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RESEARCH ARTICLE

Root, flow and order polytopes with connections to toric geometry

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Abstract
In this paper, we study the class of polytopes which can be obtained by taking the convex hull of some subset of the points $\{e_i - e_j \mid i \neq j\} \cup \{\pm e_i\}$ in $\mathbb{R}^n$, where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. Such a polytope can be encoded by a quiver Q with vertices $V \subseteq \{v_1, \dots, v_n\} \cup \{\star\}$, where each edge $v_j \to v_i$ or $\star \to v_i$ or $v_i \to \star$ gives rise to the point $e_i - e_j$ or e_i or $-e_i$, respectively; we denote the corresponding polytope as $\mathrm{Root}(Q)$. These polytopes have been studied extensively under names such as *edge polytope* and *root polytope*. We show that if the quiver Q is strongly-connected, then the root polytope $\mathrm{Root}(Q)$ is *reflexive* and *terminal*; we moreover give a combinatorial description of the facets of $\mathrm{Root}(Q)$. We also show that if Q is planar, then $\mathrm{Root}(Q)$ is (integrally equivalent to) the polar dual of the *flow polytope* of the planar dual quiver $Q^\vee$. Finally, we consider the case that Q comes from the Hasse diagram of a finite ranked poset P and show in this case that $\mathrm{Root}(Q)$ is polar dual to (a translation of) a *marked order polytope*. We then go on to study the toric variety $Y(\mathcal{F}_Q)$ associated to the face fan $\mathcal{F}_Q$ of $\mathrm{Root}(Q)$. If Q comes from a ranked poset P, we give a combinatorial description of the Picard group of $Y(\mathcal{F}_Q)$, in terms of a new *canonical ranked extension* of P, and we show that $Y(\mathcal{F}_Q)$ is a small partial desingularisation of the Hibi projective toric variety $Y_{\mathcal{O}(P)}$ of the *order polytope* $\mathcal{O}(P)$. We show that $Y(\mathcal{F}_Q)$ has a small crepant toric resolution of singularities $Y(\widehat{\mathcal{F}}_Q)$ and, as a consequence that the Hibi toric variety $Y_{\mathcal{O}(P)}$ has a small resolution of singularities for any ranked poset P. These results have applications to mirror symmetry [61].

Contents

1. Introduction

We define a *root polytope* to be the convex hull of some subset of the points $\{e_i - e_j \mid i \neq j\} \cup \{\pm e_i\}$ in $\mathbb{R}^n$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Root polytopes and their variants have been studied extensively in the literature, under different names. In 2002, Ohsugi and Hibi [55] introduced the *edge polytope* $\mathcal{P}_G$ of a directed graph G on vertices $v_1, \ldots, v_n$, defined as the convex hull of the vertices $\{e_i - e_j \mid v_i \to v_j \text{ an edge in } G\}$; they studied which orientations of the complete graph give rise to a Gorenstein Ehrhart ring $A(\mathcal{P}_G)$. There was subsequent work studying the roots of Ehrhart polynomials of $\mathcal{P}_G$ [45] and when a directed graph yields a smooth Fano polytope $\mathcal{P}_G$ [33]. In 2009, Postnikov [59] studied the edge polytope of a directed graph G on vertices $v_1, \ldots, v_n$ in which edges can only connect $v_i \to v_j$ if $i < j$; he called it the *root polytope* because its vertices $e_i - e_j$ can be identified with positive roots in the type A_{n-1} root system. He studied the volume and triangulations of the root polytopes associated to bipartite graphs. There has been much subsequent work on these polytopes, connecting them to subdivision algebras [46], subword complexes [18] and R-systems [20], studying them in other types [4] and computing their faces [63] and h^*-vectors [41].

In this paper, we adopt the term 'root polytope' for the slightly broader class of polytopes whose vertices can be any subset of the points $\{e_i - e_j \mid i \neq j\} \cup \{\pm e_i\}$ in $\mathbb{R}^n$. Our motivation for this work is mirror symmetry, more specifically, the study of certain reflexive polytopes which come from the *starred quiver* encoding a Laurent polynomial superpotential for Schubert varieties [61]. However, the beautiful properties of these reflexive polytopes hold in a very general setting, so we decided to present this material in a self-contained paper, which can be read independently of [61]. Some of our results are closely related to results that have appeared before; we have done our best to include relevant citations where appropriate.

Definition 1.1. Let Q be a quiver with vertices $\mathcal{V}_\bullet \sqcup \mathcal{V}_\star$ (where $\mathcal{V}_\bullet = \{v_1, \ldots, v_n\}$ for $n \geq 1$ and $\mathcal{V}_\star = \{\star_1, \ldots, \star_\ell\}$ for $\ell \geq 0$ are called the *(normal) vertices* and *starred vertices*, respectively), and arrows $\mathrm{Arr}(Q) \subseteq (\mathcal{V}_\bullet \times \mathcal{V}_\bullet) \sqcup (\mathcal{V}_\bullet \times \mathcal{V}_\star) \sqcup (\mathcal{V}_\star \times \mathcal{V}_\bullet)$. We will always assume that the underlying graph of any quiver is connected and has no loops. If $\ell > 0$, we call Q a *starred quiver*.

If any construction of a quiver results in duplicate arrows we remove these. If it results in an arrow between two starred vertices, then we identify these vertices into one vertex and remove the arrow.

Definition 1.2. Let Q be a quiver or starred quiver as in Definition 1.1 with arrows $\mathrm{Arr}(Q)$ and vertices $\{v_1, \ldots, v_n\} \cup \{\star_1, \ldots, \star_\ell\}$. We associate a point $u_a \in \mathbb{R}^n$ to each arrow a as follows:

- if $a : v_i \to v_j$, $u_a := e_j - e_i$;
- if $a : \star_i \to v_j$, $u_a := e_j$; and
- if $a : v_i \to \star_j$, $u_a := -e_i$.

We then define the *root polytope* to be

$$\mathrm{Root}(Q) = \mathrm{Conv}\{u_a \mid a \in \mathrm{Arr}(Q)\} \subset \mathbb{R}^n,$$

the convex hull of all the points u_a.

See Figure 1 for examples.

Remark 1.3. If Q has more than one starred vertex, we will often identify all starred vertices, obtaining a related quiver $\overline{Q}$ with a unique starred vertex. Clearly, $\mathrm{Root}(Q) = \mathrm{Root}(\overline{Q})$.

Definition 1.4. We say that a quiver Q is *strongly-connected* if there is an oriented path from any vertex to any other vertex. And we say that a starred quiver Q is *strongly-connected* if, after identifying all starred vertices, the resulting quiver $\overline{Q}$ is strongly-connected.

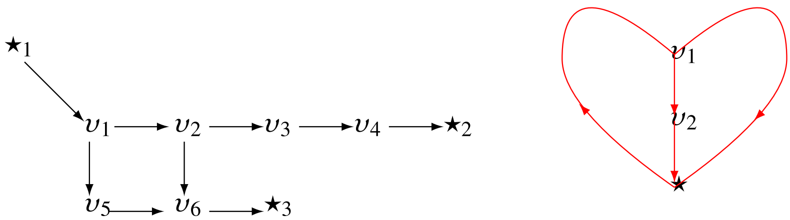

Figure 1. *Two starred quivers Q and Q'. The root polytope* $\mathrm{Root}(Q)$ *equals* $\mathrm{Conv}\{e_1, -e_4, -e_6, e_2 - e_1, e_5 - e_1, e_3 - e_2, e_6 - e_2, e_4 - e_3, e_6 - e_5\}$, *which has f-vector* $(9, 34, 70, 84, 57, 18)$. *Meanwhile,* $\mathrm{Root}(Q')$ *equals* $\mathrm{Conv}\{e_2 - e_1, -e_2, e_1, -e_1\}$, *and is a quadrilateral.*

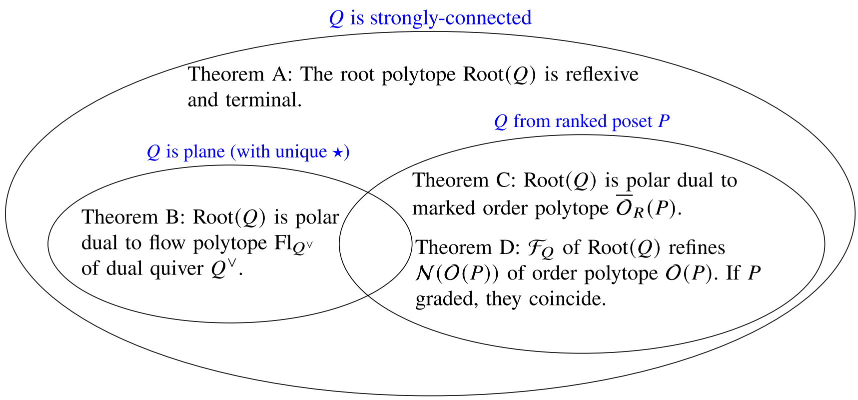

Figure 2. *Overview of our main results concerning root, flow and order polytopes: they appear as Theorem 2.31, Theorem 3.4, Theorem 4.10, Theorem 4.15.*

We now explain our main results. See Figure 2 for an overview.

Theorem A. *Let Q be a strongly-connected quiver or starred quiver. Then the root polytope* $\mathrm{Root}(Q)$ *is reflexive and terminal (cf. Definition 2.4 and Definition 2.5). That is, the polar dual of* $\mathrm{Root}(Q)$ *is a lattice polytope, and the only lattice points of* $\mathrm{Root}(Q) \cap \mathbb{Z}^n$ *are the origin and the vertices.*

Theorem A appears as Theorem 2.31 and Proposition 2.15. Note that Theorem 2.31 also includes an explicit description of the facets of $\mathrm{Root}(Q)$ in terms of *facet arrow-labelings* of the quiver. The statement that a strongly-connected quiver Q gives rise to a reflexive and terminal root polytope $\mathrm{Root}(Q)$ already appeared in [33, Proposition 1.4] without a proof; [33] asserts that the proof is the same as in the case of symmetric directed graphs, citing [45, Proposition 4.2] and [55, Lemma 1.2]. We provide an independent proof of Theorem A for completeness.

Our next main results relate special cases of root polytopes to *flow polytopes* [1] (see Definition 3.1) and to *(marked) order polytopes* [68, 5]. These are two classes of polytopes which have been extensively studied in toric geometry and combinatorics.

Theorem B. *Let Q be a strongly-connected starred quiver with vertices $\{v_1, \dots, v_n\} \cup \{\star\}$ which is planar (and comes with a given plane embedding), and let $Q^\vee$ be the dual quiver (which is plane, connected and acyclic) cf. Definition 3.2. Then the root polytope* $\mathrm{Root}(Q)$ *is integrally equivalent to the polar dual of the flow polytope* $\mathrm{Fl}_{Q^\vee}$.

Theorem B appears later as Theorem 3.4, though we have restated it using Lemma 3.3. Note that Theorem A and Theorem B imply that the flow polytope associated to any plane acyclic quiver is reflexive; this was previously shown more generally, without the planarity assumption, in [1].

We now consider the case that Q is a starred quiver that comes from a poset.

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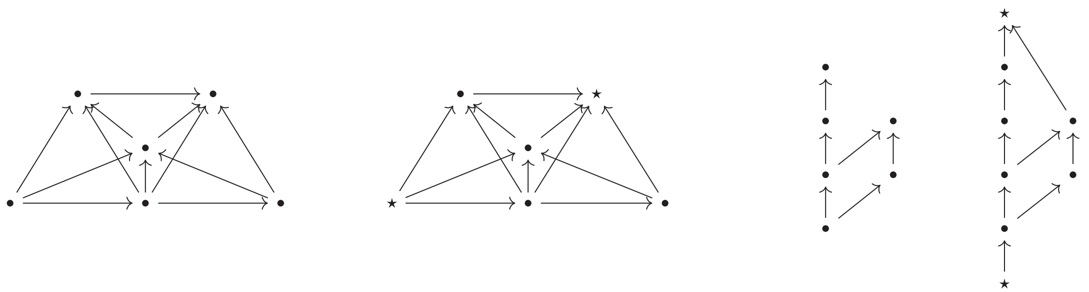

Figure 3. An acyclic quiver and the corresponding starred quiver; the Hasse diagram of a ranked poset and the starred quiver associated to its bounded extension. Note that the bounded extension is not ranked.

Definition 1.5. A *starred poset* is a finite poset P with a decomposition $P = P_\bullet \sqcup P_\star$ such that $P_\star$ contains the minimal and maximal elements of P, and no two elements of $P_\star$ are related by a covering relation. We associate a starred quiver $Q := Q_{(P_\bullet, P_\star)}$ to P by letting Q be the Hasse diagram of P, with all edges of the Hasse diagram directed upward, from smaller to larger, and the starred vertices of Q given by $P_\star$.

When we associate a starred quiver to a poset, we will always assume that poset is connected. Note that the *bounded extension* of an arbitrary poset, defined below, will always have these properties.

Definition 1.6. Let P be a finite poset. The *bounded extension* $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$ of P is the poset containing P where new elements $\hat{0}$ and $\hat{1}$ are adjoined such that $\hat{0}$ is the unique minimal and $\hat{1}$ the unique maximal element of $\hat{P}$. The associated starred quiver $Q_{\hat{P}}$ is constructed as in Definition 1.5 where $\hat{P} = P \sqcup P_\star$ with $P_\star = \{\hat{0}, \hat{1}\}$.

We note that the root polytopes associated to quivers $Q_{\hat{P}}$ were studied in [32] where it was proved that they are always terminal and reflexive – a special case of Theorem A above.

Theorem C. *Let $Q := Q_{(P_\bullet, P_\star)}$ be a starred quiver that comes from a starred poset $P = P_\bullet \sqcup P_\star$ as in Definition 1.5. Suppose that P is ranked with rank function R (cf. Definition 4.5). Then the root polytope* $\mathrm{Root}(Q)$ *of the starred quiver Q is polar dual to the (translated) marked order polytope $\overline{\mathcal{O}}_R(P)$ from Proposition 4.8. It follows that $\overline{\mathcal{O}}_R(P)$ is reflexive.*

Theorem C appears as Theorem 4.10. The fact that $\overline{\mathcal{O}}_R(P)$ is reflexive was known earlier, see [19, Proposition 3.1 and Theorem 3.4]. The related statement that the order polytope of a poset P is Gorenstein if and only if P is graded goes back to [31, 67].

The following result relates the root polytope to the (ordinary) order polytope and appears as Theorem 4.15.

