium control law is given by $\hat{\pi}$.

Proof. The proof is analogous to that of Theorem 4.1 by Björk and Murgoci [1] (see Appendix). □

3.1. Solution to the optimization problem We derive the optimal time-consistent investment strategy and the value function for problem (3.2). In this case, the function f is given by (3.3), that is, $f(t,x,\lambda,G,H) = \frac{1}{2}(H - G^2) - (\mu_1 x + \mu_2)G$, hence,

$$\begin{gathered} f_x = -\mu_1 G, \quad f_{xy} = -\mu_1, \quad f_{yy} = -1, \\ f_t = f_{xx} = f_{xz} = f_{yz} = f_{zz} = 0, \quad f_\lambda = f_{\lambda\lambda} = f_{\lambda z} = f_{\lambda y} = 0. \end{gathered}$$

Thus,

$$U_1(f,G,H) = -2\mu_1 G_x - (G_x)^2, \quad U_2(f,G,H) = -(G_\lambda)^2.$$

According to Theorem 3.2, equation (3.4) can be rewritten as

$$\begin{aligned} \inf_{\pi\in R}\bigg[& W_t + (W_x + \mu_1 G)\Big\{(r-\kappa)x + \Big(\mu - r - \int_{z>-1} z\nu(dz)\Big)\pi + \mathrm{NC}(t,\lambda) + \kappa AL(t,\lambda)\Big\} \\ & + W_\lambda\theta(t,\lambda) + \frac{1}{2}\sigma^2\pi^2\{2\mu_1 G_x + (G_x)^2 + W_{xx}\} + \frac{1}{2}\eta^2(t,\lambda)\{W_{\lambda\lambda} + (G_\lambda)^2\} \\ & + \int_{z>-1}\mu_1\pi z G(t,x+\pi z,\lambda)\nu(dz) + \frac{1}{2}\int_{z>-1}\{G(t,x+\pi z,\lambda) - G(t,x,\lambda)\}^2\nu(dz) \\ & + \int_{z>-1}\{W(t,x+\pi z,\lambda) - W(t,x,\lambda)\}\nu(dz)\bigg] = 0. \end{aligned} \tag{3.7}$$

By the linear structure of the dynamics of $X^\pi(t)$ and the boundary conditions, it is natural to guess that

$$W(t,x,\lambda) = A(t)x^2 + B(t,\lambda)x + C(t,\lambda) \tag{3.8}$$

with $A(T) = -\mu_1, B(T,\lambda) = -\mu_2 + \mu_1 Da(\lambda), C(T,\lambda) = \mu_2 Da(\lambda)$;

$$G(t,x,\lambda) = \bar{a}(t)x + b(t,\lambda)$$

with $\bar{a}(T) = 1, b(T,\lambda) = -Da(\lambda)$; and

$$H(t,x,\lambda) = q(t)x^2 + l(t,\lambda)x + d(t,\lambda)$$

with $q(T) = 1, l(T,\lambda) = -2Da(\lambda), d(T,\lambda) = (Da(\lambda))^2$.

Substituting $W(t,x,\lambda), G(t,x,\lambda)$ and their corresponding derivatives into (3.7) yields

$$\begin{aligned}
&\inf_{\pi\in R}\Big[A_t x^2 + B_t x + C_t \\
&\quad + \{(2A(t)+\mu_1\bar{a}(t))x + B(t,\lambda)+\mu_1 b(t,\lambda)\}\{(r-\kappa)x + \mathrm{NC}(t,\lambda)+\kappa AL(t,\lambda)\} \\
&\quad + (B_\lambda x + C_\lambda)\theta(t,\lambda) + \frac{1}{2}\{2A(t)+2\mu_1\bar{a}(t)+\bar{a}^2(t)\}\Big(\sigma^2 + \int_{z>-1} z^2\nu(dz)\Big)\pi^2 \\
&\quad + \frac{1}{2}\eta^2(t,\lambda)(B_{\lambda\lambda}x + C_{\lambda\lambda} + b_\lambda^2) \\
&\quad + \{(2A(t)+\mu_1\bar{a}(t))x + B(t,\lambda)+\mu_1 b(t,\lambda)\}(\mu-r)\pi\Big] = 0.
\end{aligned} \tag{3.9}$$

Differentiating with respect to π and setting the derivative to zero, we obtain the value of the equilibrium control law

$$\begin{aligned}
\hat{\pi}(t,x,\lambda) &= -\bar{\beta}\frac{(2A(t)+\mu_1\bar{a}(t))x + B(t,\lambda)+\mu_1 b(t,\lambda)}{2A(t)+2\mu_1\bar{a}(t)+\bar{a}^2(t)}, \\
\bar{\beta} &= \frac{\mu-r}{\sigma^2 + \int_{z>-1} z^2\nu(dz)}.
\end{aligned}$$

Instead of analysing the above equality directly, we make the assumption that

$$\hat{\pi}(t,x,\lambda) = k_1(t)x + k_2(t,\lambda), \tag{3.10}$$

where

$$k_1(t) = -\bar{\beta}\frac{2A(t)+\mu_1\bar{a}(t)}{2A(t)+2\mu_1\bar{a}(t)+\bar{a}^2(t)}, \tag{3.11}$$

$$k_2(t,\lambda) = -\bar{\beta}\frac{B(t,\lambda)+\mu_1 b(t,\lambda)}{2A(t)+2\mu_1\bar{a}(t)+\bar{a}^2(t)}, \tag{3.12}$$

with $k_1(T)=\mu_1\bar{\beta}$, $k_2(T,\lambda)=\mu_2\bar{\beta}$. Then, inserting (3.10) into (3.9), we obtain

$$\begin{aligned}
&A_t x^2 + B_t x + C_t \\
&\quad + [(2A(t)+\mu_1\bar{a}(t))x + B(t,\lambda)+\mu_1 b(t,\lambda)][(r-\kappa)x + \mathrm{NC}(t,\lambda)+\kappa AL(t,\lambda)] \\
&\quad + (B_\lambda x + C_\lambda)\theta(t,\lambda) + \tfrac{1}{2}\eta^2(t,\lambda)(B_{\lambda\lambda}x + C_{\lambda\lambda} + b_\lambda^2) \\
&\quad + \tfrac{1}{2}(\mu-r)[(2A(t)+\mu_1\bar{a}(t))k_1(t)x^2 + (B(t,\lambda)+\mu_1 b(t,\lambda))(2k_1(t)x + k_2(t,\lambda))] = 0.
\end{aligned}$$

Equating coefficients of x^2, x and constant terms to zero, we get

$$A_t + \{2A(t)+\mu_1\bar{a}(t)\}(r-\kappa) + \tfrac{1}{2}(\mu-r)\{2A(t)+\mu_1\bar{a}(t)\}k_1(t) = 0, \tag{3.13}$$

$$\begin{aligned}
&B_t + \{B(t,\lambda)+\mu_1 b(t,\lambda)\}(r-\kappa) + \{2A(t)+\mu_1\bar{a}(t)\}\{\mathrm{NC}(t,\lambda)+\kappa AL(t,\lambda)\} \\
&\quad + B_\lambda\theta(t,\lambda) + \tfrac{1}{2}B_{\lambda\lambda}\eta^2(t,\lambda) + (\mu-r)\{B(t,\lambda)+\mu_1 b(t,\lambda)\}k_1(t) = 0,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&C_t + \{B(t,\lambda)+\mu_1 b(t,\lambda)\}\{\mathrm{NC}(t,\lambda)+\kappa AL(t,\lambda)\} + C_\lambda\theta(t,\lambda) \\
&\quad + \tfrac{1}{2}(C_{\lambda\lambda}+b_\lambda^2)\eta^2(t,\lambda) + \tfrac{1}{2}(\mu-r)\{B(t,\lambda)+\mu_1 b(t,\lambda)\}k_2(t,\lambda) = 0.
\end{aligned} \tag{3.15}$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

Similarly, substituting $G(t, x, \lambda)$ and (3.10) into (3.5) leads to

$$\bar{a}_t x + b_t + \bar{a}(t)[(r-\kappa)x + \mathrm{NC}(t,\lambda) + \kappa AL(t,\lambda) + (\mu - r)(k_1(t)x + k_2(t,\lambda))] + b_\lambda \theta(t,\lambda) + \tfrac{1}{2} b_{\lambda\lambda}\eta^2(t,\lambda) = 0.$$

