A NOTE ON KÖTHE SPACES

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1. Let E be a locally compact space which can be expressed as the union of an increasing sequence of compact subsets K_n $(n = 1, 2, \ldots)$ and let μ be a positive Radon measure on E. Ω is the space of equivalence classes of locally integrable functions on E. We denote the equivalence class of a function f by $\hat{f}$ and if $\hat{f}$ is an equivalence class then f denotes any function belonging to $\hat{f}$. Provided with the topology defined by the sequence of seminorms

$$N_n(\hat{f}) = \int_{K_n} |\hat{f}|\, d\mu \qquad (n = 1, 2, \ldots;\ \hat{f} \in \Omega),$$

Ω is a Fréchet space. The dual of Ω is the space Φ of equivalence classes of measurable, p.p. bounded functions vanishing outside a compact subset of E. For a subset Γ of Ω, the collection Λ of all $\hat{f} \in \Omega$, such that for each $\hat{g} \in \Gamma$ the product fg is integrable, is called a Köthe space and Γ is said to be the defining set of Λ. The Köthe space $\Lambda^{\times}$ which has Λ as a defining set is called the associated Köthe space of Λ. Λ and $\Lambda^{\times}$ are put into duality by the bilinear form

$$\langle \hat{f}, \hat{g} \rangle = \int fg\, d\mu \qquad (\hat{f} \in \Lambda, \hat{g} \in \Lambda^{\times}).$$

If g is a non-negative locally integrable function, L_g^1 denotes the Köthe space having $\hat{g}$ alone as its defining set and L_g^{∞} denotes the associated Köthe space of L_g^1.

2. In [2], the topological properties of the dual pair $\langle \Lambda, \Lambda^{\times} \rangle$ are obtained from those of the pair $\langle L_g^1, L_g^{\infty} \rangle$. However, [2] has a slight error in the following statements.

(i) $\hat{f}$ belongs to L_g^1 if and only if $f\chi_A$ is $g\,d\mu$-integrable and $(f\chi_B)^{\wedge} \in \Omega$;

(ii) $\hat{h}$ belongs to L_g^{∞} if and only if h/g is p.p. bounded on A with respect to $g\,d\mu$ and $(h\chi_B)^{\wedge} \in \Phi$.

Here $A = \{x \colon x \in E,\ g(x) > 0\}$, $B = E - A$ and χ_A, χ_B are the characteristic functions of A, B respectively.

In [7], we pointed out this error and observed that it invalidated some proofs in [2]. We then introduced the spaces $\mathfrak{L}_g^1$ and $\mathfrak{L}_g^{\infty}$. Although these spaces serve the purpose of showing that all results in [2] are valid if in the modified proofs we use the pair $\langle \mathfrak{L}_g^1, \mathfrak{L}_g^{\infty} \rangle$ instead of the pair $\langle L_g^1, L_g^{\infty} \rangle$, the construction of the spaces $\mathfrak{L}_g^1$ and $\mathfrak{L}_g^{\infty}$ is rather artificial because these spaces are not, strictly speaking, Köthe spaces. Recently, S. Goes and R. Welland [3] gave a " correct characterization " of L_g^1 and L_g^{∞} and showed that with it the proofs in [2] go through, essentially unchanged.

3. The purpose of this note is to give a basically different proof of the correct characterization of L_g^1 and L_g^{∞}. It seems to us that our proof is perhaps simpler than the proof given in [3]. Besides, in the course of the proof we obtain an additional property of the space L_g^1. From this property one can see quite clearly that the duality of the pair $\langle L_g^1, L_g^{\infty} \rangle$ is a combination of the duality of the pair $\langle \Omega, \Phi \rangle$ and the duality of the pair $\langle L^1(E, g\,d\mu), L^{\infty}(E, g\,d\mu) \rangle$.

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PROPOSITION 1. *Provided with the topology defined by the countable system of seminorms*

$$N_g(\hat{f}) = \int |f| g \, d\mu, \quad N_n(\hat{f}) = \int_{K_n} |f| \, d\mu \qquad (n = 1, 2, \ldots, \hat{f} \in L^1_g),$$

L^1_g *is a Fréchet space whose dual can be identified with* $\Phi + \hat{g}L^\infty(E, d\mu)$, *where* $\hat{g}L^\infty(E, d\mu) = \{(g\gamma)^\wedge : \hat{\gamma} \in L^\infty(E, d\mu)\}$.

The canonical bilinear form is

$$\langle \hat{f}, \hat{h} \rangle = \int f h \, d\mu \qquad (\hat{f} \in L^1_g, \hat{h} \in \Phi + \hat{g}L^\infty(E, d\mu)).$$

Proof. Let $(\hat{f}_n)$ be a Cauchy sequence in L^1_g. Then $(\hat{f}_n)$ is a Cauchy sequence of the Fréchet space Ω and hence has a limit $\hat{f}_0 \in \Omega$. By the diagonal process we can extract from (f_n) a subsequence, still denoted by (f_n), converging to f_0 p.p. with respect to $d\mu$ on each K_n. Thus (f_n) converges to f_0 p.p. with respect to $d\mu$ (and *a fortiori* with respect to $g\,d\mu$) on E. On the other hand, since $\int |f_m - f_n| g \, d\mu$ tends to 0 as m, n tend to ∞, and $L^1(E, g\,d\mu)$ is complete, there exists a function f_0' such that $\int |f_n - f_0'| g \, d\mu$ tends to 0 as n tends to ∞. Since we can extract from (f_n) a subsequence converging to f_0' p.p. with respect to $g\,d\mu$, it is not difficult to see that $f_0 = f_0'$ p.p. with respect to $g\,d\mu$. Therefore $\int |f_n - f_0| g \, d\mu$ approaches 0 as n tends to ∞. Thus $\hat{f}_0$ is the limit of the Cauchy sequence $(\hat{f}_n)$.