Theorem D. *Let P be a finite ranked poset, and let $Q := Q_{\hat{P}}$ be the starred quiver associated to the bounded extension $\hat{P}$ of P (as in Definition 1.6). Then the face fan $\mathcal{F}_Q$ of the root polytope* $\mathrm{Root}(Q)$ *of the starred quiver Q refines the (inner) normal fan $\mathcal{N}(\mathcal{O}(P))$ of the order polytope $\mathcal{O}(P)$ of P: the two fans have the same set of rays, and each maximal cone of $\mathcal{N}(\mathcal{O}(P))$ is a union of maximal cones of $\mathcal{F}_{Q_P}$. And if P is a graded poset, then $\mathcal{F}_{Q_P}$ coincides with $\mathcal{N}(\mathcal{O}(P))$.*

Remark 1.7. Note that it is possible for P to be a ranked poset, but for $\hat{P}$ to fail to be a ranked poset, see Figure 3. Thus, the hypotheses of Theorem C and Theorem D are subtly different.

In light of Theorem B and Theorem C, it is interesting to consider the situation where P is a poset with a planar embedding. This setting was studied by Meszaros-Morales-Striker [49, Theorem 3.14], who showed that a flow polytope $\mathrm{Fl}(Q)$ is (integrally equivalent to) the order polytope $\mathcal{O}(P)$ of a poset P exactly when Q is a planar embedding of P; this result was extended to marked order polytopes in [44].

We next turn our attention to the toric variety $Y(\mathcal{F}_Q)$ associated to the face fan $\mathcal{F}_Q$ of the root polytope. When Q is strongly connected, Theorem A implies that $Y(\mathcal{F}_Q)$ is a Gorenstein Fano variety with at

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most terminal singularities. The fact that $Y(\mathcal{F}_Q)$ is very close to being smooth is further underlined by the following result, which appears as Theorem 5.4.

Theorem E. *Let Q be a strongly-connected starred quiver, and let $\mathcal{F}_Q$ be the face fan of the root polytope* Root(Q). *There exists a refinement $\widehat{\mathcal{F}}_Q$ of $\mathcal{F}_Q$ such that the resulting morphism $Y(\widehat{\mathcal{F}}_Q) \to Y(\mathcal{F}_Q)$ is a small crepant toric desingularisation.*

When our quiver Q comes from a ranked poset P as in Theorem D, there is a third fan to consider. Namely, Theorem D tells us that the normal fan $\mathcal{N}(\mathcal{O}(P))$ of the order polytope is refined by $\mathcal{F}_Q$, while Theorem E tells us that $\mathcal{F}_Q$ is refined by $\widehat{\mathcal{F}}_Q$. All three fans share the same set of rays. The toric variety associated to $\mathcal{N}(\mathcal{O}(P))$ is the so-called Hibi projective toric variety $Y_{\mathcal{O}(P)}$ associated to the poset P, and the two results join together to give a small desingularisation of $Y_{\mathcal{O}(P)}$ that goes via $Y(\mathcal{F}_Q)$,

$$Y(\widehat{\mathcal{F}}_Q) \longrightarrow Y(\mathcal{F}_Q) \longrightarrow Y_{\mathcal{O}(P)}.$$

In other words, as a corollary of Theorem D and Theorem E, any Hibi toric variety associated to a ranked poset has a small toric desingularisation.

Our final main result is a combinatorial description of the Picard group of $Y(\mathcal{F}_Q)$ in the case where Q comes from a ranked poset P. Recall that the poset P has its bounded extension $\hat{P}$ with one unique maximal element. It also has a 'maximal' extension $P_{\max}$ where each maximal element m is covered by a separate adjoined element $\hat{1}_m$. In between these two extensions, we define a new *canonical extension* $\bar{P}$ in Definition 5.13, and we prove the following theorem, which appears as Theorem 5.18 and Proposition 5.25.

Theorem F. *Let P be a ranked poset, let $Y(\mathcal{F}_Q)$ be the toric variety associated to the quiver $Q = Q_{\bar{P}}$, and let $Y(\widehat{\mathcal{F}}_Q)$ be its desingularisation from Theorem E. The Picard rank of $Y(\mathcal{F}_Q)$ is equal to the number of maximal elements in the canonical extension $\bar{P}$ of P, and the Picard rank of $Y(\widehat{\mathcal{F}}_Q)$ is equal to the number of maximal elements in P.*

Note that the Hibi toric variety $Y_{\mathcal{O}(P)}$ has Picard rank equal to 1 (see [52, Section 2.3]), coinciding with the number of maximal elements in $\hat{P}$. We find that the three toric varieties $Y_{\mathcal{O}(P)}, Y(\mathcal{F}_Q)$ and $Y(\widehat{\mathcal{F}}_Q)$ constructed out of P relate naturally to the three extensions $\hat{P}$, $\bar{P}$ and $P_{\max}$ of the ranked poset P.

At the heart of all of our results are the reflexive polytopes Root(Q) associated to strongly-connected starred quivers. Reflexive polytopes were introduced by Batyrev [8] in the study of mirror symmetry for toric varieties, and as mentioned earlier, this work was motivated by our concurrent work on mirror symmetry for Schubert varieties and their toric degenerations [61]; see Section 5 for more details. However, the results of this paper are combinatorial in nature. While some of the results presented here have appeared before (sometimes in special cases or without proofs), we hope that this exposition will be useful for illuminating the connections between root polytopes, flow polytopes and (marked) order polytopes, and their roles within toric geometry.

The structure of this paper is as follows. In Section 2, we prove that if Q is strongly-connected, Root(Q) is reflexive and terminal. We also describe the facets of Root(Q). In Section 3, we consider the case that Q has a planar embedding, and we show that in this case, Root(Q) is (integrally equivalent to the) polar dual to the flow polytope of the planar dual quiver $Q^\vee$. In Section 4, we consider the case that Q comes from a ranked poset, and we relate Root(Q) to a marked order polytope. We also relate the face fan $\mathcal{F}_Q$ of Q to the (inner) normal fan of the corresponding order polytope. Finally, in Section 5, we discuss the connection to mirror symmetry and toric geometry: we show that when Q is a strongly-connected starred quiver, there is a small toric desingularisation $Y(\widehat{\mathcal{F}}_Q)$ of the toric variety $Y(\mathcal{F}_Q)$, and we compute the Picard group of $Y(\mathcal{F}_Q)$. We give a particularly explicit description in the case where Q comes from a ranked poset.

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2. Root polytopes

In this section, we show that when Q is a quiver or starred quiver that is strongly connected, the polytope $\mathrm{Root}(Q)$ is reflexive and terminal. In this case, we also describe the facets of $\mathrm{Root}(Q)$ in terms of certain labelings of the arrows of Q.

2.1. Preliminaries

Definition 2.1. We say that two integral polytopes $\mathbf{P}_1 \subset \mathbb{R}^n$ and $\mathbf{P}_2 \subset \mathbb{R}^m$ are *integrally equivalent* if there is an affine transformation $\phi : \mathbb{R}^n \to \mathbb{R}^m$ whose restriction to $\mathbf{P}_1$ is a bijection $\phi : \mathbf{P}_1 \to \mathbf{P}_2$ that preserves the lattice; that is, ϕ is a bijection between $\mathbb{Z}^n \cap \mathrm{aff}(\mathbf{P}_1)$ and $\mathbb{Z}^m \cap \mathrm{aff}(\mathbf{P}_2)$, where $\mathrm{aff}(\cdot)$ denotes the affine span. The map ϕ is then an *integral equivalence*.

We note that integrally equivalent polytopes (sometimes called *isomorphic* or *unimodularly equivalent*) have the same Ehrhart polynomials and hence the same volume.

Definition 2.2. Suppose that $\mathbf{P} \subset \mathbb{R}^n$ is a lattice polytope of full dimension n which contains the origin in its interior. Then the *polar dual polytope* of $\mathbf{P}$ is

$$\mathbf{P}^* := \{y \in (\mathbb{R}^n)^* \mid x \cdot y \geq -1 \text{ for all } x \in \mathbf{P}\}. \tag{2.1}$$

Remark 2.3. There are two common definitions given for the polar dual. Our convention in (2.1) is consistent with the conventions of Polymake. The definition of polar dual from [69] is

$$\mathbf{P}^\Delta := \{y \in (\mathbb{R}^n)^* \mid x \cdot y \leq 1 \text{ for all } x \in \mathbf{P}\}. \tag{2.2}$$

The polytope $\mathbf{P}^\Delta$ is simply the negative of $\mathbf{P}^*$.

Definition 2.4. A *reflexive polytope of dimension n* is a lattice polytope of full dimension n such that its polar dual is also a lattice polytope (i.e., it is bounded and has vertices with integer coordinates).

More generally, we will use the word *reflexive* for any polytope that is integrally equivalent to a reflexive polytope.

Definition 2.5. A lattice polytope $\mathbf{P} \subset \mathbb{Z}^n$ is called *terminal* if $\mathbf{0}$ and the vertices are the only lattice points in $\mathbf{P} \cap \mathbb{Z}^n$ (with $\mathbf{0}$ in the interior).

2.2. Examples of root polytopes

Recall the definition of root polytope from Definition 1.2. We can associate natural starred quivers and thereby root polytopes to both acyclic quivers and to posets in the following way.

Example 2.6 (Strongly-connected quiver from an acyclic quiver). We can get a strongly-connected starred quiver from any acyclic quiver: we simply designate each sink and source as a starred vertex; see Figure 3.

Example 2.7 (Strongly-connected quiver from a poset). As in Definition 1.5 and Definition 1.6, we can get a strongly-connected starred quiver from a starred poset (the underlying quiver will be connected as long as, for example, the poset has a unique minimal or unique maximal element) or from the bounded extension of any finite poset P. The latter construction is illustrated at the right of Figure 3.

Lemma 2.8. *Let Q be a quiver with no starred vertices. Choose one of its vertices, say v_1, and let $Q_\star$ be the starred quiver obtained from Q by replacing v_1 with a starred vertex $\star$. Then* $\mathrm{Root}(Q)$ *is integrally equivalent to* $\mathrm{Root}(Q_\star)$.

Proof. If Q has normal vertices $\{v_1, \dots, v_n\}$ but no starred vertices, then $\mathrm{Root}(Q)$ lies in the hyperplane $x_1 + \cdots + x_n = 0$. The map $\phi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ which sends $(x_1, x_2, \dots, x_n) \mapsto (x_2, \dots, x_n)$ restricts to an integral equivalence mapping $\mathrm{Root}(Q)$ to $\mathrm{Root}(Q_\star)$. □

Whenever the underlying graph of Q has only normal vertices $\{v_1, \ldots, v_m\}$ but no starred vertices, the root polytope lies in the hyperplane $x_1 + \cdots + x_m = 0$. If there is at least one starred vertex, we have the following lemma.

Lemma 2.9. *Let Q be a starred quiver with n normal vertices $\{v_1, \ldots, v_n\}$. Then* $\mathrm{Root}(Q)$ *is full-dimensional in $\mathbb{R}^n$.*

Proof. Recall that we assume that the underlying graph of any quiver is connected. Since Q is starred, it has at least one starred vertex. We may identify the starred vertices by Remark 1.3 and assume that Q has a unique starred vertex that we call $\star$. Now for any normal vertex v_i, we have a path in the underlying graph of $\overline{Q}$ that starts at the starred vertex $\star$ and ends at v_i. Let $(a_1, \ldots, a_k)$ be the associated sequence of arrows of Q. Consider the element of $\mathbb{R}^n$ given by the signed sum $v_\pi := \sum_{i=1}^{k} \varepsilon_i u_{a_i}$ of vertices of $\mathrm{Root}(Q)$, where $\varepsilon_i = 1$ if a_i is oriented in the direction of the path π and $\varepsilon_i = -1$ otherwise. In terms of the standard basis of $\mathbb{R}^n$, we have that $v_\pi = e_i$. Thus, e_i lies in the span of $\mathrm{Root}(Q)$. This holds for all i and proves that $\mathrm{Root}(Q)$ is full-dimensional. □

Lemma 2.10. *Let Q be a starred quiver with arrows* $\mathrm{Arr}(Q)$ *and vertices $\{v_1, \ldots, v_n\} \cup \{\star\}$. Then each u_a for $a \in$* $\mathrm{Arr}(Q)$ *is a vertex of* $\mathrm{Root}(Q)$*, and the vertices of* $\mathrm{Root}(Q)$ *are in bijection with the arrows* $\mathrm{Arr}(Q)$.

Definition 2.11. Let $A^{(n)} := \mathrm{Conv}\{\{\pm e_1, \ldots, \pm e_n\} \sqcup \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}\} \subset \mathbb{R}^n$.

Note that by Lemma 2.8, $A^{(n)}$ is isomorphic to the *root polytope* of type A in $\mathbb{R}^{n+1}$ (which is defined as the convex hull of all points $\pm(e_i - e_j)$ and hence lies in the hyperplane $\sum x_i = 0$).

Proof of Lemma 2.10. Note that each of the $(2n) + n(n-1)$ points in Definition 2.11 is a vertex of $A^{(n)}$; we can easily see this by choosing for each u in S an appropriate linear functional which is maximized at u. In particular, $\pm e_1$ is a vertex because the linear functional $\lambda : (x_1, \ldots, x_n) \mapsto \pm(10x_1 + x_2 + \cdots + x_n)$ is maximized at $\pm e_1$, while $\pm(e_1 - e_2)$ is a vertex because the linear functional $\lambda : (x_1, \ldots, x_n) \mapsto \pm(10x_1 - 10x_2)$ is maximized at $\pm(e_1 - e_2)$.

But now $\{u_a \mid a \in \mathrm{Arr}(Q)\}$ is a subset of the vertices of $A^{(n)}$, and hence, each u_a must be a vertex of $\mathrm{Root}(Q)$. □

Remark 2.12. Let Q be any (possibly starred) quiver. If Q has multiple starred vertices, then we may identify all of the starred vertices to obtain a starred quiver with a single starred vertex as in Remark 1.3. If Q has no starred vertices, then we may apply Lemma 2.8 to obtain a starred quiver with one starred vertex. In either case, the root polytope of the new quiver is integrally equivalent to $\mathrm{Root}(Q)$. In light of this observation and Lemma 2.9, we may at times prefer to work with quivers which have precisely one starred vertex.

Any properties that hold for root polytopes of starred quivers with a unique starred vertex and no multiple arrows, if they are invariant under integral equivalence, also hold in general.

Proposition 2.13. *Let Q be a (possibly starred) quiver with normal vertices $\mathcal{V}_\bullet = \{v_1, \ldots, v_n\}$. Then the only possible lattice points of* $\mathrm{Root}(Q)$ *are* $\mathbf{0}$ *and its vertices u_a for $a \in$* $\mathrm{Arr}(Q)$.