To ensure the above equality, we require

$$\bar{a}_t + \bar{a}(t)(r-\kappa) + \bar{a}(t)(\mu - r)k_1(t) = 0 \tag{3.16}$$

and

$$b_t + \bar{a}(t)(\mu - r)k_2(t,\lambda) + \bar{a}(t)[\mathrm{NC}(t,\lambda) + \kappa AL(t,\lambda)] + b_\lambda\theta(t,\lambda) + \tfrac{1}{2} b_{\lambda\lambda}\eta^2(t,\lambda) = 0. \tag{3.17}$$

From (3.16) and the boundary condition $\bar{a}(T) = 1$, we obtain

$$\bar{a}(t) = \exp\Big[\int_t^T \{r - \kappa + (\mu - r)k_1(u)\}\, du\Big]. \tag{3.18}$$

In addition, inserting $H(t, x, \lambda)$ and (3.10) into (3.6) yields

$$q_t x^2 + l_t x + d_t + (2q(t)x + l(t,\lambda))[(r-\kappa)x + \mathrm{NC}(t,\lambda) + \kappa AL(t,\lambda) + (\mu - r)(k_1(t)x + k_2(t,\lambda))] + (l_\lambda x + d_\lambda)\theta(t,\lambda) + q(t)\beta(k_1(t)x + k_2(t,\lambda))^2 + \tfrac{1}{2}\eta^2(t,\lambda)(l_{\lambda\lambda}x + d_{\lambda\lambda}) = 0,$$

where $\beta = (\mu - r)/\bar{\beta}$.

Combining the coefficients of x^2, we get

$$q_t + 2q(t)(r-\kappa) + 2q(t)(\mu - r)k_1(t) + q(t)\beta k_1^2(t) = 0,$$

with the boundary condition $q(T) = 1$. Hence

$$q(t) = \exp\Big[\int_t^T \{2(r-\kappa) + 2(\mu - r)k_1(u) + \beta k_1^2(u)\} du\Big]. \tag{3.19}$$

On the other hand, from the fact that

$$\begin{aligned} W(t,x,\lambda) &= \tfrac{1}{2}\{H(t,x,\lambda) - G^2(t,x,\lambda)\} - (\mu_1 x + \mu_2)G(t,x,\lambda) \\ &= \tfrac{1}{2}\{q(t) - 2\mu_1\bar{a}(t) - \bar{a}^2(t)\}x^2 + \tfrac{1}{2}\{l(t,\lambda) - 2\bar{a}(t)b(t,\lambda) \\ &\quad - 2\mu_1 b(t,\lambda) - 2\mu_2\bar{a}(t)\}x + \tfrac{1}{2}\{d(t,\lambda) - b^2(t,\lambda) - 2\mu_2 b(t,\lambda)\}, \end{aligned}$$

and the conjecture (3.8), we obtain

$$2A(t) = q(t) - 2\mu_1\bar{a}(t) - \bar{a}^2(t). \tag{3.20}$$

Using (3.11) and (3.18)–(3.20), we find the integral equation

$$\begin{aligned} k_1(t) &= -\bar{\beta}\,\frac{q(t) - \mu_1\bar{a}(t) - \bar{a}^2(t)}{q(t)} \\ &= -\bar{\beta}\Big[1 - \exp\Big\{-\int_t^T \beta k_1^2(u)\, du\Big\} \\ &\quad - \mu_1 \exp\Big\{-\int_t^T [r - \kappa + (\mu - r)k_1(u) + \beta k_1^2(u)]\, du\Big\}\Big]. \end{aligned} \tag{3.21}$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

Next, we derive the expression for $k_2(t,\lambda)$. From (3.12) and (3.20), we get

$$B(t,\lambda) = -\mu_1 b(t,\lambda) - \frac{1}{\bar\beta} q(t) k_2(t,\lambda).$$

The partial derivatives are

$$\begin{aligned}
B_t &= -\mu_1 b_t - \frac{1}{\bar\beta} q(t) \frac{\partial k_2(t,\lambda)}{\partial t} + \frac{1}{\bar\beta} q(t) k_2(t,\lambda)[2(r-\kappa) + 2(\mu - r)k_1(t) + \beta k_1^2(t)],\\
B_\lambda &= -\mu_1 b_\lambda - \frac{1}{\bar\beta} q(t) \frac{\partial k_2(t,\lambda)}{\partial \lambda},\\
B_{\lambda\lambda} &= -\mu_1 b_{\lambda\lambda} - \frac{1}{\bar\beta} q(t) \frac{\partial^2 k_2(t,\lambda)}{\partial \lambda^2}.
\end{aligned}$$

Substituting these into (3.14), collecting the similar terms and using (3.17) yields

$$\begin{aligned}
&\frac{\partial k_2(t,\lambda)}{\partial t} + \theta(t,\lambda)\frac{\partial k_2(t,\lambda)}{\partial \lambda} + \frac{1}{2}\eta^2(t,\lambda)\frac{\partial^2 k_2(t,\lambda)}{\partial \lambda^2}\\
&\quad - p(t)k_2(t,\lambda) - \bar\beta\Big[1 - \exp\Big\{-\int_t^T \beta k_1^2(u)\,du\Big\}\Big][\mathrm{NC}(t,\lambda) + \kappa AL(t,\lambda)] = 0,
\end{aligned}$$

where

$$\begin{aligned}
p(t) &= r - \kappa + (\mu - r)k_1(t) + \beta k_1^2(t)\\
&\quad + \mu_1(\mu - r)\bar\beta \exp\Big\{-\int_t^T [r - \kappa + (\mu - r)k_1(u) + \beta k_1^2(u)]\,du\Big\}.
\end{aligned}$$

From the boundary condition $k_2(T,\lambda) = \mu_2\bar\beta$, we can state the Feynman–Kac representation for $k_2(t,\lambda)$ as follows:

$$\begin{aligned}
k_2(t,\lambda) = \mathbb{E}_{t,\lambda}\Big[&\mu_2\bar\beta \exp\Big(-\int_t^T p(u)\,du\Big) - \int_t^T \bar\beta \exp\Big(-\int_t^s p(u)\,du\Big)\\
&\times\Big\{1 - \exp\Big(-\int_s^T \beta k_1^2(u)\,du\Big)\Big\}\{\mathrm{NC}(s,\lambda) + \kappa AL(s,\lambda)\}\,ds\Big].
\end{aligned}$$

Combining (2.3) and (2.4), and applying Fubini's theorem, the Markov property of the mortality intensity and the law of iterated expectations lead to

$$\begin{aligned}
k_2(t,\lambda) &= \mu_2\bar\beta \exp\Big(-\int_t^T p(u)\,du\Big) - \mathbb{E}_{t,\lambda}[Da(\lambda(T))]\int_t^T \bar\beta \exp\Big(-\int_t^s p(u)\,du - \rho(T-s)\Big)\\
&\quad\times\Big\{1 - \exp\Big(-\int_s^T \beta k_1^2(u)\,du\Big)\Big\}\{m(s) + \kappa M(s)\}\,ds. \qquad (3.22)
\end{aligned}$$

Remark 3.3. Denote $a(t,\lambda) \triangleq \mathbb{E}_{t,\lambda}[Da(\lambda(T))]$. Based on the Theorem 1 in Heath and Schweizer [6], $a(t,\lambda)$ on $[0,T'] \times \mathbb{R}^+$ satisfies the partial differential equation

$$\frac{\partial a(t,\lambda)}{\partial t} + \theta(t,\lambda)\frac{\partial a(t,\lambda)}{\partial \lambda} + \frac{1}{2}\eta^2(t,\lambda)\frac{\partial^2 a(t,\lambda)}{\partial \lambda^2} = 0,$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

with boundary condition

$$a(T,\lambda) = Da(\lambda). \qquad \diamond$$

We combine the above results in the following theorem.

THEOREM 3.4. *For the mean–variance problem (3.2), the equilibrium control is expressed as*

$$\hat{\pi}(t,x,\lambda) = k_1(t)x + k_2(t,\lambda),$$

where $k_1(t)$ and $k_2(t,\lambda)$ are given by (3.21) and (3.22), respectively. Moreover, the corresponding equilibrium value function is expressed as $V(t,x,\lambda) = A(t)x^2 + B(t,\lambda)x + C(t,\lambda)$, where the functions A, B and C are given by the expressions in (3.13)–(3.15), respectively.