Let $\varphi(E)$ be the space of all continuous functions on E having compact support. When provided with the topology defined by the seminorm

$$N_g(f) = \int |f| g \, d\mu \qquad (f \in \varphi(E)),$$

the dual of $\varphi(E)$ is $[L^1(E, g\,d\mu)]'$ because $\varphi(E)$ is dense in $L^1(E, g\,d\mu)$. When provided with the topology defined by the sequence of seminorms

$$N_n(f) = \int_{K_n} |f| \, d\mu \qquad (n = 1, 2, \ldots; f \in \varphi(E)),$$

the dual of $\varphi(E)$ is Φ because $\varphi(E)$ is dense in Ω. Hence, when provided with the topology defined by N_g together with N_n $(n = 1, 2, \ldots)$, the dual of $\varphi(E)$ is $[L^1(E, g\,d\mu)]' + \Phi$, the last topology of $\varphi(E)$ being the projective limit of the first two topologies. (For details of the proof of this assertion, we refer the reader to [[5], p. 292], proof of Proposition (3).) It is easy to see that the dual of L^1_g is the same as that of $\varphi(E)$. Thus we only need to show that the dual $[L^1(E, g\,d\mu)]'$ of $L^1(E, g\,d\mu)$ can be identified with $\hat{g}L^\infty(E, d\mu)$. There are two ways of showing this.

First Method. The dual of $L^1(E, g\,d\mu)$ is $L^\infty(E, g\,d\mu)$. Since we want to take the canonical bilinear form as an integral with respect to $d\mu$ instead of $g\,d\mu$, we have to use $\hat{g}L^\infty(E, g\,d\mu)$ as dual. It is not difficult to see that a subset of $A = \{x : x \in E, g(x) > 0\}$ has measure 0 with respect to $g\,d\mu$ if and only if it has measure 0 with respect to $d\mu$. From this it follows that $\hat{g}L^\infty(E, g\,d\mu) = \hat{g}L^\infty(E, d\mu)$.

Second Method. We denote the equivalence class with respect to $g\,d\mu$ of a function h by $\tilde{h}$. It is clear that each element of $\hat{g}L^\infty(E, d\mu)$ generates a continuous linear form on $L^1(E, g\,d\mu)$. On the other hand, given any continuous linear form F on $L^1(E, g\,d\mu)$, there exists $\tilde{h}$ such that for each $\tilde{f} \in L^1(E, g\,d\mu)$ the product fh is $g\,d\mu$-integrable ([1], p. 85) and

$$F(\tilde{f}) = \int fhg\,d\mu = \int (gh)f\,d\mu \qquad (\tilde{f} \in L^1(E, g\,d\mu)).$$

Now, for any $\hat{k} \in L^1(E, d\mu)$, define

$$\begin{aligned} f(x) &= 0 \qquad (x \in B), \\ &= k(x)/g(x) \qquad (x \in A). \end{aligned}$$

Then $\tilde{f} \in L^1(E, g\,d\mu)$. Therefore fh is $g\,d\mu$-integrable, i.e. $kh\chi_A$ is $d\mu$-integrable. Since this is true for every $\hat{k} \in L^1(E, d\mu)$, $(h\chi_A)^\wedge \in L^\infty(E, d\mu)$ ([1], p. 85). Hence $(gh)^\wedge \in \hat{g}L^\infty(E, d\mu)$. We have thus proved that the dual of $L^1(E, g\,d\mu)$ can be identified with $\hat{g}L^\infty(E, d\mu)$.

PROPOSITION 2. *The associated Köthe space L_g^∞ of L_g^1 is equal to its dual $\Phi + \hat{g}L^\infty(E, d\mu)$.*

Proof. It is clear that $\Phi + \hat{g}L^\infty(E, d\mu) \subset L_g^\infty$. Conversely, let $\hat{h} \in L_g^\infty$. For each $n = 1, 2, \ldots$, define

$$\begin{aligned} h_n(x) &= h(x) \qquad (|h(x)| < n, x \in K_n), \\ &= n \qquad (|h(x)| \geqq n, x \in K_n), \\ &= 0 \qquad (x \notin K_n). \end{aligned}$$

Then $L_n : \hat{f} \to \int fh_n\,d\mu$ $(\hat{f} \in L_g^1)$ is a continuous linear form on L_g^1 because $|\int fh_n\,d\mu| \leqq nN_n(\hat{f})$. On the other hand, fh_n tends to fh everywhere as n tends to ∞, $|fh_n| \leqq |fh|$ and fh is $d\mu$-integrable. Therefore, by the Lebesgue dominated convergence theorem, $\int fh_n\,d\mu$ tends to $\int fh\,d\mu$ as n tends to ∞. Since L_g^1 is a Fréchet space, by the Banach–Steinhaus theorem (Cf. e.g. [[4], p. 69]), $L : \hat{f} \to \int fh\,d\mu$ is also a continuous linear form on L_g^1 and $\hat{h} \in \Phi + \hat{g}L^\infty(E, d\mu)$, Thus $L_g^\infty = \Phi + \hat{g}L^\infty(E, d\mu)$.

NOTE. The technique we use here is quite standard in the theory of Köthe spaces and this is the reason for our including the second method in the proof of Proposition 1. Before Köthe and Toeplitz introduced their perfect sequence spaces in 1934 [6], Banach ([1], p. 85) had used this technique to show that $[L^1(E, d\mu)]^\times = L^\infty(E, d\mu)$. (It is obvious that $(L^\infty)^\times = L^1$.)

The original version of the second method in the proof of Proposition 1 was criticized by the referee as confusing. I am grateful to the referee for this helpful criticism.

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