Proof. Choose a lattice point $p = (p_1, \ldots, p_n) \in \mathrm{Root}(Q)$. We can write it as

$$p = \sum_{a \in \mathrm{Arr}(Q)} m_a u_a, \tag{2.3}$$

where each $m_a \geq 0$, and $\sum_a m_a = 1$. If we then replace each u_a by its expression in terms of e_i's, we get

$$p = (p_1, \ldots, p_n) = \sum_{i=1}^{n} (\mathrm{in}(i) - \mathrm{out}(i)) e_i, \tag{2.4}$$

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where $\mathrm{in}(i) = \sum_a m_a$, where the sum is over all arrows a pointing toward v_i, and $\mathrm{out}(i) = \sum_a m_a$, where the sum is over all arrows a pointing away from v_i.

Assume that p is not one of the vertices u_a. Then we must have that $m_a < 1$ for all arrows a. Since $\sum_a m_a = 1$, we must have $0 \leq \mathrm{in}(i) \leq 1$ and $0 \leq \mathrm{out}(i) \leq 1$ for all i. Moreover, if $\mathrm{in}(i) = 1$ (respectively, $\mathrm{out}(i) = 1$), then $\mathrm{out}(i) = 0$ (respectively, $\mathrm{in}(i) = 0$). Since each coordinate $p_i \in \mathbb{Z}$, it then follows that $p_i \in \{0, 1, -1\}$ for all $1 \leq i \leq n$.

Now fix some i, and suppose that $p_i = 1$ (the proof in the case where $p_i = -1$ is analogous, so we omit it). Then $\mathrm{in}(i) = 1$ and $\mathrm{out}(i) = 0$. Since $\sum_a m_a = 1$, this implies that for each arrow a not pointing to v_i, we have that $m_a = 0$.

Since $\mathrm{in}(i) = 1$, we know there exists some arrow a pointing to v_i such that $m_a > 0$. If the only such arrow(s) a starts from a starred vertex $\star$, then we have that p has the form u_a, where a is the arrow from $\star$ to v_i, which contradicts our assumption that p does not have the form u_a. Therefore, we must have an arrow $a : v_k \to v_i$ with $m_a > 0$; recall that we also know that $m_a < 1$. But then $p_k = \mathrm{in}(k) - \mathrm{out}(k) = 0 - m_a \notin \mathbb{Z}$. This is a contradiction. □

The previous proposition immediately implies the following result.

Corollary 2.14. *Let Q be a (possibly starred) quiver with normal vertices $\mathcal{V}_\bullet = \{v_1, \ldots, v_n\}$. If $\mathbf{0}$ lies in the relative interior of* $\mathrm{Root}(Q)$*, then* $\mathrm{Root}(Q)$ *is terminal.*

Proposition 2.15. *Let Q be a (possibly starred) quiver, and let Q be strongly-connected. Then the polytope* $\mathrm{Root}(Q)$ *is terminal: in particular, it contains $\mathbf{0}$ in its relative interior.*

Proof. By Remark 2.12, we can assume that Q has arrows $\mathrm{Arr}(Q)$ and vertices $\mathcal{V} = \{v_1, \ldots, v_n\} \cup \{\star\}$. Choose any $v_h \in \mathcal{V}$, and let p be a path from $\star$ to v_h, with arrows $a_1, \ldots, a_k$. Note that the sum of the corresponding points $u_{a_1} + \cdots + u_{a_k}$ equals e_h, and hence, $\frac{1}{k} e_h$ lies in $\mathrm{Root}(Q)$. Similarly, if we choose a path from v_h to $\star$, with arrows $a'_1, \ldots, a'_\ell$, we have $u_{a'_1} + \cdots + u_{a'_\ell} = -e_h$, and hence, $-\frac{1}{\ell} e_h$ lies in $\mathrm{Root}(Q)$. Since $\frac{1}{k} e_h$ and $-\frac{1}{\ell} e_h$ lie in $\mathrm{Root}(Q)$, so does $\mathbf{0}$.

The above argument works for any $v_h \in \mathcal{V}$, so for each $h \in \{1, 2, \ldots, n\}$, the interval $[-\mu e_h, \mu e_h]$ lies in $\mathrm{Root}(Q)$, for some small $\mu > 0$. This implies that $\mathbf{0}$ lies in the interior of $\mathrm{Root}(Q)$. And now by Corollary 2.14, $\mathrm{Root}(Q)$ is terminal. □

2.3. *Faces of root polytopes*

We next turn our attention to faces of $\mathrm{Root}(Q)$, where Q is a starred quiver. We will start by identifying each point u of $\mathrm{Root}(Q)$ with a $\bullet$*-labeling* $L : \mathcal{V}_\bullet \to \mathbb{R}$ of Q. (Abusing notation, we will also identify a $\bullet$-labeling $L' : \mathcal{V}_\bullet \to \mathbb{R}$ with a linear functional, which we evaluate on points u of $\mathrm{Root}(Q)$ by taking the dot product.) Our main result is Theorem 2.31, which characterizes the facets of the root polytope of a strongly-connected starred quiver; as a consequence, we also prove that this polytope is reflexive.

Note that in the case that Q is an acyclic quiver, the faces of $\mathrm{Root}(Q)$ were characterized in [63]. While we can always associated a strongly-connected starred quiver to an acyclic quiver (cf. Example 2.6), a strongly-connected starred quiver may have cycles.

To describe the faces and facets of $\mathrm{Root}(Q) \subset \mathbb{R}^n$, we will use two types of coordinates.

Definition 2.16 (Vertex coordinates and arrow coordinates). Let Q be a starred quiver as in Definition 1.1. We define a $\bullet$*-labeling* to be a map $L : \mathcal{V}_\bullet \to \mathbb{R}$ from the set of normal vertices $\mathcal{V}_\bullet = \{v_1, \ldots, v_n\}$ to $\mathbb{R}$; if $L(v_i) = c_i$, we identify L with the point $c_1 e_1 + \ldots c_n e_n = (c_1, \ldots, c_n) \in \mathbb{R}^n$. And we define an *arrow labeling* to be a map $M : \mathrm{Arr}(Q) \to \mathbb{R}$. We associate to each $\bullet$-labeling L an *arrow labeling* $M_L : \mathrm{Arr}(Q) \to \mathbb{R}$ as follows:

1. if $a : v_i \to v_j$, $M(a) = L(v_j) - L(v_i)$;
2. if $a : \star_i \to v_j$, $M(a) = L(v_j)$; and
3. if $a : v_i \to \star_j$, $M(a) = -L(v_i)$.

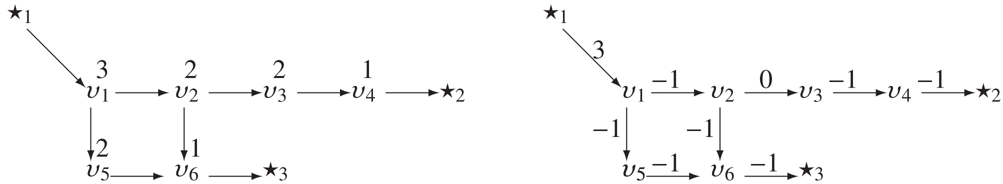

Figure 4. *A $\bullet$-labeling and the corresponding arrow labeling.*

We may think of each starred vertex as getting labeled by a 0; that is, we may extend L to a labeling of $\mathcal{V}_\star$ by setting $L(\star_i) = 0$, and then Cases (2) and (3) above are special cases of (1).

See Figure 4 for an example of a $\bullet$-labeling of the quiver Q from Figure 1, together with the corresponding arrow labeling.

We now give a simple characterisation of the maps $M : \mathrm{Arr}(Q) \to \mathbb{R}$ that arise from $\bullet$-labelings.

Definition 2.17. Let Q be a (possibly starred) quiver, and let $\overline{Q}$ be the related quiver where all starred vertices are identified; see Remark 1.3. Let $M : \mathrm{Arr}(Q) \to \mathbb{R}$ be an arrow labeling of Q. We call M a 0*-sum arrow labeling* if $\sum_{a\in\pi} \varepsilon_a M(a) = 0$ whenever π is a closed path in the underlying graph of $\overline{Q}$, where $\varepsilon_a = 1$ for arrows a pointing in the direction of the path, and $\varepsilon_a = -1$ for all other arrows. We call M *nontrivial* if there is at least one arrow a with $M(a) \neq 0$. We use the notation $\mathbf{M}_{Q,\mathbb{R}}$ to denote the vector space of 0-sum arrow labelings, and $\mathbf{M}_Q$ for its sublattice of $\mathbb{Z}$-valued 0-sum arrow labelings.

We can go from a 0-sum arrow labeling to a vertex labeling of Q. Moreover, for strongly connected quivers, there is a simpler description of 0-sum arrow labelings.

Lemma 2.18. *Let Q be a starred quiver. If L is a $\bullet$-labeling of Q, then the corresponding arrow labeling M_L is a* 0*-sum arrow labeling. Conversely, let $M : \mathrm{Arr}(Q) \to \mathbb{R}$ be a* 0*-sum arrow labeling. Then $M = M_L$ for some $\bullet$-labeling $L : \mathcal{V}_\bullet \to \mathbb{R}^n$.*

If Q is a strongly connected starred quiver, then $M : \mathrm{Arr}(Q) \to \mathbb{R}$ is a 0*-sum arrow labeling if and only if for each oriented path π in Q between two starred vertices, we have $\sum_{a\in\pi} M(a) = 0$.*

Proof. If L is a $\bullet$-labeling of Q, then it follows from the definitions that the corresponding arrow labeling M_L is a 0-sum arrow labeling.

For the converse, let M be a 0-sum arrow labeling, and let $v \in \mathcal{V}_\bullet$. We may assume that Q has a unique starred vertex $\star$ by identifying all starred vertices to one. Since the underlying graph of Q is connected, as we assume throughout, we may now choose an (unoriented) path p from $\star$ to v. Let $L(v) := \sum_{a\in p} \varepsilon_a M(a)$, where $\varepsilon_a = 1$ if the arrow a is pointing in the direction of the path p and otherwise $\varepsilon_a = -1$. The 0-sum condition implies that $L(v)$ depends only on the vertex $v \in \mathcal{V}_\bullet$ and not on the choice of the path p. It is straightforward that $M_L = M$.

Now suppose Q is a strongly-connected starred quiver with a 0-sum arrow labeling M. Clearly, if π is any oriented path from a starred vertex to a starred vertex, then we have $\sum_{a\in\pi} M(a) = 0$. (Note that π gives rise to a closed oriented path in $\overline{Q}$.) We now check the converse. Let $v \in \mathcal{V}_\bullet$. Since Q is strongly-connected, there exists an *oriented* path p from a starred vertex to v. Let us define a vertex labeling L by $L(v) := \sum_{a\in p} M(a)$. To see that this is well defined, consider any other oriented path p' from a (possibly different) starred vertex to v. Since Q is strongly-connected, we also have a path p'' from v to a third starred vertex. Since the concatenations of both p with p'' and of p' with p'' are paths from a starred vertex to a starred vertex, our assumption on M now says that $\sum_{a\in p} M(a) + \sum_{a\in p''} M(a) = 0$ and also $\sum_{a\in p'} M(a) + \sum_{a\in p''} M(a) = 0$. This implies that $\sum_{a\in p} M(a) = \sum_{a\in p'} M(a)$, and therefore, L is well defined. Again, it is straightforward that $M = M_L$. It follows from this that M is a 0-sum arrow labeling as defined in Definition 2.17. □

Remark 2.19. The map given in Lemma 2.18 from the space of vertex coordinates $\mathbb{R}^{\mathcal{V}_\bullet}$ to $\mathbf{M}_{Q,\mathbb{R}}$ (sending a $\bullet$-labeling L to its associated 0-sum arrow-labeling M_L) defines a vector space isomorphism

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$\Psi : \mathbb{R}^{\mathcal{V}_\bullet} \to \mathbf{M}_{Q,\mathbb{R}}$ that is moreover an integral equivalence. This follows directly from its construction in Definition 2.16 and the construction of the inverse in the proof of Lemma 2.18.

Definition 2.20. Given a 0-sum arrow labeling M of Q, we let $M_{\min} = \min\{M(a) \mid a \in \mathrm{Arr}(Q)\}$ be the minimal arrow label occurring in M.

Lemma 2.21. *Let Q be any starred quiver. Let $M = M_L$ be a 0-sum arrow labeling of Q associated to the $\bullet$-labeling $L \in \mathbb{R}^{\mathcal{V}_\bullet}$. Let $F(M) = \{a \in \mathrm{Arr}(Q) \mid M(a) = M_{\min}\}$ be the set of arrows with label $M_{\min}$. Consider the linear functional on* Root(Q) *defined by taking the dot product with L. This linear functional is minimized precisely at the face of* Root(Q) *containing vertices $\{u_a \mid a \in F(M)\}$; moreover, the dot product $L \cdot u_a = M_{\min}$ for all $a \in F(M)$.*

Proof. Let us compute the dot product of L with an arbitrary vertex $u_a = (u_1, \ldots, u_n)$ of Root(Q). Suppose that $a : v_i \to v_j$ joins two normal vertices of Q. Then $u_j = 1$, $u_i = -1$, and $u_h = 0$ for all other h. Therefore, the dot product $L \cdot u_a$ of L with u_a is $L(v_j) - L(v_i) = M(a)$; compare Definition 2.16.(1).

Now suppose that $a : \star \to v_j$ joins a starred vertex to a normal vertex. Then $u_j = 1$ and $u_h = 0$ for all other h. Therefore, $L \cdot u_a = L(v_j) = M(a)$; compare Definition 2.16.(2). Similarly, if $a : v_i \to \star$ joins a normal vertex to a starred vertex, we have $u_i = -1$ and $u_h = 0$ for all other h, and the dot product $L \cdot u_a = -L(v_i) = M(a)$, by Definition 2.16.(3).

In all cases, the dot product $L \cdot u_a$ of L with the vertex u_a is $M(a) \geq M_{\min}$, with equality attained precisely at the vertices $\{u_a \mid a \in F(M)\}$. This is a nonempty set of vertices by construction, and it defines a face of Root(Q) because of the linear functional L. ◻

We will assume throughout the remainder of this section that starred quivers are strongly-connected. We will also identify all starred vertices to a single vertex $\star$ as in Remark 2.12.

Remark 2.22. For any 0-sum arrow labeling M of a strongly-connected starred quiver Q, we always have $M_{\min} \leq 0$, since the sum of the arrow labels in each path from a starred vertex to a starred vertex is 0. Thus, if Q is strongly connected and M is nontrivial, then $M_{\min} < 0$.