REMARK 3.5. From Björk et al. [2], the above integral equation satisfied by $k_1(t)$ admits a unique solution. We can use the analogous recursive algorithm numerically for the determination of $k_1(t)$. Construct a sequence $k_1^{(i)}(t) \in C[0,T]$ as follows:

$$\begin{aligned} k_1^{(0)}(t) &\equiv 1, \\ k_1^{(n)}(t) &= -\bar{\beta}\Big[1 - \exp\Big\{-\int_t^T \beta(k_1^{(n-1)}(u))^2\,du\Big\} \\ &\qquad - \mu_1 \exp\Big\{-\int_t^T (r-\kappa+(\mu-r)k_1^{(n-1)}(u) + \beta(k_1^{(n-1)}(u))^2)\,du\Big\}\Big], \end{aligned}$$

$n = 1, 2, \ldots$. This sequence converges to $k_1(t)$ in $C[0,T]$. $\diamond$

3.2. Special case

We consider the case where $\mu_1 = 0$, that is, the risk aversion coefficient is constant.

By setting $\mu_1 = 0$, we see that $k_1(t) \equiv 0$ is the root of the integral equation (3.21). Simplifying the expression (3.22) for $k_2(t,\lambda)$, we get $k_2(t,\lambda) = \mu_2\bar{\beta}e^{-(r-\kappa)(T-t)}$. Hence, the equilibrium control is given by

$$\hat{\pi}(t,x,\lambda) = \mu_2\bar{\beta}e^{-(r-\kappa)(T-t)}.$$

Additionally, from (3.13), (3.14) and (3.16) and their corresponding boundary conditions, we obtain

$$\begin{aligned} A(t) &\equiv 0, \\ \bar{a}(t) &= e^{(r-\kappa)(T-t)}, \\ B(t,\lambda) &= -\mu_2 e^{(r-\kappa)(T-t)}. \end{aligned}$$

Substituting the values of $\bar{a}(t)$ and $k_2(t,\lambda)$ into (3.17), and by applying the Feynman–Kac formula, we get

$$b(t,\lambda) = \mu_2(\mu-r)(T-t)\bar{\beta} + \mathbb{E}_{t,\lambda}[Da(\lambda(T))]\Big[\int_t^T e^{(r-\kappa-\rho)(T-s)}\{m(s)+\kappa M(s)\}\,ds - 1\Big].$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

Moreover, integration by parts yields

$$b(t,\lambda) = \mu_2(\mu - r)(T - t)\bar{\beta} + \mathbb{E}_{t,\lambda}[Da(\lambda(T))]\Big[\frac{r-\rho}{\kappa+\rho-r} - \frac{\kappa}{\kappa+\rho-r} M(t) e^{(r-\kappa-\rho)(T-t)} - \frac{r-\rho}{\kappa+\rho-r}\int_t^T e^{(r-\kappa-\rho)(T-s)} m(s)\,ds\Big].$$

Similarly, from (3.15) we get

$$\begin{aligned} C(t,\lambda) &= \mu_2 \mathbb{E}_{t,\lambda}[Da(\lambda(T))]\Big[1 - \int_t^T e^{(r-\kappa-\rho)(T-s)}\{m(s) + \kappa M(s)\}\,ds\Big] \\ &\quad + \frac{1}{2}\mathbb{E}_{t,\lambda}\Big[\int_t^T b_\lambda^2(s,\lambda)\eta^2(s,\lambda)\,ds\Big] - \frac{\mu_2^2(\mu-r)(T-t)\bar{\beta}}{2} \\ &= \frac{1}{2}\mathbb{E}_{t,\lambda}\Big[\int_t^T b_\lambda^2(s,\lambda)\eta^2(s,\lambda)\,ds\Big] - \mu_2 b(t,\lambda) + \frac{\mu_2^2(\mu-r)(T-t)\bar{\beta}}{2}. \end{aligned}$$

Hence,

$$\begin{aligned} V(t,x,\lambda) &= -\mu_2 e^{(r-\kappa)(T-t)} x + C(t,\lambda), \\ G(t,x,\lambda) &= e^{(r-\kappa)(T-t)} x + b(t,\lambda), \end{aligned}$$

and the variance is given by

$$\begin{aligned} \mathbb{V}ar_{t,x,\lambda}[X^{\hat{\pi}}(T) - Da(\lambda(T))] &= 2(V(t,x,\lambda) + \mu_2 G(t,x,\lambda)) \\ &= \mathbb{E}_{t,\lambda}\Big[\int_t^T b_\lambda^2(s,\lambda)\eta^2(s,\lambda)\,ds\Big] + \mu_2^2(\mu-r)(T-t)\bar{\beta}. \end{aligned}$$

We observe that the optimal time-consistent investment strategy is independent of the current wealth x and the current mortality intensity λ. Therefore, from an economic point of view, the equilibrium solution for the constant risk aversion is economically unreasonable, as stated by Björk et al. [2] and Li and Li [13].

4. Generalized mean–variance optimization problem

In this section, we solve a generalized problem which is to minimize the MV objective of the terminal debt $X^\pi(T) - Da(\lambda(T))$ along with the contribution risk SC^2 on the interval $[0,T]$. Minimization of the contribution risk has been explored, for example, in Josa-Fombellida and Rincón-Zapatero [9–11], Ngwira and Gerrard [15] and Delong et al. [4]. Thus, we consider an optimization problem

$$\begin{aligned} J_1(t,x,\lambda,\pi,\mathrm{SC}) &= \mathbb{E}_{t,x,\lambda}\Big[\int_t^T \mathrm{SC}^2(u)\,du\Big] + \frac{1}{2}\mathbb{V}ar_{t,x,\lambda}[X^\pi(T) - Da(\lambda(T))] \\ &\quad - (\mu_1 x + \mu_2)\mathbb{E}_{t,x,\lambda}[X^\pi(T) - Da(\lambda(T))]. \end{aligned} \tag{4.1}$$

In this case, the wealth process $X^{\pi,\mathrm{SC}}(t)$ evolves as

$$\begin{aligned} dX^{\pi,\mathrm{SC}}(t) &= \pi(t-)\Big(\mu\,dt + \sigma\,dW(t) + \int_{z>-1} z\widetilde{M}(dt,dz)\Big) + (X^{\pi,\mathrm{SC}}(t) - \pi(t))r\,dt \\ &\quad + (\mathrm{NC}(t,\lambda) + \mathrm{SC}(t))\,dt. \end{aligned} \tag{4.2}$$

Moreover, the *admissible strategy* $\Pi = \{\pi(t), \mathrm{SC}(t)\}_{0\le t\le T}$ is defined as follows:

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

(i) For all $t \in [0, T]$, $\pi(t)$ and $\mathrm{SC}(t)$ are predictable mappings with respect to $\mathcal{F}_t$.
(ii) $\int_0^T (\pi^2(t) + \mathrm{SC}^2(t))\,dt < \infty$ almost everywhere.
(iii) The stochastic differential equation (4.2) has a unique solution X^{Π} on $[0, T]$.

The set of all admissible strategies is denoted by $\mathcal{B}$.

As in the previous section, we define the equilibrium control law and equilibrium value function similarly and provide a verification theorem including the supplementary cost rate.

Definition 4.1. We say that an admissible control $\hat{\Pi}$ is an equilibrium control law if for any fixed real numbers $\pi, \mathrm{SC}, h > 0$ and $(t, x, \lambda) \in [0, T] \times R \times R^+$,

$$\limsup_{h\to 0} \frac{J_1(t, x, \lambda, \hat{\Pi}) - J_1(t, x, \lambda, \Pi_h)}{h} \le 0,$$

where the control law Π_h is given by

$$\Pi_h(s, y, \lambda) = \begin{cases} (\pi, \mathrm{SC}) & \text{for } t \le s < t + h, \quad y \in R,\ \lambda \in R^+, \\ \hat{\Pi}(s, y, \lambda) & \text{for } t + h \le s \le T, \quad y \in R,\ \lambda \in R^+. \end{cases}$$

The corresponding equilibrium value function, V_1, is defined by

$$V_1(t, x, \lambda) = J_1(t, x, \lambda, \hat{\Pi}).$$

Theorem 4.2. *Suppose that there exist three real-valued functions $\widetilde{W}(t, x, \lambda), G^*(t, x, \lambda), H^*(t, x, \lambda) \in C^{1,2,2}([0, T] \times R \times R^+)$ satisfying the following extended HJB system: for all $(t, x, \lambda) \in [0, T] \times R \times R^+$,*

$$\begin{aligned} \inf_{(\pi, SC)\in R^2} \Big[& SC^2 + \widetilde{W}_t + (\widetilde{W}_x + \mu_1 G^*)\Big\{rx + \Big(\mu - r - \int_{z>-1} z\nu(dz)\Big)\pi + \mathrm{NC}(t, \lambda) + \mathrm{SC}\Big\} \\ & + \widetilde{W}_\lambda \theta(t, \lambda) + \frac{1}{2}\sigma^2\pi^2\{2\mu_1 G^*_x + (G^*_x)^2 + \widetilde{W}_{xx}\} + \frac{1}{2}\eta^2(t, \lambda)\{\widetilde{W}_{\lambda\lambda} + (G^*_\lambda)^2\} \\ & + \frac{1}{2}\int_{z>-1} [G^*(t, x + \pi z, \lambda) - G^*(t, x, \lambda)]^2 \nu(dz) \\ & + \int_{z>-1} \mu_1 \pi z G^*(t, x + \pi z, \lambda)\nu(dz) \\ & + \int_{z>-1} \{\widetilde{W}(t, x + \pi z, \lambda) - \widetilde{W}(t, x, \lambda)\}\nu(dz)\Big] = 0, \end{aligned} \tag{4.3}$$

$$\begin{gathered} \widetilde{W}(T, x, \lambda) = -(\mu_1 x + \mu_2)(x - Da(\lambda)), \\ \mathcal{A}^{\hat{\Pi}} G^*(t, x, \lambda) = 0 \\ G^*(T, x, \lambda) = x - Da(\lambda), \end{gathered} \tag{4.4}$$

$$\begin{gathered} \mathcal{A}^{\hat{\Pi}} H^*(t, x, \lambda) = 0, \\ H^*(T, x, \lambda) = (x - Da(\lambda))^2, \end{gathered} \tag{4.5}$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

where

$$\mathcal{A}^{\Pi}\phi(t,x,\lambda) = \phi_t + \phi_x\Big[rx + \Big(\mu - r - \int_{z>-1} z\nu(dz)\Big)\pi + \mathrm{NC}(t,\lambda) + \mathrm{SC}\Big] + \phi_\lambda\theta(t,\lambda)$$
$$+ \frac{1}{2}\phi_{xx}\sigma^2\pi^2 + \frac{1}{2}\phi_{\lambda\lambda}\eta^2(t,\lambda) + \int_{z>-1}[\phi(t,x+\pi z,\lambda) - \phi(t,x,\lambda)]\nu(dz),$$

and

$$\hat{\pi} = \arg\inf_{\pi\in R}\Big\{(\widetilde{W}_x + \mu_1 G^*)\Big(\mu - r - \int_{z>-1} z\nu(dz)\Big)\pi + \frac{1}{2}\sigma^2\pi^2[2\mu_1 G_x^* + (G_x^*)^2 + \widetilde{W}_{xx}]$$
$$+ \int_{z>-1}\mu_1\pi z G^*(t,x+\pi z,\lambda)\nu(dz)$$
$$+ \frac{1}{2}\int_{z>-1}[G^*(t,x+\pi z,\lambda) - G^*(t,x,\lambda)]^2\nu(dz)$$
$$+ \int_{z>-1}[\widetilde{W}(t,x+\pi z,\lambda) - \widetilde{W}(t,x,\lambda)]\nu(dz)\Big\}, \tag{4.6}$$

$$\widehat{\mathrm{SC}} = \arg\inf_{SC\in R}\{\mathrm{SC}^2 + (\widetilde{W}_x + \mu_1 G^*)\mathrm{SC}\}. \tag{4.7}$$

Then $V_1(t,x,\lambda) = \widetilde{W}(t,x,\lambda)$, $G^*(t,x,\lambda) = \mathbb{E}_{t,x,\lambda}[X^{\hat{\Pi}}(T) - Da(\lambda(T))]$ *and* $H^*(t,x,\lambda) = \mathbb{E}_{t,x,\lambda}[(X^{\hat{\Pi}}(T) - Da(\lambda(T)))^2]$, *and the equilibrium control law is given by* $\hat{\Pi}$.

Proof. The proof is similar to that of Theorem 3.2. □

4.1. Solution to the generalized optimization problem In this subsection, we try to find the time-consistent strategies and the value function for the generalized problem (4.1). Similarly to Section 3, we guess that

$$\widetilde{W}(t,x,\lambda) = \widetilde{A}(t)x^2 + \widetilde{B}(t,\lambda)x + \widetilde{C}(t,\lambda) \tag{4.8}$$

with $\widetilde{A}(T) = -\mu_1, \widetilde{B}(T,\lambda) = -\mu_2 + \mu_1 Da(\lambda), \widetilde{C}(T,\lambda) = \mu_2 Da(\lambda)$;

$$G^*(t,x,\lambda) = \tilde{a}(t)x + \tilde{b}(t,\lambda)$$

with $\tilde{a}(T) = 1, \tilde{b}(T,\lambda) = -Da(\lambda)$; and $H^*(t,x,\lambda) = \tilde{q}(t)x^2 + \tilde{l}(t,\lambda)x + \tilde{d}(t,\lambda)$ with $\tilde{q}(T) = 1, \tilde{l}(T,\lambda) = -2Da(\lambda), \tilde{d}(T,\lambda) = (Da(\lambda))^2$.

Substituting $\widetilde{W}(t,x,\lambda), G^*(t,x,\lambda)$ and their corresponding derivatives into (4.6) and (4.7), we obtain the values of the equilibrium control law $\hat{\pi}$ and SC as follows:

$$\hat{\pi}(t,x,\lambda) = -\bar{\beta}\frac{(2\widetilde{A}(t) + \mu_1\tilde{a}(t))x + \widetilde{B}(t,\lambda) + \mu_1\tilde{b}(t,\lambda)}{2\widetilde{A}(t) + 2\mu_1\tilde{a}(t) + \tilde{a}^2(t)},$$
$$\widehat{\mathrm{SC}}(t,x,\lambda) = -\frac{(2\widetilde{A}(t) + \mu_1\tilde{a}(t))x + \widetilde{B}(t,\lambda) + \mu_1\tilde{b}(t,\lambda)}{2}.$$

Instead of analysing the above equalities directly, we make the following assumptions:

$$\hat{\pi}(t,x,\lambda) = k_1(t)x + k_2(t,\lambda), \tag{4.9}$$
$$\widehat{\mathrm{SC}}(t,x,\lambda) = c_1(t)x + c_2(t,\lambda), \tag{4.10}$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

where

$$k_1(t) = -\bar{\beta}\frac{2\widetilde{A}(t) + \mu_1\tilde{a}(t)}{2\widetilde{A}(t) + 2\mu_1\tilde{a}(t) + \tilde{a}^2(t)}, \tag{4.11}$$

$$k_2(t,\lambda) = -\bar{\beta}\frac{\widetilde{B}(t,\lambda) + \mu_1\tilde{b}(t,\lambda)}{2\widetilde{A}(t) + 2\mu_1\tilde{a}(t) + \tilde{a}^2(t)}, \tag{4.12}$$

$$c_1(t) = -\frac{2\widetilde{A}(t) + \mu_1\tilde{a}(t)}{2}, \tag{4.13}$$

$$c_2(t,\lambda) = -\frac{\widetilde{B}(t,\lambda) + \mu_1\tilde{b}(t,\lambda)}{2}, \tag{4.14}$$

with $k_1(T) = \mu_1\bar{\beta}, k_2(T,\lambda) = \mu_2\bar{\beta}$, $c_1(T) = \mu_1/2$ and $c_2(T,\lambda) = \mu_2/2$.

Then, by substitution of (4.8)–(4.10) into (4.3), we obtain

$$\begin{aligned}
&\widetilde{A}_t x^2 + \widetilde{B}_t x + \widetilde{C}_t + [(2\widetilde{A}(t) + \mu_1\tilde{a}(t))x + \widetilde{B}(t,\lambda) + \mu_1\tilde{b}(t,\lambda)][rx + \mathrm{NC}(t,\lambda)] \\
&\quad + (\widetilde{B}_\lambda x + \widetilde{C}_\lambda)\theta(t,\lambda) + \tfrac{1}{2}\eta^2(t,\lambda)(\widetilde{B}_{\lambda\lambda}x + \widetilde{C}_{\lambda\lambda} + \tilde{b}_\lambda^2) \\
&\quad + \tfrac{1}{2}(\mu - r)[(2\widetilde{A}(t) + \mu_1\tilde{a}(t))k_1(t)x^2 \\
&\quad + (\widetilde{B}(t,\lambda) + \mu_1\tilde{b}(t,\lambda))(2k_1(t)x + k_2(t,\lambda))] - [c_1(t)x + c_2(t,\lambda)]^2 = 0.
\end{aligned}$$