Lemma 2.23 follows from the definition of 0-sum arrow labeling, Remark 2.22 and Lemma 2.21.

Lemma 2.23. *Let M be a nontrivial 0-sum arrow labeling of a strongly connected starred quiver Q. Let $M' : \mathrm{Arr}(Q) \to \mathbb{R}$ be defined by setting $M'(a) = \frac{1}{|M_{\min}|} M(a)$ for all arrows $a \in \mathrm{Arr}(Q)$. Then M' is a 0-sum arrow labeling with $M'_{\min} = -1$, and both M and M' are minimized at the same set of vertices of* Root(Q).

Definition 2.24. We say that a 0-sum arrow labeling M of Q is a *face arrow-labeling* if $M_{\min} = -1$.

If we want to understand faces of Root(Q), then by Lemma 2.23, we can restrict our attention to face arrow-labelings of Q.

Proposition 2.25. *Consider a face F of* Root(Q) *with vertices $\{u_a | a \in S\}$ for some $S \subseteq \mathrm{Arr}(Q)$. Then this face is minimized by a linear functional L such that $M := M_L$ is a face arrow-labeling, and $S = \{a \in \mathrm{Arr}(Q) \mid M(a) = -1\}$.*

Proof. Since F is a face, it is minimized by some linear functional $L : \mathcal{V}_\bullet \to \mathbb{R}$. Letting $r \in \mathbb{R}$ be the minimal value attained, we have that $L \cdot u_a = r$ for all $a \in S$, and $L \cdot u_a > r$ for $a \notin S$. By Proposition 2.15, we know that 0 lies in the interior of Root(Q), so $L \cdot 0 = 0$. Therefore, r must be negative; by multiplying all entries of L by $\frac{1}{|r|}$, we can assume that $L \cdot u_a = -1$ for $a \in S$, and $L \cdot u_a > -1$ for $a \notin S$.

Now L gives rise to a 0-sum arrow labeling $M := M_L$. Consider a vertex u_a with $a \in S$; suppose $a : v_i \to v_j$. Then $M(a) = L(v_j) - L(v_i)$ and also $-1 = L \cdot u_a = L \cdot (e_j - e_i) = L(v_j) - L(v_i)$, which implies that $M(a) = -1$. Similarly, if $a \in S$ is an arrow connecting a starred vertex with a normal vertex, $M(a) = -1$.

For any u_a with $a \notin S$, we have $L \cdot u_a > -1$, so the same argument shows that $M(a) > -1$. Therefore, M is a face arrow-labeling. ◻

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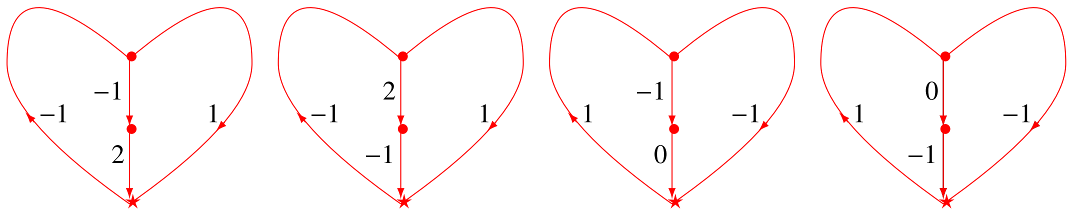

Figure 5. *A starred quiver and its four facet arrow-labelings.*

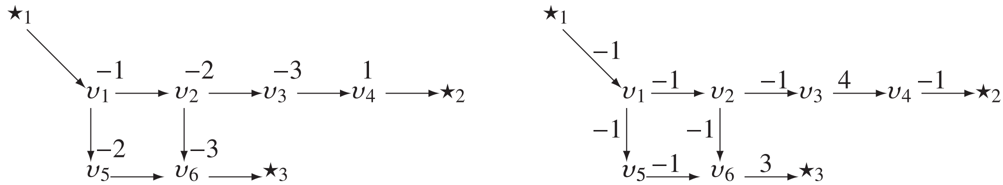

Figure 6. *A ●-labeling and the corresponding arrow labeling.*

Definition 2.26. Let Q be a strongly connected starred quiver. We say that a face arrow-labeling M of Q is a *facet arrow-labeling* if the set $F(M) = \{a \in \mathrm{Arr}(Q) \mid M(a) = -1\}$ is maximal by inclusion among all face arrow-labelings.

In other words, a facet arrow-labeling is an arrow labeling $M : \mathrm{Arr}(Q) \to \mathbb{R}_{\geq -1}$ such that

1. $M_{\min} = -1$, and for each (oriented) path p in Q from a starred vertex to a starred vertex, we have $\sum_{a \in p} M(a) = 0$;
2. There is no $M' : \mathrm{Arr}(Q) \to \mathbb{R}_{\geq -1}$ satisfying (1) such that $F(M')$ properly contains $F(M)$.

Figure 5 shows the four facet arrow-labelings of a particular starred quiver. And the arrow labeling from the right of Figure 6 is a facet arrow-labeling.

Corollary 2.27. *Every facet F of* $\mathrm{Root}(Q)$ *is defined by (minimized by) a linear functional L such that M_L is a facet arrow-labeling; moreover, the vertices of F are precisely the vertices $\{u_a \mid M_L(a) = -1\}$.*

Proof. The fact that M_L is a face arrow-labeling and that F has vertices $\{u_a \mid M_L(a) = -1\}$ comes from Proposition 2.25. Additionally, M_L must be a facet arrow-labeling (i.e., the set $F(M_L) = \{a \in \mathrm{Arr}(Q) \mid M_L(a) = -1\}$ must be maximal by inclusion among all face arrow-labelings) because otherwise, there would be another face arrow-labeling M defining a face of $\mathrm{Root}(Q)$ whose vertices are a superset of the vertices of F. □

Our next goal is to show that facet arrow-labelings $M : \mathrm{Arr}(Q) \to \mathbb{R}_{\geq -1}$ take on only integer values. This requires a little preparation.

Definition 2.28. Given a facet arrow-labeling $M : \mathrm{Arr}(Q) \to \mathbb{R}_{\geq -1}$, we define a *facet component* of Q (with respect to M) to be a connected subquiver C of Q in which $M(a) = -1$ for all $a \in \mathrm{Arr}(C)$, and such that C is maximal by inclusion with this property.

Note that every normal vertex and every starred vertex of Q is part of some facet component of Q (with respect to M); note also that a facet component can be made up of only one single vertex and no arrows.

Lemma 2.29. *Let $M : \mathrm{Arr}(Q) \to \mathbb{R}_{\geq -1}$ be a facet arrow-labeling. Then each facet component of Q (with respect to M) contains some starred vertex.*

Proof. Let L be the ●-labeling of Q such that $M_L = M$. Suppose that there is a facet component C with no starred vertex. Consider all arrows a of Q which are *adjacent* to C (i.e. such that $a \notin \mathrm{Arr}(C)$, but a

shares a vertex with C), and choose the arrow $\tilde{a}$ such that $M(\tilde{a})$ is minimal. We will use $\tilde{a}$ to produce a new $\bullet$-labeling L'.

Let $\ell := M(\tilde{a})$. Clearly, $\ell > -1$. Suppose without loss of generality that $\tilde{a}$ points toward C (the case where $\tilde{a}$ points away from C is similar). Then for each vertex v of C, set $L'(v) = L(v) - (\ell + 1)$. For all other vertices v of Q, we set $L'(v) = L(v)$. Now let us consider how $M' := M_{L'}$ differs from $M = M_L$. Clearly, the only arrows a in which $M'(a) \neq M(a)$ are those arrows a which are adjacent to C. Among these arrows,

- if a points toward C, then $M'(a) = M(a) - (\ell + 1)$.
- if a points away from C, then $M'(a) = M(a) + (\ell + 1)$.

By our assumptions, each arrow a adjacent to C satisfies $M(a) \geq \ell > -1$, and hence, $M'(a) \geq -1$. Moreover, $M'(\tilde{a}) = -1$.

In summary, we have produced a new M' whose set of arrows labeled by -1 strictly contains the set of arrows labeled by -1 in M. So M is not a facet arrow-labeling, which is a contradiction. □

Proposition 2.30. *Let $M : \mathrm{Arr}(Q) \to \mathbb{R}_{\geq -1}$ be a facet arrow-labeling. Then M must be integral: for each $a \in \mathrm{Arr}(Q)$, $M(a) \in \mathbb{Z}$.*

Proof. Let $M : \mathrm{Arr}(Q) \to \mathbb{R}_{\geq -1}$ be a facet arrow-labeling. Write $M = M_L$ where L is the corresponding $\bullet$-labeling. Let C be a facet component of Q (with respect to M). By Lemma 2.29, C contains a starred vertex. Now, using rules (1), (2), (3) of Definition 2.16, and the fact that each arrow $a \in \mathrm{Arr}(C)$ satisfies $M(a) = -1$, we see that for each vertex v of C, we must have $L(v) \in \mathbb{Z}$. Since every vertex of Q lies in some facet component, we must have $L(v) \in \mathbb{Z}$ for *all* vertices of Q. Therefore, $M_L(a) \in \mathbb{Z}$ for all arrows of Q. □

The following is the main result of this section. While this result is stated for strongly-connected starred quivers, by Lemma 2.8, it also applies to strongly-connected quivers.

Theorem 2.31. *Let Q be a strongly-connected starred quiver with normal vertices $\mathcal{V}_\bullet = \{v_1, \dots, v_n\}$, and consider the corresponding polytope $\mathrm{Root}(Q)$. Then the facets of $\mathrm{Root}(Q)$ are in bijection with the $\bullet$-labelings $L : \mathcal{V}_\bullet \to \mathbb{Z}_{\geq -1}$ such that M_L is a facet arrow-labeling. Moreover, the facet inequality is*

$$\{L(v_1)x_1 + L(v_2)x_2 + \cdots + L(v_n)x_n \geq -1\},$$

and $\mathrm{Root}(Q)$ is cut out by the union of these inqualities. Finally, the polytope $\mathrm{Root}(Q)$ is reflexive.

Example 2.32. Consider again the starred quiver from Figure 1. The polytope defined there has facets given in Table 1, where, for example, the first row in the table encodes the facet inequality

$$1 + 3x_1 + 2x_2 + 2x_3 + x_4 + 2x_5 + x_6 \geq 0$$

(which corresponds to the $\bullet$-labeling and arrow labeling in Figure 4), and the fourth row in the table encodes the facet inequality

$$1 - x_1 - 2x_2 - 3x_3 + x_4 - 2x_5 - 3x_6 \geq 0$$

(which corresponds to the $\bullet$-labeling and arrow labeling in Figure 6).

Remark 2.33. The statement that $\mathrm{Root}(Q)$ is reflexive also appears in [33, Proposition 4.2], which additionally says that $\mathrm{Root}(Q)$ is terminal. [33] does not provide a proof but asserts it can be done similarly to [45, Proposition 3.2] (which proved the reflexive property in the case of symmetric directed graphs) and [55, Lemma 1.2] (which discusses when the origin lies in the interior). Another approach to reflexivity is to observe that the matrix whose rows encode the coordinates of the vertices of $\mathrm{Root}(Q)$ is a *totally unimodular* matrix (an integer matrix all of whose minors are ± 1 or 0) [23]; see also [39, Theorem 29]. By a version of the Hoffman-Kruskal Theorem [36, 62], this implies that the polar dual of $\mathrm{Root}(Q)$, if it is nonempty, is integral.

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Table 1. *The inequalities defining the polytope* $\mathrm{Root}(Q)$*, where* Q *is the quiver from Figure 1 and Figure 6.*

	x_1	x_2	x_3	x_4	x_5	x_6
1	3	2	2	1	2	1
1	3	2	1	0	2	1
1	3	2	1	1	2	1
1	-1	-2	-3	1	-2	-3
1	-1	-2	-3	1	2	1
1	-1	2	1	1	2	1
1	-1	2	1	1	-2	1
1	-1	-2	-3	1	-2	1
1	-1	-2	2	1	-2	1
1	-1	2	2	1	-2	1
1	-1	2	2	1	2	1
1	-1	-2	2	1	2	1
1	-1	-2	2	1	-2	-3
1	-1	-2	-3	-4	-2	1
1	-1	2	1	0	-2	1
1	-1	2	1	0	2	1
1	-1	-2	-3	-4	2	1
1	-1	-2	-3	-4	-2	-3

Proof of Theorem 2.31. By Corollary 2.27, each facet F of $\mathrm{Root}(Q)$ is minimized by a linear functional $L : \mathcal{V}_\bullet \to \mathbb{R}_{\geq -1}$ such that M_L is a facet arrow-labeling, and the vertices of F are precisely the vertices $\{u_a \mid M_L(a) = -1\}$. Let $\mathcal{L}$ be the set of linear functionals corresponding to facets of $\mathrm{Root}(Q)$. Note that for $L \in \mathcal{L}$, $L \cdot u_a = M_L(a)$, so the inequality cutting out this facet is $L \cdot (x_1, \dots, x_n) \geq -1$; that is, $\{L(v_1)x_1 + L(v_2)x_2 + \cdots + L(v_n)x_n \geq -1\}$. And by Proposition 2.30, each $L(v_i) \in \mathbb{Z}$.

Therefore, using the correspondence between the i-dimensional faces of a polytope and the $(n-i-1)$-dimensional faces of its polar dual, we see that the vertices of the polar dual $\mathrm{Root}(Q)^*$ are the points $\{(L(v_1), \dots, L(v_n)) \mid L \in \mathcal{L}\}$. In particular, they are integral. We already showed in Proposition 2.15 that the origin is in the interior of $\mathrm{Root}(Q)$, so $\mathrm{Root}(Q)$ is reflexive. □

3. The planar setting: relation to flow polytopes

Let Q be a connected quiver with no oriented cycles. One can associate to each quiver a reflexive polytope called a *flow polytope*, as shown in [1]. Flow polytopes are an interesting class of polytopes that are closely connected to toric geometry, as we explain in Section 5, as well as representation theory [6] and gentle algebras [11]. There has also been a great deal of work on their volumes [6, 47, 48, 15, 29] and faces [3]. In this section, we show that if Q is a strongly-connected quiver with a unique starred vertex which is planar, then its root polytope $\mathrm{Root}(Q)$ is polar dual to the flow polytope of the dual quiver. This gives a new proof that the flow polytope associated to a plane acyclic quiver is reflexive.