Equating coefficients of x^2, x and constant terms to zero, we get

$$\widetilde{A}_t - 2c_1(t)r - (\mu - r)k_1(t)c_1(t) - c_1^2(t) = 0, \tag{4.15}$$

$$\begin{aligned}
&\widetilde{B}_t + \widetilde{B}_\lambda\theta(t,\lambda) + \tfrac{1}{2}\widetilde{B}_{\lambda\lambda}\eta^2(t,\lambda) - 2c_1(t)\mathrm{NC}(t,\lambda) \\
&\quad - 2[r + (\mu - r)k_1(t) + c_1(t)]c_2(t,\lambda) = 0,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&\widetilde{C}_t + \widetilde{C}_\lambda\theta(t,\lambda) + \tfrac{1}{2}(\widetilde{C}_{\lambda\lambda} + \tilde{b}_\lambda^2)\eta^2(t,\lambda) \\
&\quad - [2\mathrm{NC}(t,\lambda) + (\mu - r)k_2(t,\lambda)]c_2(t,\lambda) - c_2^2(t,\lambda) = 0.
\end{aligned} \tag{4.17}$$

Similarly, we plug $G^*(t,x,\lambda)$, (4.9) and (4.10) into (4.4), thereby obtaining

$$\begin{aligned}
&\tilde{a}_t x + \tilde{b}_t + \tilde{a}(t)[rx + \mathrm{NC}(t,\lambda) + c_1(t)x + c_2(t,\lambda) + (\mu - r)(k_1(t)x + k_2(t,\lambda))] \\
&\quad + \tilde{b}_\lambda\theta(t,\lambda) + \tfrac{1}{2}\tilde{b}_{\lambda\lambda}\eta^2(t,\lambda) = 0.
\end{aligned}$$

To ensure the above equality, we require

$$\tilde{a}_t + \tilde{a}(t)r + \tilde{a}(t)(\mu - r)k_1(t) + \tilde{a}(t)c_1(t) = 0, \tag{4.18}$$

$$\begin{aligned}
&\tilde{b}_t + \tilde{a}(t)[(\mu - r)k_2(t,\lambda) + \mathrm{NC}(t,\lambda) + c_2(t,\lambda)] + \tilde{b}_\lambda\theta(t,\lambda) \\
&\quad + \tfrac{1}{2}\tilde{b}_{\lambda\lambda}\eta^2(t,\lambda) = 0.
\end{aligned} \tag{4.19}$$

From (4.18) and the boundary condition $\tilde{a}(T) = 1$, we obtain

$$\tilde{a}(t) = \exp\bigg(\int_t^T \{r + (\mu - r)k_1(u) + c_1(u)\}\,du\bigg). \tag{4.20}$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

Then, inserting $H^*(t,x,\lambda)$, (4.9) and (4.10) into (4.5) yields

$$
\begin{aligned}
&\tilde{q}_t x^2 + \tilde{l}_t x + \tilde{d}_t + (2\tilde{q}(t)x + \tilde{l}(t,\lambda))[rx + \mathrm{NC}(t,\lambda) \\
&\quad + c_1(t)x + c_2(t,\lambda) + (\mu - r)(k_1(t)x + k_2(t,\lambda))] + (\tilde{l}_\lambda x + \tilde{d}_\lambda)\theta(t,\lambda) \\
&\quad + \beta\tilde{q}(t)(k_1(t)x + k_2(t,\lambda))^2 + \tfrac{1}{2}\eta^2(t,\lambda)(\tilde{l}_{\lambda\lambda}x + \tilde{d}_{\lambda\lambda}) = 0.
\end{aligned}
$$

Collecting the coefficients of x^2, we see that

$$
\tilde{q}_t + 2\tilde{q}(t)[r + (\mu - r)k_1(t) + c_1(t)] + \beta\tilde{q}(t)k_1^2(t) = 0
$$

with the boundary condition $\tilde{q}(T) = 1$. Hence, $\tilde{q}(t) = \exp(\int_t^T \psi(u)\,du)$, where

$$
\psi(u) = 2r + 2(\mu - r)k_1(u) + 2c_1(u) + \beta k_1^2(u).
$$

On the other hand, the dynamics of the fund process $X^{\hat{\pi},\widehat{\mathrm{SC}}}$ under the optimal strategy satisfies the SDE

$$
\begin{aligned}
dX_t^{\hat{\pi},\widehat{\mathrm{SC}}} &= [rX_t^{\hat{\pi},\widehat{\mathrm{SC}}} + (\mu - r)(k_1(t)X_t^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(t,\lambda)) + \mathrm{NC}(t,\lambda) + c_1(t)X_t^{\hat{\pi},\widehat{\mathrm{SC}}} + c_2(t,\lambda)]\,dt \\
&\quad + \sigma[k_1(t)X_t^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(t,\lambda)]\,dW_t + [k_1(t)X_{t-}^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(t,\lambda)]\int_{z>-1} z\widetilde{M}(dt,dz).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d(X_t^{\hat{\pi},\widehat{\mathrm{SC}}})^2 &= 2[r + (\mu - r)k_1(t) + c_1(t)](X^{\hat{\pi},\widehat{\mathrm{SC}}_t})^{2dt + 2X_t^{\hat{\pi},\widehat{\mathrm{SC}}}[(\mu-r)k_2(t,\lambda)+\mathrm{NC}(t,\lambda)} \\
&\quad + c_2(t,\lambda)]\,dt + \beta[k_1(t)X_t^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(t,\lambda)]^2 dt + 2\sigma X_t^{\hat{\pi},\widehat{\mathrm{SC}}}[k_1(t)X_t^{\hat{\pi},\widehat{\mathrm{SC}}} \\
&\quad + k_2(t,\lambda)]\,dW_t + 2X_{t-}^{\hat{\pi},\widehat{\mathrm{SC}}}[k_1(t)X_{t-}^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(t,\lambda)]\int_{z>-1} z\widetilde{M}(dt,dz) \\
&\quad + [k_1(t)X_{t-}^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(t,\lambda)]^2 \int_{z>-1} z^2\widetilde{M}(dt,dz).
\end{aligned}
$$

Hence, by rearranging the terms we obtain

$$
\begin{aligned}
(X_s^{\hat{\pi},\widehat{\mathrm{SC}}})^2 &= \exp\Big(\int_t^s \psi(u)\,du\Big)(X_t^{\hat{\pi},\widehat{\mathrm{SC}}})^2 + 2\int_t^s e^{\int_u^s \psi(v)\,dv} X_u^{\hat{\pi},\widehat{\mathrm{SC}}}[(\mu - r)k_2(u,\lambda) + \mathrm{NC}(u,\lambda) \\
&\quad + c_2(u,\lambda) + \beta k_1(u)k_2(u,\lambda)]\,du + \int_t^s e^{\int_u^s \psi(v)\,dv}\beta k_2^2(u,\lambda)\,du \\
&\quad + 2\int_t^s e^{\int_u^s \psi(v)\,dv} X_u^{\hat{\pi},\widehat{\mathrm{SC}}}\sigma[k_1(u)X_u^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(u,\lambda)]\,dW_u \\
&\quad + 2\int_t^s\int_{z>-1} e^{\int_u^s \psi(v)\,dv} X_{u-}^{\hat{\pi},\widehat{\mathrm{SC}}}[k_1(u)X_{u-}^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(u,\lambda)]z\widetilde{M}(du,dz) \\
&\quad + \int_t^s\int_{z>-1} e^{\int_u^s \psi(v)\,dv}[k_1(u)X_{u-}^{\hat{\pi},\widehat{\mathrm{SC}}} + k_2(u,\lambda)]^2 z^2\widetilde{M}(du,dz)
\end{aligned}
$$

for all $s \geq t$. Following the similar derivation of Lemma 4.2 by Delong et al. [4], we conclude that the above three local martingales are in fact martingales. Therefore,

$$
\begin{aligned}
\mathbb{E}_{t,x,\lambda}[(X_s^{\hat{\pi},\widehat{\mathrm{SC}}})^2] &= e^{\int_t^s \psi(u)\,du}x^2 + 2\mathbb{E}_{t,x,\lambda}\Big\{\int_t^s e^{\int_u^s \psi(v)\,dv} X_u^{\hat{\pi},\widehat{\mathrm{SC}}}[(\mu - r)k_2(u,\lambda) + \mathrm{NC}(u,\lambda) \\
&\quad + c_2(u,\lambda) + \beta k_1(u)k_2(u,\lambda)]\,du\Big\} + \mathbb{E}_{t,x,\lambda}\Big\{\int_t^s e^{\int_u^s \psi(v)\,dv}\beta k_2^2(u,\lambda)\,du\Big\}.
\end{aligned}
$$