Definition 3.1. Let $Q = (Q_0, Q_1)$ be a connected quiver which is *acyclic*; that is, it has no oriented cycles. We denote elements of $\mathbb{R}^{Q_1}$ by $(r_a)_{a \in Q_1}$. We let V_Q be the subspace of $\mathbb{R}^{Q_1}$ obtained by imposing for each vertex $q \in Q_0$ the relation

$$\sum_{a \to q} r_a = \sum_{b \leftarrow q} r_b, \tag{3.1}$$

where the first sum is over all arrows with target q, and the second sum is over all arrows with source q. Then the *flow polytope* Fl_Q is defined to be

$$\mathrm{Fl}_Q = \{(r_a)_{a \in Q_1} \mid r_a \geq -1\} \subset V_Q.$$

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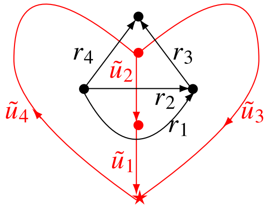

Figure 7. *A plane acyclic quiver Q (in black, with arrows labeled r_i) and the dual starred quiver $Q^\vee$ (in red, with arrows labeled $\tilde{u}_i$). In V_Q (see Definition 3.1), an element (r_1, r_2, r_3, r_4) satisfies $r_1 + r_2 = r_3$, $r_1 + r_2 + r_4 = 0$ and $r_3 + r_4 = 0$, with one relation for each vertex of Q. In $\mathbf{N}_{Q^\vee,\mathbb{R}}$ (see Definition 3.6), the elements $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4$ satisfy $\tilde{u}_1 + \tilde{u}_2 - \tilde{u}_3 = 0$, $\tilde{u}_1 + \tilde{u}_2 + \tilde{u}_4 = 0$ and $\tilde{u}_3 + \tilde{u}_4 = 0$, with one relation for each cycle of $Q^\vee$.*

The flow polytope as defined above is also known as the canonical weight flow polytope. See Remark 3.10 for a comparison to the original definition of [1].

A *plane quiver* is a quiver which is properly embedded into the plane $\mathbb{R}^2$; that is, no two arrows cross each other.

Definition 3.2. Let $Q = (Q_0, Q_1)$ be an acyclic connected plane quiver. Let Q_2 be the set of bounded regions of $\mathbb{R}^2 \setminus Q$. We construct the *dual starred quiver* $Q^\vee = (Q^\vee_\bullet \sqcup \{\star\}, Q^\vee_1)$ by placing one normal vertex in each bounded region of $\mathbb{R}^2 \setminus Q$ so that $Q^\vee_\bullet \cong Q_2$, and placing a starred vertex $\star$ in the unbounded region of $\mathbb{R}^2 \setminus Q$. For each arrow $q \in Q_1$ separating two regions of $\mathbb{R}^2 \setminus Q$, we also place a dual arrow $a^\vee \in Q^\vee_1$ which crosses the original arrow a from 'left to right'; see Figure 7.

The following lemma is easy to check and well known.

Lemma 3.3. *The map in Definition 3.2 gives a bijection between acyclic connected plane quivers and strongly-connected plane starred quivers which have a unique starred vertex.*

The following is the main result of this section.

Theorem 3.4. *Let $Q = (Q_0, Q_1)$ be a plane acyclic connected quiver and let $Q^\vee$ be the dual starred quiver, as in Definition 3.2. Then the root polytope* $\mathrm{Root}(Q^\vee)$ *is integrally equivalent to the polar dual of the flow polytope* Fl_Q.

Corollary 3.5. *Let Q be a connected plane acyclic quiver. Then Fl_Q is reflexive.*

Proof. This follows from Theorem 3.4 and Theorem 2.31. □

The corollary reproves in the plane case the result of [1] that (canonical weight) flow polytopes are reflexive.

Before proving Theorem 3.4, we start by giving a description of the root polytope of a starred quiver in terms of arrow coordinates.

Definition 3.6. Let Q be a starred quiver with vertices $\mathcal{V}_\bullet \sqcup \mathcal{V}_\star$ (where $\mathcal{V}_\bullet = \{v_1, \ldots, v_n\}$ and $\mathcal{V}_\star = \{\star_1, \ldots, \star_\ell\}$), and arrows $Q_1 = \mathrm{Arr}(Q)$. Let $\mathbf{N}_{Q,\mathbb{R}} = \mathbb{R}^{Q_1}/U$ where, in terms of the standard basis $\{\delta_a\}$ of $\mathbb{R}^{Q_1}$,

$$U = \langle \sum_{a \in \pi} \varepsilon_a \delta_a \mid \pi \text{ an undirected path that is closed, or begins and ends in a starred vertex}\rangle_{\mathbb{R}}. \tag{3.2}$$

Here, $\varepsilon_a = 1$ for arrows $a \in Q$ pointing in the direction of the path π and -1 for all other arrows.

Note that $\mathbf{N}_{Q,\mathbb{R}}$ is dual to the vector space of 0-sum arrow-labelings $\mathbf{M}_{Q,\mathbb{R}}$ as constructed in Definition 2.17. In particular, the dual of the $\mathbb{Z}$-lattice $\mathbf{M}_Q \subset \mathbf{M}_{Q,\mathbb{R}}$ is a $\mathbb{Z}$-lattice in $\mathbf{N}_{Q,\mathbb{R}}$ that we may denote by $\mathbf{N}_Q$. This gives $\mathbf{N}_{Q,\mathbb{R}}$ an integral structure.

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For each $a \in Q_1$, let $\tilde{u}_a$ denote the image of δ_a in $\mathbf{N}_{Q,\mathbb{R}} = \mathbb{R}^{Q_1}/U$. We define

$$\widetilde{\mathrm{Root}}(Q) := \mathrm{Conv}(\{\tilde{u}_a \mid a \in Q_1\}) \subseteq \mathbf{N}_{Q,\mathbb{R}}.$$

Compare the above definition with Definition 1.2.

Example 3.7. If we consider the 'dual starred quiver' $Q^\vee$ shown in Figure 7, then the elements $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4 \in \mathbf{N}_{Q^\vee,\mathbb{R}}$ associated to the arrows in $Q_1^\vee$ satisfy the relations $\tilde{u}_1+\tilde{u}_2-\tilde{u}_3 = 0$, $\tilde{u}_1+\tilde{u}_2+\tilde{u}_4 = 0$ and $\tilde{u}_3 + \tilde{u}_4 = 0$.

Proposition 3.8. *Let Q be a starred quiver as in Definition 3.6. The lattice polytopes $\widetilde{\mathrm{Root}}(Q)$ and $\mathrm{Root}(Q)$ are integrally equivalent.*

Proof. Recall the isomorphism $\Psi : \mathbb{R}^{\mathcal{V}_\bullet} \to \mathbf{M}_{Q,\mathbb{R}}$ from Remark 2.19 sending a $\bullet$-labeling L to its associated 0-sum arrow-labeling M_L. Let us write $(\mathbb{R}^{\mathcal{V}_\bullet})^* = \mathbb{R}^n$; this is the vector space containing $\mathrm{Root}(Q)$. Recall that we paired $L \in \mathbb{R}^{\mathcal{V}_\bullet}$ (viewed as an element of $\mathbb{R}^n$) with $u_a \in \mathbb{R}^n$ using the dot product, $\langle L, u_a\rangle = L \cdot u_a$, and we had $L \cdot u_a = M_L(a)$. Thus, we have

$$\langle L, \Psi^*(\tilde{u}_a)\rangle = \langle \Psi(L), \tilde{u}_a\rangle = \langle M_L, \tilde{u}_a\rangle = M_L(a) = \langle L, u_a\rangle,$$

for any $L \in \mathbb{R}^{\mathcal{V}_\bullet}$. The dual isomorphism $\Psi^* : \mathbf{N}_{Q,\mathbb{R}} \to \mathbb{R}^n$ therefore sends the vertex $\tilde{u}_a$ of $\widetilde{\mathrm{Root}}(Q)$ to the vertex u_a of $\mathrm{Root}(Q)$. Thus, it defines an integral equivalence between $\widetilde{\mathrm{Root}}(Q)$ and $\mathrm{Root}(Q)$. □

Remark 3.9. For any strongly-connected starred quiver Q, the polar dual of the 'root polytope' $\widetilde{\mathrm{Root}}(Q) \subset \mathbf{N}_{Q,\mathbb{R}}$ is precisely the convex hull of the set of facet arrow-labelings of Q (which are indeed lattice points in the dual space $\mathbf{M}_{Q,\mathbb{R}}$). This is the translation of Theorem 2.31 from $\mathrm{Root}(Q)$ to $\widetilde{\mathrm{Root}}(Q)$.

Proof of Theorem 3.4. We know from Proposition 3.8 that $\mathrm{Root}(Q^\vee)$ is integrally equivalent to the polytope $\mathbf{P} := \widetilde{\mathrm{Root}}(Q^\vee) = \mathrm{Conv}(\{\tilde{u}_{\bar{a}} \mid \bar{a} \in Q_1^\vee\})$ in $\mathbf{N}_{Q^\vee,\mathbb{R}} = \mathbf{N}_{Q,\mathbb{R}}$. The polar dual $\mathbf{P}^*$ of $\mathbf{P}$ is

$$\{r \in (\mathbf{N}_{Q^\vee,\mathbb{R}})^* \mid \langle r, x\rangle \geq -1 \text{ for all } x \in \mathbf{P}\} = \{r \in \mathbf{M}_{Q^\vee,\mathbb{R}} \mid \langle r, x\rangle \geq -1 \text{ for each vertex } x \text{ of } \mathbf{P}\}.$$

Thus, we get one inequality of $\mathbf{P}^*$ for each vertex $\tilde{u}_{\bar{a}}$ of $\mathbf{P}$, and this inequality is $\langle r, \tilde{u}_{\bar{a}}\rangle \geq -1$. In other words, the elements of $\mathbf{P}^*$ are 0-sum arrow labelings $(r_{\bar{a}})_{\bar{a}\in Q_1^\vee}$ satisfying $r_{\bar{a}} \geq -1$ for all arrows $\bar{a} \in Q_1^\vee$.

Let us refer to a closed path in the underlying graph of $Q^\vee$ that bounds a connected region of $\mathbb{R}^2 \setminus Q^\vee$ as a 'minimal' closed path for $Q^\vee$. Then the 0-sum conditions for $(r_{\bar{a}})_{\bar{a}\in Q_1^\vee}$ coming from such minimal closed paths generate all of the 0-sum conditions of $\mathbf{M}_{Q^\vee,\mathbb{R}}$ from Definition 2.17. To complete the proof, note that we obtain an isomorphism from $\mathbf{M}_{Q^\vee,\mathbb{R}}$ to V_Q by the map sending $(r_{\bar{a}})_{\bar{a}\in Q_1^\vee}$ to $(r_a)_{a\in Q_1}$ (where $r_{\bar{a}} = r_a$ if a is the arrow in Q_1 dual to the arrow $\bar{a} \in Q_1^\vee$). Namely, the 0-sum conditions for $(r_{\bar{a}})_{\bar{a}\in Q_1^\vee}$ coming from minimal closed paths translate precisely to the relations (3.1) for $(r_a)_{a\in Q_1}$ that define V_Q. This isomorphism is clearly an integral equivalence, and moreover, the inequalities $r_{\bar{a}} \geq -1$ describing $\mathbf{P}^*$ translate to the inequalities $r_a \geq -1$ defining Fl_Q. Thus, Fl_Q is integrally equivalent to the polar dual of $\mathrm{Root}(Q^\vee)$. □

Remark 3.10. The original definition of flow polytope (associated to the *canonical weight* $\theta = \theta^c$) from [1] looks slightly different but is easily seen to be equivalent to Definition 3.1. Following [1], let $Q = (Q_0, Q_1)$ be a connected quiver with no oriented cycles, and for each vertex q, set

$$\theta(q) := \#\{\text{arrows with source } q\} - \#\{\text{arrows with target } q\}.$$

We denote elements of $\mathbb{R}^{Q_1}$ by $(R_a)_{a\in Q_1}$. Let V_Q be the subspace of $\mathbb{R}^{Q_1}$ obtained by imposing for each vertex $q \in Q_0$ the relation

$$\theta(q) + \sum_{a\to q} R_a = \sum_{b\leftarrow q} R_b.$$

We then define the flow polytope $\Delta(Q)$ as

$$\Delta(Q) = \{(R_a)_{a\in Q_1} \mid R_a \geq 0\} - \mathbf{1} \subset V_Q,$$

where $\mathbf{1}$ is the all-one vector. If we set $r_a = R_a - 1$, we see that $\mathrm{Fl}_Q = \Delta(Q)$.

Remark 3.11. Theorem 3.4 does not extend to the nonplanar case; it fails already for quivers obtained from the complete bipartite graph $K_{3,3}$, where the polar dual of the flow polytope is no longer a root polytope.

4. The ranked poset setting: relation to (marked) order polytopes

In this section, we specialize to the setting where our starred quiver Q comes from a poset. We will prove Theorem C and Theorem D, relating root polytopes to marked order polytopes and order polytopes.

4.1. Background on order polytopes and marked order polytopes

Order polytopes associated to finite posets were investigated in [68]. Given a poset $P = \{v_1, \ldots, v_n\}$ (where we identify P with its set of points), we let $\mathbb{R}^P$ denote the set of all functions $f : P \to \mathbb{R}$. Recall that $\hat{P}$ denotes the *bounded extension* of P, obtained by adjoining a new minimum element $\hat{0}$ and a new maximum element $\hat{1}$.

Definition 4.1. The *order polytope* $\mathcal{O}(P)$ of the poset P is the subset of $\mathbb{R}^P$ defined by the conditions

$$\begin{aligned} 0 \leq f(v_i) \leq 1 \qquad & \text{for all } v_i \in P \\ f(v_i) \leq f(v_j) \qquad & \text{if } v_i \leq v_j \text{ in } P. \end{aligned}$$

Equivalently, we can use $\hat{P}$ and define the polytope $\hat{\mathcal{O}}(P)$ to be the set of functions $g \in \mathbb{R}^{\hat{P}}$ satisfying

$$\begin{aligned} & g(\hat{0}) = 0 \text{ and } g(\hat{1}) = 1 \\ & g(v_i) \leq g(v_j) \text{ if } v_i \leq v_j \text{ in } \hat{P}. \end{aligned}$$

The linear map $\hat{\mathcal{O}}(P) \to \mathcal{O}(P)$ obtained by projection to $\mathbb{R}^P$ is a bijection.