Moreover, from

$$\begin{aligned}\widetilde{W}(t,x,\lambda) &= \mathbb{E}_{t,x,\lambda}\Big[\int_t^T \widehat{\mathrm{SC}}^2(s)\,ds\Big] + \frac{1}{2}(H^*(t,x,\lambda) - G^{*2}(t,x,\lambda)) - (\mu_1 x + \mu_2)G^*(t,x,\lambda)\\ &= \mathbb{E}_{t,x,\lambda}\Big[\int_t^T (c_1(s)X_s^{\hat{\pi},\widehat{\mathrm{SC}}} + c_2(s,\lambda_s))^2\,ds\Big] + \frac{1}{2}(\tilde{q}(t)x^2 + \tilde{l}(t,\lambda)x + \tilde{d}(t,\lambda)\\ &\quad - \tilde{a}^2(t)x^2 - 2\tilde{a}(t)\tilde{b}(t,\lambda)x - \tilde{b}^2(t,\lambda)) - (\mu_1 x + \mu_2)(\tilde{a}(t)x + \tilde{b}(t,\lambda))\end{aligned}$$

and equation (4.8), by comparing the coefficients of x^2, we get

$$2\widetilde{A}(t) + 2\mu_1\tilde{a}(t) + \tilde{a}^2(t) = \exp\Big(\int_t^T \psi(u)\,du\Big) + 2\int_t^T c_1^2(s)\exp\Big(\int_t^s \psi(u)\,du\Big)\,ds. \quad (4.21)$$

Hence, combining (4.11), (4.13), (4.20) and (4.21), we obtain the following integral equation system:

$$\begin{aligned}k_1(t) &= -\bar{\beta}\Big[1 - \Big(\Big\{\exp\Big(\int_t^T 2\{r + (\mu - r)k_1(u) + c_1(u)\}\,du\Big)\\ &\qquad + \mu_1\exp\Big(\int_t^T \{r + (\mu - r)k_1(u) + c_1(u)\}\,du\Big)\Big\}\\ &\qquad \times\Big\{\exp\Big(\int_t^T \psi(u)\,du\Big) + 2\int_t^T c_1^2(s)\exp\Big(\int_t^s \psi(u)\,du\Big)\,ds\Big\}^{-1}\Big)\Big], \quad (4.22)\end{aligned}$$

$$c_1(t) = \frac{k_1(t)}{2\bar{\beta}}\Big[\exp\Big(\int_t^T \psi(u)\,du\Big) + 2\int_t^T c_1^2(s)\exp\Big(\int_t^s \psi(u)\,du\Big)\,ds\Big]. \quad (4.23)$$

Next, we will calculate the values of $k_2(t,\lambda)$ and $c_2(t,\lambda)$ provided that $k_1(t)$ and $c_1(t)$ are known. Equations (4.11)–(4.14) imply

$$k_2(t,\lambda) = c_1^{-1}(t)k_1(t)c_2(t,\lambda) \quad (4.24)$$

and $\widetilde{B}(t,\lambda) = -2c_2(t,\lambda) - \mu_1\tilde{b}(t,\lambda)$. The partial derivatives corresponding to $\widetilde{B}(t,\lambda)$ are

$$\begin{aligned}\widetilde{B}_t &= -2\frac{\partial c_2}{\partial t}(t,\lambda) - \mu_1\tilde{b}_t,\\ \widetilde{B}_\lambda &= -2\frac{\partial c_2}{\partial \lambda}(t,\lambda) - \mu_1\tilde{b}_\lambda,\\ \widetilde{B}_{\lambda\lambda} &= -2\frac{\partial^2 c_2}{\partial \lambda^2}(t,\lambda) - \mu_1\tilde{b}_{\lambda\lambda}.\end{aligned}$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

Substituting these into (4.16) and using (4.19), we get

$$\frac{\partial c_2}{\partial t}(t,\lambda) + \theta(t,\lambda)\frac{\partial c_2}{\partial \lambda}(t,\lambda) + \frac{1}{2}\eta^2(t,\lambda)\frac{\partial^2 c_2}{\partial \lambda^2}(t,\lambda)$$
$$+ \Big\{c_1(t) - \frac{\mu_1}{2}\tilde{a}(t)\Big\}\mathrm{NC}(t,\lambda) + \alpha(t)c_2(t,\lambda) = 0,$$

where $\alpha(t) = r + (\mu - r)k_1(t) + c_1(t) - \mu_1\tilde{a}(t)/2 - \{\mu_1(\mu - r)\tilde{a}(t)c_1^{-1}(t)k_1(t)\}/2$. Hence, by the Feynman–Kac formula and assumption (2.4), we arrive at

$$\begin{aligned} c_2(t,\lambda) = {} & \frac{\mu_2}{2}\exp\Big(\int_t^T \alpha(u)\,du\Big) \\ & + \int_t^T \Big(c_1(s) - \frac{\mu_1}{2}\tilde{a}(s)\Big)\exp\Big(\int_t^s \alpha(u)\,du\Big) \\ & - \rho(T-s)m(s)\,ds\mathbb{E}_{t,\lambda}[Da(\lambda(T))]. \end{aligned} \tag{4.25}$$

We summarize the above results in the following theorem.

Theorem 4.3. *For the generalized optimization problem* (4.1)*, the equilibrium strategies are given by*

$$\hat{\pi}(t,x,\lambda) = k_1(t)x + k_2(t,\lambda),$$
$$\widehat{\mathrm{SC}}(t,x,\lambda) = c_1(t)x + c_2(t,\lambda),$$

where $k_1(t)$ and $c_1(t)$ satisfy the integral equation system (4.22) *and* (4.23)*, and the values of $k_2(t,\lambda)$ and $c_2(t,\lambda)$ are given by* (4.24) *and* (4.25)*. Moreover, the corresponding equilibrium value function is given by $V_1(t,x,\lambda) = \widetilde{A}(t)x^2 + \widetilde{B}(t,\lambda)x + \widetilde{C}(t,\lambda)$, where the functions $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are given by the expressions in* (4.15)–(4.17)*, respectively.*

4.2. Special case We now investigate the case of the generalized optimization problem (4.1) with constant risk aversion. By setting $\mu_1 = 0$, we get $k_1(t) = 0$ and $c_1(t) = 0$ as the roots of the integral equation system derived in Theorem 4.3. Moreover, from (4.14)–(4.16), (4.18) and their corresponding boundary conditions, we obtain

$$\widetilde{A}(t) = 0, \quad \widetilde{B}(t,\lambda) = -\mu_2 e^{r(T-t)}, \quad \tilde{a}(t) = e^{r(T-t)}.$$

Furthermore, equations (4.12) and (4.14) respectively simplify to

$$c_2(t) = \frac{\mu_2}{2}e^{r(T-t)}, \quad k_2(t) = \mu_2\bar{\beta}e^{-r(T-t)},$$

so the optimal time-consistent strategies are given by

$$\hat{\pi}(t) = \mu_2\bar{\beta}e^{-r(T-t)}, \quad \widehat{\mathrm{SC}}(t) = \frac{\mu_2}{2}e^{r(T-t)},$$

which are independent of the current wealth x and the current mortality intensity λ.

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

Substituting the values of $\tilde{a}(t), c_2(t)$ and $k_2(t)$ into (4.17) and (4.19), and by applying the Feynman–Kac formula, we get

$$\tilde{b}(t,\lambda) = \mathbb{E}_{t,\lambda}[Da(\lambda(T))]\Big\{\int_t^T e^{(r-\rho)(T-s)}m(s)\,ds - 1\Big\} + \mu_2(\mu - r)(T-t)\bar{\beta} + \frac{\mu_2}{4r}(e^{2r(T-t)} - 1)$$

and

$$\widetilde{C}(t,\lambda) = \frac{1}{2}\mathbb{E}_{t,\lambda}\Big[\int_t^T \tilde{b}_\lambda^2(s,\lambda)\eta^2(s,\lambda)\,ds\Big] - \mu_2\tilde{b}(t,\lambda) + \frac{\mu_2^2(\mu - r)(T-t)\bar{\beta}}{2} + \frac{\mu_2^2}{8r}(e^{2r(T-t)} - 1).$$

Hence,

$$V_1(t,x,\lambda) = -\mu_2 e^{r(T-t)}x + \widetilde{C}(t,\lambda),$$
$$G^*(t,x,\lambda) = e^{r(T-t)}x + \tilde{b}(t,\lambda),$$

and the variance is given by

$$\mathbb{V}ar_{t,x,\lambda}[X^{\hat{\pi}}(T) - Da(\lambda(T))] = 2\Big[V_1(t,x,\lambda) + \mu_2 G^*(t,x,\lambda) - \int_t^T \widehat{\mathrm{SC}}^2(s)\,ds\Big] = \mathbb{E}_{t,\lambda}\Big[\int_t^T \tilde{b}_\lambda^2(s,\lambda)\eta^2(s,\lambda)\,ds\Big] + \mu_2^2(\mu - r)(T-t)\bar{\beta}.$$

5. Numerical results

In this section, we provide numerical examples to illustrate our results. We consider the cohort of workers who join the plan at the age of 45 and retire at the age of 65, so that $t = 0, T = 20$. The maximum future lifetime is taken to be 100 years, and $T' = 55$. We assume that the mortality intensity follows the Vasicek model

$$d\lambda(t) = 0.078\,282\lambda(t)\,dt + 0.001\,606\,d\overline{W}(t),$$

which has been considered by Jalen and Mamon [8] and Qian et al. [16]. The financial market parameters are $r = 0.05, \mu = 0.1, \sigma = 0.2, D = 1000, \rho = 0.08, m(t) = 1/20$, $\kappa = 0$ and $\nu = 0$.

Figures 1–5 show the graphical results of the optimal investment strategy in Theorem 3.4. In Figure 1 we observe that $k_1(t)$ is increasing with t, and for a fixed t, the value of $k_1(t)$ when $\mu_1 = 0.6$ is larger than that when $\mu_1 = 0.3$. We plot the function $k_2(t,\lambda)$ for $\mu_1 = 0.3$ and $\mu_2 = 0.5$ in Figure 2. The changes in $k_2(t,\lambda)$ with respect to t and λ are illustrated in Figures 3 and 4, respectively. In Figure 5 we show the relationship between the optimal investment amount and the wealth level, when $t = 0$ and $\lambda = 0.001\,217$.

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

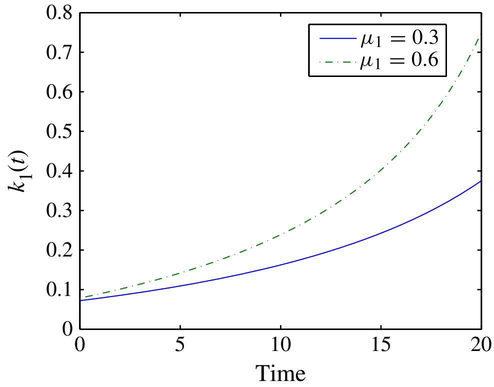

FIGURE 1. $k_1(t)$ for $\mu_1 = 0.3, 0.6$.

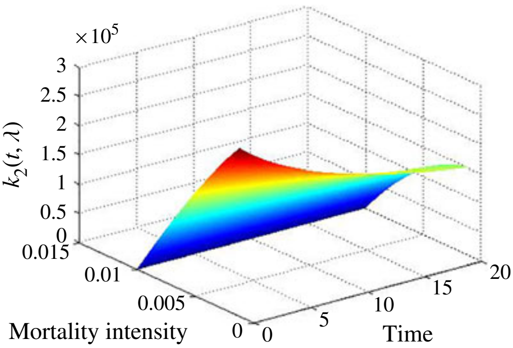

FIGURE 2. $k_2(t, \lambda)$ for $\mu_1 = 0.3, \mu_2 = 0.5$.

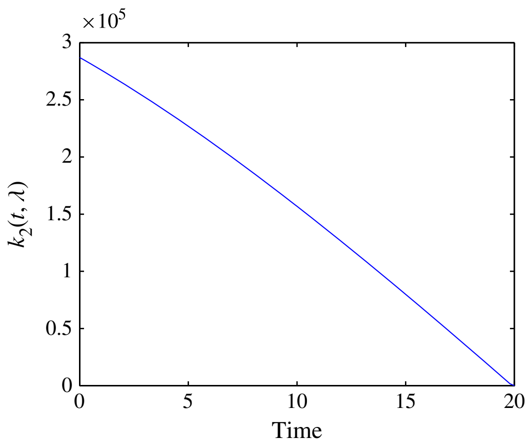

FIGURE 3. Variation of $k_2(t, \lambda)$ with t for $\lambda = 0.001\,217$.

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

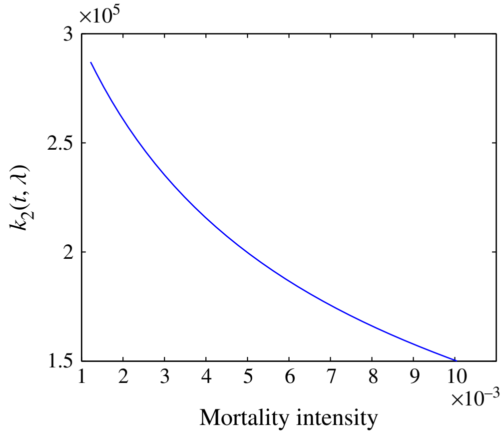

FIGURE 4. Variation of $k_2(t, \lambda)$ with λ for $t = 0$.

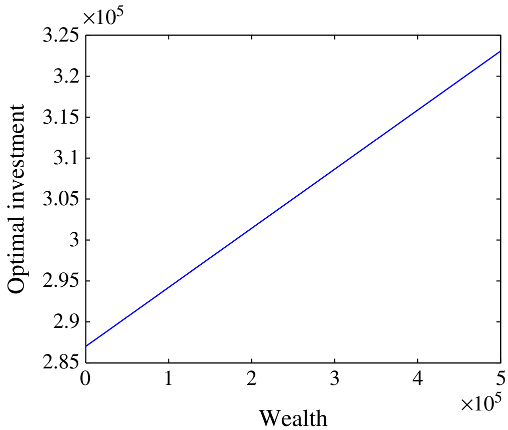

FIGURE 5. Relationship between the optimal investment amount and wealth X_t for $t = 0$, $\lambda = 0.001\,217$.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (11171164 and 11001136).

Appendix

PROOF OF THEOREM 3.2. (i) We first show that $W(t, x, \lambda) = J(t, x, \lambda, \hat{\pi})$.

In light of (3.4), we see that

$$\mathcal{A}^{\hat{\pi}} W(t, x, \lambda) - \mathcal{A}^{\hat{\pi}} f(t, x, \lambda, G, H) + f_y(t, x, \lambda, G, H)\mathcal{A}^{\hat{\pi}} G(t, x, \lambda) \\ + f_z(t, x, \lambda, G, H)\mathcal{A}^{\hat{\pi}} H(t, x, \lambda) = 0.$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

Since G and H satisfy (3.5) and (3.6), the above equation can be rewritten as

$$\mathcal{A}^{\hat{\pi}} W(t,x,\lambda) - \mathcal{A}^{\hat{\pi}} f(t,x,\lambda,G,H) = 0.$$

Hence, from the boundary condition and Dynkin's theorem,

$$W(t,x,\lambda) = f(t,x,\lambda,G,H). \tag{A.1}$$

Again, from (3.5) and (3.6) and the boundary conditions, we know that $G(t,x,\lambda)$ and $H(t,x,\lambda)$ are martingales, and consequently

$$G(t,x,\lambda) = \mathbb{E}_{t,x,\lambda}[X^{\hat{\pi}}(T) - Da(\lambda(T))],$$
$$H(t,x,\lambda) = \mathbb{E}_{t,x,\lambda}[(X^{\hat{\pi}}(T) - Da(\lambda(T)))^2].$$

Inserting the above results into (A.1), we obtain

$$W(t,x,\lambda) = J(t,x,\lambda,\hat{\pi}). \tag{A.2}$$

(ii) We now show that $\hat{\pi}$ is indeed an equilibrium control law. For any $h > 0$ and an arbitrary $\pi \in \mathcal{A}$, we construct a control law π_h as in Definition 3.1. From the fact that $J(t,x,\lambda,\pi_h) = f(t,x,\lambda,y^{\pi_h}(t,x,\lambda),z^{\pi_h}(t,x,\lambda))$,

$$\begin{aligned} &J(t,x,\lambda,\pi_h) \\ &\quad= \mathbb{E}_{t,x,\lambda}[J(t+h,X^{\pi_h}_{t+h},\lambda_{t+h},\pi_h)] \\ &\qquad - \mathbb{E}_{t,x,\lambda}[f(t+h,X^{\pi_h}_{t+h},\lambda_{t+h},y^{\pi_h}(t+h,X^{\pi_h}_{t+h},\lambda_{t+h}),z^{\pi_h}(t+h,X^{\pi_h}_{t+h},\lambda_{t+h}))] \\ &\qquad + f(t,x,\lambda,\mathbb{E}_{t,x,\lambda}[y^{\pi_h}(t+h,X^{\pi_h}_{t+h},\lambda_{t+h})],\mathbb{E}_{t,x,\lambda}[z^{\pi_h}(t+h,X^{\pi_h}_{t+h},\lambda_{t+h})]). \end{aligned}$$