Recall that a *(lower) order ideal* or *down-set* of P is a subset I of P such that if $v_j \in I$ and $v_i \leq v_j$, then $v_i \in I$. And a *filter* (or *dual order ideal* or *up-set*) of P is a subset I of P such that if $v_i \in I$ and $v_j \geq v_i$, then $v_j \in I$. Let $\chi_I : P \to \mathbb{R}$ denote the characteristic function of I; that is,

$$\chi_I(v_i) = \begin{cases} 1, & v_i \in I \\ 0, & v_i \notin I. \end{cases}$$

Proposition 4.2. *[68] The vertices of $\mathcal{O}(P)$ are in bijection with the filters of P: namely, the vertices of $\mathcal{O}(P)$ are the characteristic functions χ_I of filters I of P.*

The facets of $\mathcal{O}(P)$ are in bijection with the cover relations of $\hat{P}$: namely, if $v_i \lessdot v_j$ is a cover relation in $\hat{P}$, then the corresponding facet consists of those $g \in \mathcal{O}(P)$ satisfying $g(v_i) = g(v_j)$.

There is a natural generalization of order polytope which first appeared in [5].

Definition 4.3. Let $P = P_\bullet \sqcup P_\star$ be a *starred poset* as in Definition 1.5. We write $P_\bullet = \{v_1, \ldots, v_n\}$ and $P_\star = \{\star_1, \ldots, \star_\ell\}$. A pair (P, m) is called a *marked poset* if $m : P_\star \to \mathbb{R}$ is an order-preserving map on $P_\star$, called a *marking*.

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Definition 4.4. [5] The *marked order polytope* $\mathcal{O}_m(P)$ of the marked poset $(P = P_\bullet \sqcup P_\star, m)$ is the subset of $\mathbb{R}^{P_\bullet}$ defined by the conditions

$$\begin{aligned} m(\star_i) &\le f(v_j) \quad \text{if } \star_i \le v_j \text{ where } \star_i \in P_\star, v_j \in P_\bullet \\ f(v_i) &\le m(\star_j) \quad \text{if } v_i \le \star_j \text{ where } v_i \in P_\bullet, \star_j \in P_\star, \\ f(v_i) &\le f(v_j) \quad \text{if } v_i \le v_j \text{ where } v_i, v_j \in P_\bullet. \end{aligned}$$

4.2. The relation to marked order polytopes

In what follows, we will be particularly interested in *ranked* and *graded* posets.

Definition 4.5. Let P be a finite poset with bounded extension $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$. We say that a poset P is *ranked* if for each $v \in P$, each maximal chain in $P \cup \hat{0}$ from $\hat{0}$ to v has the same length. And we say that P is *graded* if all maximal chains in $\hat{P} = P \cup \hat{0} \cup \hat{1}$ from $\hat{0}$ to $\hat{1}$ have the same length. Given a ranked poset P, for $v \in P$, we let $\mathrm{rank}(v)$ denote the length of any maximal chain from $\hat{0}$ to v. For brevity, we will sometimes denote $\mathrm{rank}(v)$ by $R(v)$.

Remark 4.6. Note that every graded poset P is ranked, but the notion of ranked poset is more general.

Definition 4.7. Let P be a starred poset as in Definition 1.5, and suppose that P is ranked with rank function R. The rank function gives rise to a marking $m : P_\star \to \mathbb{Z}$ by letting $m(\star_i) = R(\star_i)$ for all $\star_i \in P_\star$. We let $\mathcal{O}_R(P)$ denote the marked order polytope associated to the marking coming from R.

[19] showed that certain marked order polytopes are reflexive.

Proposition 4.8 [19, Proposition 3.1 and Theorem 3.4]. *Consider the marked order polytope $\mathcal{O}_R(P)$ associated to (P, R), where $P = P_\bullet \sqcup P_\star$ is ranked with rank function R.*[1] *Then $\mathcal{O}_R(P)$ contains a unique interior lattice point $\mathbf{u} = (u_v)_{v \in P_\bullet}$, where $u_v = R(v)$. Moreover, if we let $\overline{\mathcal{O}}_R(P) := \mathcal{O}_R(P) - \mathbf{u}$, then $\overline{\mathcal{O}}_R(P)$ is a reflexive polytope.*

Example 4.9. Let P be the starred poset with Hasse diagram shown at the left of Figure 8, identified with its corresponding starred quiver Q as in Definition 1.5. The ranks of the elements of P are indicated at the right of Figure 8. By definition, $\mathcal{O}_R(P)$ is defined by the inequalities

$$0 \le f(v_1) \le f(v_2) \le f(v_3) \le f(v_4) \le 5, \qquad f(v_2) \le f(v_6), \qquad f(v_1) \le f(v_5) \le f(v_6) \le 4.$$

Clearly, $\mathcal{O}_R(P)$ has an interior lattice point $\mathbf{u} = (u_1, \dots, u_6)$ with $u_i = \mathrm{rank}(v_i)$ (i.e., $u_1 = 1, u_2 = 2, u_3 = 3, u_4 = 4, u_5 = 2, u_6 = 3$). Now if we let $F(v_i) = f(v_i) - \mathrm{rank}(v_i)$, or equivalently substitute $f(v_i) = F(v_i) + \mathrm{rank}(v_i)$ into the above inequalities, we see that $\overline{\mathcal{O}}_R(P)$ is cut out by the inequalities

$$\begin{array}{lll} F(v_1) \ge -1, & F(v_2) - F(v_1) \ge -1, & F(v_3) - F(v_2) \ge -1, \\ F(v_4) - F(v_3) \ge -1, & -F(v_4) \ge -1, & F(v_6) - F(v_2) \ge -1, \\ F(v_5) - F(v_1) \ge -1, & F(v_6) - F(v_5) \ge -1, & -F(v_6) \ge -1. \end{array}$$

From these inequalities, we see that the polar dual of $\overline{\mathcal{O}}_R(P)$ is the polytope with vertices

$$\{e_1, e_2 - e_1, e_3 - e_2, e_4 - e_3, -e_4, e_6 - e_2, e_5 - e_1, e_6 - e_5, -e_6\}.$$

This is exactly the root polytope $\mathrm{Root}(Q)$ of our starred quiver Q!

More generally, the polar dual to $\overline{\mathcal{O}}_R(P)$ is the root polytope of the associated starred quiver.

[1] We note that [19] used a definition of ranked that is slightly more general than the one we use here. [19] defines a ranked poset to be a poset P together with a rank function $R : P \to \mathbb{Z}$ such that for each covering relation $a \lessdot b$, we have $R(b) = R(a) + 1$. So in particular, [19] would allow two minimal elements to have different ranks.

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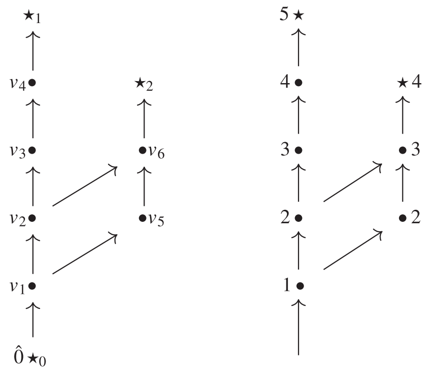

Figure 8. *At left: a marked poset, drawn as a starred quiver. At right: the marked poset with elements labeled by their ranks.*

Theorem 4.10. *Let $P = P_\bullet \sqcup P_\star$ be a starred poset, and let $Q := Q_{(P_\bullet, P_\star)}$ be the starred quiver defined in Definition 1.5. Suppose further that P is ranked with rank function R. Then the polytope $\overline{\mathcal{O}}_R(P)$ from Proposition 4.8 is polar dual to the root polytope* Root(Q).

One consequence of Theorem 4.10 is a new proof that Root(Q) is reflexive in this setting.

Proof. We know from Proposition 4.8 that $\overline{\mathcal{O}}_R(P) = \mathcal{O}_R(P) - \mathbf{u}$ is reflexive. By definition, it has one inequality for each cover relation $a \lessdot b$ in the poset (where at least one of a, b lies in $P_\bullet$):

$$\begin{cases} F(a) + R(a) \leq F(b) + R(b) & \text{if } a, b \in P_\bullet \\ R(a) \leq F(b) + R(b) & \text{if } a \in P_\star \text{ and } b \in P_\bullet \\ F(a) + R(a) \leq R(b) & \text{if } a \in P_\bullet \text{ and } b \in P_\star. \end{cases}$$

Since our poset is ranked (and hence $R(b) = R(a) + 1$ when $a \lessdot b$), these inequalities become

$$\begin{cases} F(b) - F(a) \geq -1 & \text{if } a, b \in P_\bullet \\ F(b) \geq -1 & \text{if } a \in P_\star \text{ and } b \in P_\bullet \\ F(a) \geq -1 & \text{if } a \in P_\bullet \text{ and } b \in P_\star. \end{cases}$$

But then the polar dual has vertices given by those of the root polytope associated to the quiver from Definition 1.5, so we are done. □

Theorem 4.10 allows us to relate the face fan of the root polytope to the normal fan of the marked order polytope.

Definition 4.11. Suppose that a convex polytope $\mathbf{P}$ contains the origin in its interior. Then the *face fan* $\mathcal{F}(\mathbf{P})$ is the fan whose cones are the cones over the faces of $\mathbf{P}$.

Definition 4.12. Given a convex polytope $\mathbf{P}$ in $\mathbb{R}^n$, the *inner normal fan* $\mathcal{N}(\mathbf{P})$ of $\mathbf{P}$ is a polyhedral fan in the dual space $(\mathbb{R}^n)^*$ whose cones consist of the *normal cone* c_F to each face F of $\mathbf{P}$. That is,

$$\mathcal{N}(\mathbf{P}) = \{c_F\}_{F \in \mathrm{face}(\mathbf{P})},$$

where each normal cone c_F is defined as the set of linear functionals w such that the set of points x in $\mathbf{P}$ that *minimize* $w(x)$ contains F,

$$c_F = \{w \in (\mathbb{R}^n)^* \mid F \subset \mathrm{argmin}_{x \in \mathbf{P}}\, w(x)\}.$$

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Corollary 4.13. *Let P and Q be the ranked starred poset and its associated starred quiver from Theorem 4.10. Then the face fan $\mathcal{F}_Q := \mathcal{F}(\mathrm{Root}(Q))$ of the root polytope $\mathrm{Root}(Q)$ coincides with the normal fan of the marked order polytope $\mathcal{O}_R(P)$.*

Proof. This follows immediately from Theorem 4.10, together with the fact that shifting a polytope does not change its normal fan. □

In the next section, we will give an analogue of Corollary 4.13, but we will use the order polytope $\mathcal{O}(P)$ instead of the marked order polytope $\overline{\mathcal{O}}_R(P)$.

4.3. Relation to order polytopes

In this section, we will use the starred quiver $Q_{\hat{P}}$ associated to the bounded extension $\hat{P}$ of a finite poset P, as in Definition 1.6. We will show that the rays of the face fan $\mathcal{F}_{Q_{\hat{P}}}$ of the root polytope $\mathrm{Root}(Q_{\hat{P}})$ coincide with the rays of the (inner) normal fan $\mathcal{N}(\mathcal{O}(P))$ of the order polytope $\mathcal{O}(P)$, for any finite poset P. Additionally, if P is a ranked poset, then $\mathcal{F}_{Q_{\hat{P}}}$ refines $\mathcal{N}(\mathcal{O}(P))$: that is, each maximal cone of $\mathcal{F}_{Q_{\hat{P}}}$ is contained in a maximal cone of $\mathcal{N}(\mathcal{O}(P))$. Finally, if P is a graded poset, then the two fans agree; namely, $\mathcal{F}_{Q_{\hat{P}}}$ equals $\mathcal{N}(\mathcal{O}(P))$. The latter result also follows from the observation, found in [32, Remark 1.6], that for a graded poset P the polar dual of $\mathrm{Root}(Q_{\hat{P}})$ is a dilation of the order polytope $\mathcal{O}(P)$ up to shift.

Lemma 4.14. *Let P be an arbitrary finite poset. Then the rays of the face fan $\mathcal{F}_{Q_{\hat{P}}}$ of the root polytope $\mathrm{Root}(Q_{\hat{P}})$ of the starred quiver $Q_{\hat{P}}$ coincide with the rays of the (inner) normal fan $\mathcal{N}(\mathcal{O}(P))$ of the order polytope $\mathcal{O}(P)$; these rays are in bijection with arrows of $Q_{\hat{P}}$ or, equivalently, with cover relations in the Hasse diagram of $\hat{P}$.*

Proof. Let $P = \{v_1, \ldots, v_n\}$. It follows from Example 2.7, Definition 1.2 and Lemma 2.10 that the vertices of $\mathrm{Root}(Q_{\hat{P}})$, and hence the rays of $\mathcal{F}_{Q_{\hat{P}}}$, are in bijection with the cover relations in $\hat{P}$. Specifically,

- For each cover relation $v_i \lessdot v_j$, we have the vertex $e_j - e_i$ of $\mathrm{Root}(Q_{\hat{P}})$.
- For each cover relation $\hat{0} \lessdot v_j$, we have the vertex e_j.
- For each cover relation $v_i \lessdot \hat{1}$, we have the vertex $-e_i$.

Each vertex u of $\mathrm{Root}(Q_{\hat{P}})$ listed above gives rise to the ray $\overrightarrow{0u}$ from 0 to $e_j - e_i$ (or e_j or $-e_i$) in the face fan $\mathcal{F}_{Q_{\hat{P}}}$.

Meanwhile, the facet inequalities of $\mathcal{O}(P)$ are also in bijection with the cover relations in $\hat{P}$. If we identify $\mathbb{R}^P$ with $\mathbb{R}^n$ in the natural way, we get the following facet inequalities.

- For each cover relation $v_i \lessdot v_j$, we have the facet inequality $x_j - x_i \geq 0$. The linear functional w which takes the dot product with $(e_j - e_i)$ is minimized precisely at this facet.
- For each cover relation $\hat{0} \lessdot v_j$, we have the facet inequality $x_j \geq 0$. The linear functional w which takes the dot product with e_j is minimized precisely at this facet.
- For each cover relation $v_i \lessdot \hat{1}$, we have the facet inequality $1 - x_i \geq 0$. The linear functional w which takes the dot product with $-e_i$ is minimized precisely at this facet.

Therefore, we see that we have an identification of rays of the face fan $\mathcal{F}_{Q_{\hat{P}}}$ with rays of the (inner) normal fan $\mathcal{N}(\mathcal{O}(P))$. □

Theorem 4.15. *Let $P = \{v_1, \ldots, v_n\}$ be a finite ranked poset. Then the face fan $\mathcal{F}_{Q_{\hat{P}}}$ of the root polytope $\mathrm{Root}(Q_{\hat{P}})$ of the starred quiver $Q_{\hat{P}}$ refines the (inner) normal fan $\mathcal{N}(\mathcal{O}(P))$ of the order polytope $\mathcal{O}(P)$ of P: the two fans have the same set of rays, and each maximal cone of $\mathcal{N}(\mathcal{O}(P))$ is a union of maximal cones of $\mathcal{F}_{Q_{\hat{P}}}$. Moreover, if P is a graded poset, then $\mathcal{F}_{Q_{\hat{P}}}$ coincides with $\mathcal{N}(\mathcal{O}(P))$.*

We note that if P is not ranked, then the refinement statement of Theorem 4.15 may fail. Namely, the right-hand side image in Figure 9 shows an example where it fails.

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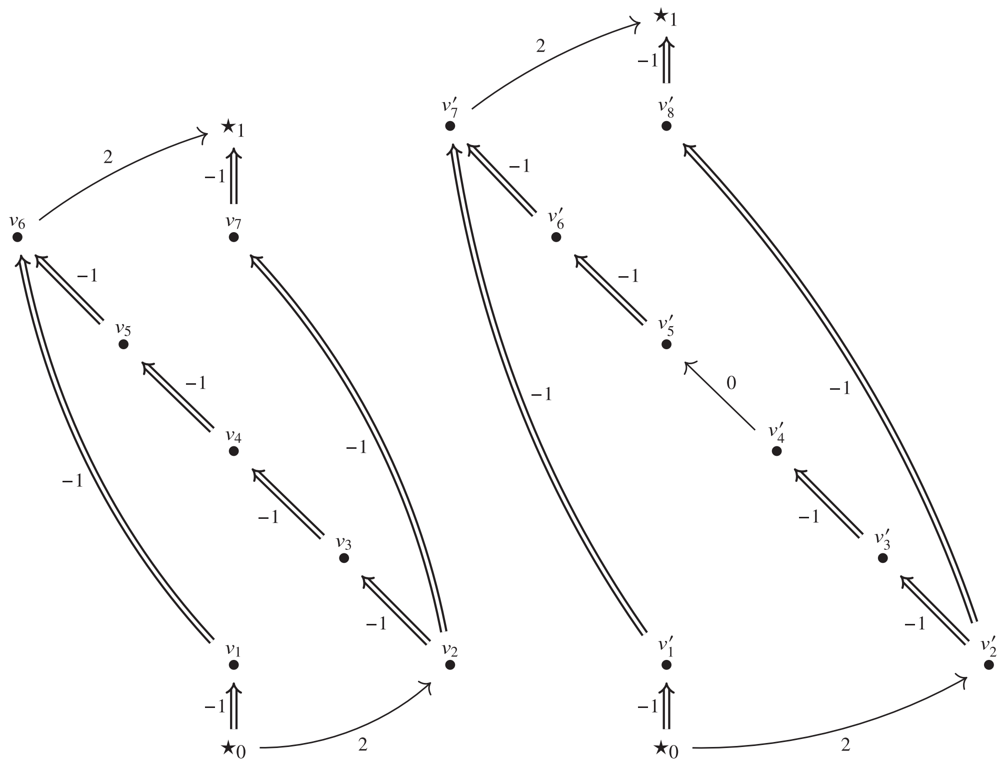

Figure 9. *The above quivers $Q_{\hat{P}}$ and $Q_{\hat{P}'}$ are constructed out of two posets, P and P', neither of which is ranked. At left: a facet arrow-labeling of $Q_{\hat{P}}$ for which the facet components C_0 of $\star_0$ and C_1 of $\star_1$ are not distinct. At right: a facet arrow-labeling of $Q_{\hat{P}'}$ whose facet component C_0 is not a filter; hence, the corresponding maximal cone of $\mathcal{F}_{Q_{\hat{P}'}}$ does not lie in a maximal cone of $\mathcal{N}(\mathcal{O}(P'))$. In both examples, the double arrows show the arrows labeled -1.*

Before proving the theorem, we first need the following lemma. Recall the facet arrow-labelings and corresponding $\bullet$-labelings from Section 2 that are used for describing the facets of root polytopes. Recall also the facet components associated to facet arrow-labelings in Definition 2.28

Lemma 4.16. *Suppose that $P = \{v_1, \dots, v_n\}$ is a ranked poset. Let M be a facet arrow-labeling of the starred quiver $Q_{\hat{P}}$, and L the corresponding $\bullet$-labeling. Consider the associated facet components C_0 and C_1 of $Q_{\hat{P}}$ containing $\star_0$ and $\star_1$, respectively.*

The facet components C_0 and C_1 are disjoint. The facet component C_0 consists of a set of vertices $S \sqcup \{\star_0\}$ along with all arrows between them, where the set S forms a lower order ideal in P. For the facet component C_1, the set of normal vertices in C_1 forms a filter in P – namely, $\{v_1, \dots, v_n\} \setminus S$.

Remark 4.17. Note that as the lemma says, the vertices of C_0 form a lower order ideal in $\hat{P}$, and the vertices of C_1 form a filter in $\hat{P}$. The lemma also says that C_0 is a full subquiver of $Q_{\hat{P}}$, though, while C_1 need not be. This reflects the asymmetry inherent in the definition of a ranked poset. Note also that C_0 and C_1 need not be distinct if P is not ranked. An example of this is shown in Figure 9 on the left-hand side.

Proof. Note that we can compute the vertex coordinates of all vertices of $Q_{\hat{P}}$ that lie in C_0 since the vertex $\star_0$ (which gets vertex-coordinate equal to 0) lies in C_0 by definition, and whenever two vertices are connected by an arrow in C_0, their vertex coordinates differ by -1.

Since P is ranked, we can use this to show for each vertex v_j in C_0 that $L(v_j) = -\operatorname{rank}(v_j)$. Namely, let $r = \operatorname{rank}(v_j)$, and choose a path in C_0 (not necessarily oriented) connecting $\star_0$ to v_j. Going from $\star_0$ to v_j along this path, if we traverse a total of s edges in the forward direction, then $s \geq r$ and we must also traverse a total of $(s - r)$ edges in the backward direction (to be able to reach the rank r element v_j). Computing vertex coordinates along the way, we get $L(v_j) = -s + (s - r) = -\operatorname{rank}(v_j)$. Note that this also implies that $\star_1$ cannot lie in C_0, since $L(\star_1) = 0$.

We can now deduce the description of C_0. Consider an element v_j of S, and suppose we have another element $v_\ell \in P$ with $v_\ell < v_j$. Note that $k := \operatorname{rank}(v_\ell) < \operatorname{rank}(v_j) = r$. Choose a maximal chain in $\hat{P}$ between $\hat{0}$ and v_ℓ. This maximal chain has $k + 1$ elements (including $\hat{0}$ and v_ℓ), and in the quiver $Q_{\hat{P}}$, it corresponds to a directed path from $\star_0$ to v_ℓ. We furthermore consider a maximal chain between v_ℓ and v_j and its associated directed path in $Q_{\hat{P}}$. The concatenation of the two paths gives a directed path in $Q_{\hat{P}}$ that goes from $\star_0$ to v_j, passes through v_ℓ, and has overall length r. For each arrow a of this path, we have an arrow label $M(a) \geq -1$, and the sum of the arrow labels equals to $L(v_j)$. Moreover, since $v_j \in S$, we know that $L(v_j) = -\operatorname{rank}(v_j) = -r$. Since our path is made up of precisely r arrows, to get the sum $-r$, we must have $M(a) = -1$ for each arrow a. Therefore, the entire path lies in the facet component C_0. In particular, it follows that v_ℓ lies in C_0. Since v_ℓ was in P, we now have $v_\ell \in S$. We see that S is an order ideal for P.

Now suppose v_i and v_j in S are connected by an arrow a, so $v_i \lessdot v_j$. Applying the argument above with $v_\ell = v_i$, we obtain a directed path in $Q_{\hat{P}}$ going from $\star_0$ to v_j and passing through v_i, that lies entirely in C_0. This path necessarily contains the arrow a and therefore a lies in C_0, as claimed.

Finally, let us consider the facet component C_1 of $\star_1$. We saw earlier that $\star_1$ cannot be in C_0, so C_0 and C_1 are distinct facet components. Moreover, by Lemma 2.29, each facet component of $Q_{\hat{P}}$ contains some starred vertex, so each vertex of $Q_{\hat{P}}$ must lie either in C_1 or in C_0. Since the vertices of S form a lower order ideal in P, the complementary set of vertices (namely, the normal vertices of C_1) must form a filter in P. ◻

Remark 4.18. If the poset P in Lemma 4.16 is not just ranked but also graded, then the proof in Lemma 4.16 extends to show that the facet component C_1 consists of the vertices $\{\star_1\} \cup \{v_1, \ldots, v_n\} \setminus S$ together with all arrows joining two elements in this set.

Proof of Theorem 4.15. The statement that the rays of the two fans coincide was already proved in Lemma 4.14. To show that each maximal cone of $\mathcal{N}(\mathcal{O}(P))$ is a union of maximal cones of $\mathcal{F}_{Q_{\hat{P}}}$, it suffices to show that each maximal cone of $\mathcal{F}_{Q_{\hat{P}}}$ is contained in a maximal cone of $\mathcal{N}(\mathcal{O}(P))$. Equivalently, we need to show that the rays of each maximal cone of $\mathcal{F}_{Q_{\hat{P}}}$ are a subset of the rays of some maximal cone of $\mathcal{N}(\mathcal{O}(P))$.

Each maximal cone c_M of $\mathcal{F}_{Q_{\hat{P}}}$ comes from a facet arrow-labeling M of $Q_{\hat{P}}$, and the rays in c_M are indexed by those arrows $a \in \operatorname{Arr}(Q_{\hat{P}})$ for which $M(a) = -1$. These are the arrows appearing in the facet components of $Q_{\hat{P}}$ (with respect to M).

By Lemma 2.29, each facet component of $Q_{\hat{P}}$ contains some starred vertex, so the vertices of $Q_{\hat{P}}$ lie in either C_1, the facet component of $\star_1$, or C_0, the facet component of $\star_0$. By Lemma 4.16, the vertices of the facet component C_0 form a lower-order ideal of P, and the vertices of C_1 form a filter of P. Therefore, each arrow indexing a ray in c_M – which is by definition an arrow appearing in a facet components of $Q_{\hat{P}}$ – connects either two vertices in the filter I or two vertices in the complement of the filter $\{v_1, \ldots, v_n\} \setminus I$. By Remark 4.18, if P is graded, then the arrows appearing in C_i (for $i = 0$ or 1) comprise *all* arrows connecting two vertices in C_i.

Meanwhile, each maximal cone c'_I of $\mathcal{N}(\mathcal{O}(P))$ corresponds to a vertex of $\mathcal{O}(P)$, which is the characteristic function χ_I of a filter I of P. The rays in c'_I correspond to the edges $v_i \lessdot v_j$ of the Hasse diagram of $\hat{P}$, where $\chi_I(v_i) = \chi_I(v_j)$. Therefore, each ray of c'_I corresponds to an arrow in $\operatorname{Arr}(Q_{\hat{P}})$ which connects either two vertices in the filter I or two vertices in the complement of the filter $\{v_1, \ldots, v_n\} \setminus I$.

This shows that the set of rays in a maximal cone c_M of $\mathcal{F}_{Q_{\hat{P}}}$ are a subset of the rays of a corresponding maximal cone c'_I of $\mathcal{N}(\mathcal{O}(P))$, with equality when P is graded. ◻

If we combine Theorem 4.15 and Theorem 2.31, we obtain the following result, which also follows from [32, Remark 1.6].

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Corollary 4.19. *Let P be a graded finite poset. Then $\mathcal{N}(\mathcal{O}(P))$ coincides with $\mathcal{F}_{Q_{\hat{P}}}$. It follows that the order polytope $\mathcal{O}(P)$ is combinatorially equivalent to the polar dual of the root polytope $Q_{\hat{P}}$.*

5. Applications in mirror symmetry and toric geometry

In this final section, we start by explaining how our previous results are related to mirror symmetry and toric geometry. We show that when Q is a strongly-connected starred quiver, the toric variety associated to the fan $\mathcal{F}_Q$ has a small toric desingularisation. We also apply our constructions to describe the Picard group for toric varieties arising from quivers, with a particular focus on the quivers coming from ranked posets.

5.1. Quiver Laurent polynomials, Newton polytopes and superpotential polytopes

In the study of mirror symmetry for Fano varieties such as partial flag varieties, quiver flag varieties and Grassmannians, there are naturally associated Laurent polynomial *superpotentials* which have the form

$$S(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}][q_1, \ldots, q_d]. \tag{5.1}$$

Given such a superpotential, there are two polytopes that one can associate: the *Newton polytope* and the *superpotential polytope*.

Definition 5.1 (Newton polytope). The *Newton polytope* $\mathrm{Newt}_S \subset \mathbb{R}^n$ is the convex hull of the exponent vectors of the Laurent monomials in $S(x_1, \ldots, x_n)$, where the exponent vector of $c(q_1, \ldots, q_d)x_1^{a_1} \ldots x_n^{a_n}$ for $c(q_1, \ldots, q_d) \in \mathbb{C}[q_1, \ldots, q_d]$ is $(a_1, \ldots, a_n)$.

The above definition defines a polytope as the convex hull of points, one for each Laurent monomial in $S(x_1, \ldots, x_n)$. However, we can define a different polytope – the superpotential polytope – by tropicalizing the superpotential. This cuts out a polytope that lies in the dual space by inequalities associated to the Laurent monomials in $S(x_1, \ldots, x_n)$. Our terminology is following [60].

Definition 5.2 (Superpotential polytope). = Let $S(x_1, \ldots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}][q_1, \ldots, q_d]$ be a Laurent polynomial with positive coefficients, and choose real numbers $\mathbf{r} = (r_1, \ldots, r_d)$. Set $\mathrm{Trop}(x_i) = X_i$ and $\mathrm{Trop}(q_i) = r_i$. We now inductively define the tropicalization $\mathrm{Trop}(S(x_1, \ldots, x_n))$ of $S(x_1, \ldots, x_n)$ by requiring that if $\mathbf{h_1}, \mathbf{h_2} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}][q_1, \ldots, q_d]$ are Laurent polynomials with positive coefficients and $a_1, a_2 \in \mathbb{R}_{>0}$, then

$$\mathrm{Trop}(a_1\mathbf{h_1} + a_2\mathbf{h_2}) = \min(\mathrm{Trop}(\mathbf{h_1}), \mathrm{Trop}(\mathbf{h_2})) \quad \text{and} \quad \mathrm{Trop}(\mathbf{h_1}\mathbf{h_2}) = \mathrm{Trop}(\mathbf{h_1}) + \mathrm{Trop}(\mathbf{h_2}).$$

We define the *superpotential polytope* $\Gamma_S^{\mathbf{r}} = \Gamma_S \subset \mathbb{R}^n$ by $\mathrm{Trop}(S(x_1, \ldots, x_n)) \geq 0$. In other words, we impose one inequality

$$(\sum_{i=1}^{d} \ell_i r_i) + (\sum_{j=1}^{n} m_j X_j) \geq 0$$

for each Laurent monomial summand $\prod_{i=1}^d q_i^{\ell_i} \prod_{j=1}^n x_j^{m_j}$ of $S(x_1, \ldots, x_n)$.