With the help of the definition of π_h, the above equation can be rewritten as

$$\begin{aligned} &J(t,x,\lambda,\pi_h) \\ &\quad= \mathbb{E}_{t,x,\lambda}[W(t+h,X^{\pi}_{t+h},\lambda_{t+h})] \\ &\qquad - \mathbb{E}_{t,x,\lambda}[f(t+h,X^{\pi}_{t+h},\lambda_{t+h},G(t+h,X^{\pi}_{t+h},\lambda_{t+h}),H(t+h,X^{\pi}_{t+h},\lambda_{t+h}))] \\ &\qquad + f(t,x,\lambda,\mathbb{E}_{t,x,\lambda}[G(t+h,X^{\pi}_{t+h},\lambda_{t+h})],\mathbb{E}_{t,x,\lambda}[H(t+h,X^{\pi}_{t+h},\lambda_{t+h})]). \end{aligned} \tag{A.3}$$

In addition, according to the extended HJB system, we have

$$\begin{aligned} &\mathcal{A}^{\pi} W(t,x,\lambda) - \mathcal{A}^{\pi} f(t,x,\lambda,G,H) + f_y(t,x,\lambda,G,H)\mathcal{A}^{\pi} G(t,x,\lambda) \\ &\quad + f_z(t,x,\lambda,G,H)\mathcal{A}^{\pi} H(t,x,\lambda) \geq 0, \quad \text{for all } \pi \in \mathcal{A}. \end{aligned}$$

Discretizing the above inequality, we obtain

$$\begin{aligned} &\mathbb{E}_{t,x,\lambda}[W(t+h,X^{\pi}_{t+h},\lambda_{t+h})] - W(t,x,\lambda) \\ &\quad - \{\mathbb{E}_{t,x,\lambda}[f(t+h,X^{\pi}_{t+h},\lambda_{t+h},G(t+h,X^{\pi}_{t+h},\lambda_{t+h}),H(t+h,X^{\pi}_{t+h},\lambda_{t+h}))] \\ &\quad - f(t,x,\lambda,G(t,x,\lambda),H(t,x,\lambda))\} \\ &\quad + \{f(t,x,\lambda,\mathbb{E}_{t,x,\lambda}[G(t+h,X^{\pi}_{t+h},\lambda_{t+h})],\mathbb{E}_{t,x,\lambda}[H(t+h,X^{\pi}_{t+h},\lambda_{t+h})]) \\ &\quad - f(t,x,\lambda,G(t,x,\lambda),H(t,x,\lambda))\} \geq o(h). \end{aligned}$$

https://doi.org/10.1017/S1446181114000212 Published online by Cambridge University Press

After simplification, it can be transformed to

$$\begin{aligned}W(t, x, \lambda)\\ &\le \mathbb{E}_{t,x,\lambda}[W(t+h, X^{\pi}_{t+h}, \lambda_{t+h})]\\ &\quad - \mathbb{E}_{t,x,\lambda}[f(t+h, X^{\pi}_{t+h}, \lambda_{t+h}, G(t+h, X^{\pi}_{t+h}, \lambda_{t+h}), H(t+h, X^{\pi}_{t+h}, \lambda_{t+h}))]\\ &\quad + f(t, x, \lambda, \mathbb{E}_{t,x,\lambda}[G(t+h, X^{\pi}_{t+h}, \lambda_{t+h})], \mathbb{E}_{t,x,\lambda}[H(t+h, X^{\pi}_{t+h}, \lambda_{t+h})]) + o(h).\end{aligned}$$

Combining the results of (A.2) and the expression for $J(t, x, \lambda, \pi_h)$ in (A.3), we obtain

$$J(t, x, \lambda, \hat{\pi}) \le J(t, x, \lambda, \pi_h) + o(h).$$

Hence,

$$\limsup_{h\to 0} \frac{J(t, x, \lambda, \hat{\pi}) - J(t, x, \lambda, \pi_h)}{h} \le 0.$$

This completes the proof of Theorem 3.2. □

References

[1] T. Björk and A. Murgoci, "A general theory of Markovian time inconsistent stochastic control problems", *SSRN eLibrary*, 2010; doi:10.2139/ssrn.1694759.

[2] T. Björk, A. Murgoci and X. Y. Zhou, "Mean-variance portfolio optimization with state-dependent risk aversion", *Math. Finance* **24** (2012) 1–24; doi:10.1111/j.1467-9965.2011.00515.x.

[3] A. J. Cairns, "Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time", *Astin Bull.* **30** (2000) 19–56; doi:10.2143/AST.30.1.504625.

[4] Ł. Delong, R. Gerrard and S. Haberman, "Mean–variance optimization problems for an accumulation phase in a defined benefit plan", *Insurance Math. Econom.* **42** (2008) 107–118; doi:10.1016/j.insmatheco.2007.01.005.

[5] L. He and Z. Liang, "Optimal investment strategy for the DC plan with the return of premiums clauses in a mean–variance framework", *Insurance Math. Econom.* **53** (2013) 643–649; doi:10.1016/j.insmatheco.2013.09.002.

[6] D. Heath and M. Schweizer, "Martingales versus PDEs in finance: an equivalence result with examples", *J. Appl. Probab.* **37** (2000) 947–957; doi:10.1239/jap/1014843075.

[7] Y. Hu, H. Jin and X. Y. Zhou, "Time-inconsistent stochastic linear–quadratic control", *SIAM J. Control Optim.* **50** (2012) 1548–1572; doi:10.1137/110853960.

[8] L. Jalen and R. Mamon., "Valuation of contingent claims with mortality and interest rate risks", *Math. Comput. Modelling* **49** (2009) 1893–1904; doi:10.1016/j.mcm.2008.10.014.

[9] R. Josa-Fombellida and J. P. Rincón-Zapatero, "Optimal risk management in defined benefit stochastic pension funds", *Insurance Math. Econom.* **34** (2004) 489–503; doi:10.1016/j.insmatheco.2004.03.002.

[10] R. Josa-Fombellida and J. P. Rincón-Zapatero, "Mean–variance portfolio and contribution selection in stochastic pension funding", *European J. Oper. Res.* **187** (2008) 120–137; doi:10.1016/j.ejor.2007.03.002.

[11] R. Josa-Fombellida and J. P. Rincón-Zapatero, "Stochastic pension funding when the benefit and the risky asset follow jump diffusion processes", *European J. Oper. Res.* **220** (2012) 404–413; doi:10.1016/j.ejor.2012.01.033.

[12] D. Li and W.-L. Ng, "Optimal dynamic portfolio selection: multiperiod mean–variance formulation", *Math. Finance* **10** (2000) 387–406; doi:10.1111/1467-9965.00100.

[13] Y. Li and Z. Li, "Optimal time-consistent investment and reinsurance strategies for mean–variance insurers with state dependent risk aversion", *Insurance Math. Econom.* **53** (2013) 86–97; doi:10.1016/j.insmatheco.2013.03.008.

[14] H. Markowitz, “Portfolio selection”, *J. Finance* **7** (1952) 77–91; doi:10.1111/j.1540-6261.1952.tb01525.x.

[15] B. Ngwira and R. Gerrard, “Stochastic pension fund control in the presence of Poisson jumps”, *Insurance Math. Econom.* **40** (2007) 283–292; doi:10.1016/j.insmatheco.2006.05.002.

[16] L. Qian, W. Wang, R. Wang and Y. Tang, “Valuation of equity-indexed annuity under stochastic mortality and interest rate”, *Insurance Math. Econom.* **47** (2010) 123–129; doi:10.1016/j.insmatheco.2010.06.005.

[17] R. H. Strotz, “Myopia and inconsistency in dynamic utility maximization”, *Rev. Econ. Stud.* **23** (1955) 165–180; doi:10.2307/2295722.

[18] Y. Zeng and Z. Li, “Optimal time-consistent investment and reinsurance policies for mean-variance insurers”, *Insurance Math. Econom.* **49** (2011) 145–154; doi:10.1016/j.insmatheco.2011.01.001.

[19] Y. Zeng, Z. Li and Y. Lai, “Time-consistent investment and reinsurance strategies for mean–variance insurers with jumps”, *Insurance Math. Econom.* **52** (2013) 498–507; doi:10.1016/j.insmatheco.2013.02.007.

[20] X. Y. Zhou and D. Li, “Continuous-time mean-variance portfolio selection: a stochastic LQ framework”, *Appl. Math. Optim.* **42** (2000) 19–33; doi:10.1007/s002450010003.