Example 5.3. Let

$$S(x_1, \ldots, x_6) = x_1 + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{x_4}{x_3} + \frac{q_1}{x_1} + \frac{x_5}{x_1} + \frac{x_6}{x_2} + \frac{x_6}{x_5} + \frac{q_2}{x_6}.$$

Then the Newton polytope is

$$\mathrm{Newt}_S = \mathrm{Conv}(e_1, e_2 - e_1, e_3 - e_2, e_4 - e_3, -e_1, e_5 - e_1, e_6 - e_2, e_6 - e_5, -e_6) \subset \mathbb{R}^6.$$

$$\begin{tikzcd} \star 1 \arrow[dr] & & & & & \\ & v_1 \arrow[r] \arrow[d] & v_2 \arrow[r] \arrow[d] & v_3 \arrow[r] & v_4 \arrow[r] & \star q_1 \\ & v_5 \arrow[r] & v_6 \arrow[r] & \star q_2 & & \end{tikzcd} \qquad S_Q = x_1 + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{x_4}{x_3} + \frac{q_1}{x_1} + \frac{x_5}{x_1} + \frac{x_6}{x_2} + \frac{x_6}{x_5} + \frac{q_2}{x_6}.$$

Figure 10. *The starred quiver Q and the Laurent polynomial S_Q. This Laurent polynomial is associated to the Schubert variety $X_{(4,2)}$ in the Grassmannian $Gr_2(\mathbb{C}^8)$ [61].*

Meanwhile, if we let $\mathbf{r} = (r_1, r_2) = (1, 1)$, then the superpotential polytope $\Gamma_S^{\mathbf{r}} \subset \mathbb{R}^6$ is cut out by the inequalities

$$\begin{array}{lll} X_1 \geq 0 & X_2 - X_1 \geq 0 & X_3 - X_2 \geq 0 \\ X_4 - X_3 \geq 0 & 1 - X_1 \geq 0 & X_5 - X_1 \geq 0 \\ X_6 - X_2 \geq 0 & X_6 - X_5 \geq 0 & 1 - X_6 \geq 0. \end{array}$$

In the context of mirror symmetry for the Fano varieties mentioned at the start of this section, many of the associated superpotentials can be read off from a strongly-connected starred quiver Q (cf. Definition 1.4). In particular, such a quiver Q gives rise to the Laurent polynomial

$$S_Q(x_1, \dots, x_n) := \sum_{a: v_i \to v_j} \frac{x_j}{x_i} + \sum_{a: \star_i \to v_j} \frac{1}{\mathrm{wt}(\star_i)} \cdot x_j + \sum_{a: v_i \to \star_j} \mathrm{wt}(\star_j) \cdot \frac{1}{x_i}, \tag{5.2}$$

where we sometimes refer to $\mathrm{wt}(\star_i)$ as a 'quantum parameter' and write $\mathrm{wt}(\star_i) = q_i$. Note that the above expression consists of one 'head over tail' Laurent monomial for each arrow of Q, with a coordinate x_i associated to each $\bullet$-vertex v_i. As an example, the superpotential from Example 5.3 comes from the quiver in Figure 10.

5.2. The toric variety $Y(\mathcal{F}_Q)$ and connections with mirror symmetry

Let us consider the Laurent polynomial $S_Q(x_1, \dots, x_n)$ coming from a starred quiver Q. Note that the Newton polytope Newt_{S_Q} of the quiver Laurent polynomial $S_Q(x_1, \dots, x_n)$ is precisely the root polytope $\mathrm{Root}(Q)$ described in Definition 1.2. Therefore, when Q is strongly-connected, we know from Theorem A that $\mathrm{Newt}_{S_Q} = \mathrm{Root}(Q)$ is reflexive and terminal. Letting $\mathcal{F}_Q$ denote the face fan of $\mathrm{Root}(Q)$, we can now associate the Gorenstein Fano toric variety $Y(\mathcal{F}_Q)$ to S_Q, and we call S_Q the mirror superpotential for $Y(\mathcal{F}_Q)$. Note that $Y(\mathcal{F}_Q)$ is not quite smooth in general, since it may have terminal singularities, and not all aspects of mirror symmetry make sense directly for $Y(\mathcal{F}_Q)$.

In the smooth case, study of the Laurent polynomial mirrors for toric Fano manifolds Y and their properties (relations to I-functions, quantum cohomology) goes back to [7, 8, 24, 26, 25, 56, 37]. More generally, if X is a smooth Fano variety with a suitable flat toric degeneration whose central fiber is $X_0 = Y(\mathcal{F}_Q)$, then S_Q may also encapsulate Gromov-Witten invariants ('quantum periods') of X; see [42, Section 4]. See also Example 5.12 and [21, Example 5.7], and more generally, [28, Conjecture 3.3 and Remark 3.4]. We note also that S_Q is automatically 'maximally mutable'; see [14, Section 5].

An early example of quiver Laurent polynomial mirrors was Givental's quiver Laurent polynomial superpotential for the full flag variety [27] and [9, 10], which gave partial flag variety analogues and related the construction to toric degenerations. These examples have also been extended to some homogeneous spaces in other Lie types, such as quadrics and maximal orthogonal Grassmannians [57, 58, 64, 65, 66]. Also Kalashnikov [40] constructed Laurent polynomial mirrors for certain quiver flag varieties using toric degenerations.

Most recently, we constructed mirrors for Grassmannian Schubert varieties that restrict to quiver-Laurent polynomials on an appropriate chart [61]; Figure 10 shows an example. Though Grassmannian

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Schubert varieties are not smooth, for the calculation of quantum periods of smooth Calabi-Yau subvarieties of a variety X, the smoothness assumption on X can be relaxed. See [51], which used Laurent polynomial mirrors of the form S_Q to study mirror symmetry for Calabi-Yau 3-folds in Gorenstein Fano Schubert varieties.

On the symplectic side, if there is a symplectic Fano manifold X with a degeneration to $Y(\mathcal{F}_Q)$ satisfying certain technical conditions, then the Laurent polynomial S_Q obtains a Floer theoretic interpretation; see [54, Theorem 1] and [12, Theorem 4.4]. Note that [54, Theorem 1] requires the existence of a small resolution of the central toric fiber. Our first result in this section will be to prove the existence of such a resolution for $Y(\mathcal{F}_Q)$; see Theorem 5.4. In this symplectic context, the existence of a degeneration to a $Y(\mathcal{F}_Q)$ (and the resulting Floer-theoretic interpretation of S_Q) then implies the existence of a non-displaceable Lagrangian torus in X; see [54, Corollary 2].

5.3. A small resolution of the toric variety $Y(\mathcal{F}_Q)$

In this section, we show that when Q is a strongly-connected starred quiver, we can resolve the singularities of $Y(\mathcal{F}_Q)$ to get a smooth toric variety without changing the rays of the fan. In other words, there is a *small* toric desingularisation of $Y(\mathcal{F}_Q)$. Such a resolution was constructed explicitly for a specific example in [10, Section 3]. We deduce its existence in our more general setting by making use of the *maximal projective crepant partial (MPCP)* desingularisation of [8], which relies on a construction from [22].

Theorem 5.4. *Consider any strongly-connected starred quiver Q. Let $\mathcal{F}_Q$ be the face fan of the root polytope* $\mathrm{Root}(Q)$. *There exists a refinement $\widehat{\mathcal{F}}_Q$ of $\mathcal{F}_Q$ such that $Y(\widehat{\mathcal{F}}_Q) \to Y(\mathcal{F}_Q)$ is a small crepant toric desingularisation.*

Proof. By Corollary 5.8, we know that $Y(\mathcal{F}_Q)$ is Gorenstein with at most terminal singularities. By [8, Theorem 2.2.24], any Gorenstein toric variety has a MPCP desingularization, which will be terminal and $\mathbb{Q}$-factorial. Since $Y(\mathcal{F}_Q)$ was already terminal, it follows that any crepant partial resolution of $Y(\mathcal{F}_Q)$ is necessarily small. Indeed, there is no way to add new rays and stay crepant if the faces of the fan polytope have no interior lattice points. Now we have a new fan $\widehat{\mathcal{F}}_Q$ which is simplicial, refines $\mathcal{F}_Q$, and has the same rays as $\mathcal{F}_Q$. Thus, the primitive ray generators are vertices u_a of the root polytope $\mathrm{Root}(Q)$. Consider now a maximal cone C of $\widehat{\mathcal{F}}_Q$, and let $\{u_a \mid a \in \mathrm{Arr}_C\}$ denote the set of primitive generators of the rays belonging to C. The convex hull of $\{u_a \mid a \in \mathrm{Arr}_C\} \cup \{0\}$ is a simplex. Consider the quiver Q' obtained from Q by removing the arrows not belonging to Arr_C. The associated matrix whose row vectors are the vertices of $\mathrm{Root}(Q')$ is square (since the cone C was simplicial) and totally unimodular; see Remark 2.33. Therefore, its determinant must be ± 1 or 0. However, it cannot be 0 since the cone was full-dimensional. Therefore, it must be ± 1, which implies that the cone C is regular. It follows that $Y(\widehat{\mathcal{F}}_Q)$ is smooth. □

Recall that a lattice polytope $\mathbf{P} \in \mathbb{R}^n$ is said to have the *integer decomposition property (IDP)* if for any $k \in \mathbb{Z}_{>0}$ and any element $v \in (k\mathbf{P}) \cap \mathbb{Z}^n$, the element v can be represented as the sum of k lattice points from $\mathbf{P}$.

Corollary 5.5. *For any strongly-connected starred quiver Q, the polytope* $\mathrm{Root}(Q)$ *has a triangulation into unimodular simplices and satisfies the integer decomposition property.*

Proof. A triangulation of $\mathrm{Root}(Q)$ into unimodular simplices arises in the proof of Theorem 5.4. Namely, for any cone of $\widehat{\mathcal{F}}_Q$, the convex hull of the primitive generators of the rays together with $\mathbf{0}$ gives a unimodular simplex. These form the simplices of the triangulation. Each of these has the IDP, and the second statement also follows. □

Remark 5.6. In fact, it is possible to show in another way that the polytope $\mathrm{Root}(Q)$ has a unimodular triangulation (and thus the IDP) whenever Q contains an oriented cycle or an oriented path π from a starred vertex to a starred vertex. Namely, in this case, the origin $\mathbf{0}$ lies in $\mathrm{Root}(Q)$ (since $\sum_{a\in\pi} u_a = \mathbf{0}$), and whenever $\mathrm{Root}(Q)$ contains the origin, it has a unimodular triangulation by [30, Theorem 3.9].

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Remark 5.7. All superpotential polytopes (see Definition 5.2) have the integer decomposition property. The normal vectors to the facets of a superpotential polytope are the vertices of the root polytope, and the matrix encoding the vertices of the root polytope is totally unimodular; hence, we can apply [30, Theorem 2.4]. This says that each superpotential polytope has a regular unimodular triangulation.

5.4. The toric variety $Y(\mathcal{F}_Q)$ when Q is planar or comes from a ranked poset

The relationship between quiver Laurent polynomials S_Q and toric geometry is particularly beautiful when the quiver Q is planar or when it comes from a ranked poset, as we will explain; the quivers we study in [61] are both.

5.4.1. When Q is planar

When the quiver Q is planar (as it is in [61]), we get an interpretation of the toric variety $Y(\mathcal{F}_Q)$ as a *toric quiver variety* for the planar dual quiver $Q^\vee$ (i.e., a *quiver moduli space* [43] for $Q^\vee$ parameterizing representations of $Q^\vee$ with dimension vector $(1, 1, \dots, 1)$). To see this, recall that the projective variety associated to the flow polytope $\mathrm{Fl}_{Q^\vee}$ for $Q^\vee$ is the quiver moduli space (for the canonical weight) associated to $Q^\vee$ [3]. This is also the toric variety associated to the normal fan $\mathcal{N}(\mathrm{Fl}_{Q^\vee})$. Since $\mathcal{F}_Q$ agrees with $\mathcal{N}(\mathrm{Fl}_{Q^\vee})$ (by Theorem B), we obtain an interpretation of the toric variety $Y(\mathcal{F}_Q)$ as a quiver moduli space for $Q^\vee$. These varieties have been studied extensively; see [34, 35, 1, 38, 2, 17, 53] as well as in [40, 13] and [52, 50] where plane quivers and planar duality play a role.

5.4.2. When Q comes from a ranked poset

Suppose that Q comes from the bounded extension $\hat{P}$ of a poset P (as in Definition 1.6), and we let the weight of the starred vertices associated to $\hat{1}$ and $\hat{0}$ be q and 1, respectively. Then the superpotential polytope $\Gamma_Q^{\mathbf{r}} := \Gamma_{S_Q}^{\mathbf{r}}$ with $\mathbf{r} = (1)$, let us denote it Γ_Q, agrees with the order polytope $\mathcal{O}(P)$ of P. For example, the quiver from Figure 10 can be identified with the Hasse diagram of a poset P on $\{v_1, \dots, v_6\} \cup \{\hat{0}, \hat{1}\}$, where the starred vertex labeled 1 is identified with $\hat{0}$, the starred vertices labeled q_1, q_2 are identified with $\hat{1}$, and the arrows point from smaller to larger elements in the poset, and the superpotential polytope Γ_Q is exactly the order polytope of P; see Example 5.3. The fact that the superpotential polytope agrees with the order polytope allows one to deduce nice properties of Γ_Q (e.g., its volume is the number of linear extensions of the poset [68]).

Given that there are two polytopes that we can associate to the quiver Laurent polynomial S_Q – namely, the Newton polytope $\mathrm{Newt}_{S_Q} = \mathrm{Root}(Q)$ of S_Q and the superpotential polytope Γ_Q – it is natural to ask how these two polytopes are related. The answer is provided by our Theorem D (see also Theorem C): when Q comes from a ranked poset P, the face fan $\mathcal{F}_Q$ of $\mathrm{Root}(Q)$ refines the (inner) normal fan $\mathcal{N}(\Gamma_Q) = \mathcal{N}(\mathcal{O}(P))$ while preserving the rays, and when P is graded, we get equality. It follows that when P is ranked, the toric variety $Y(\mathcal{F}_Q)$ of $\mathcal{F}_Q$ provides a small partial desingularization of the toric variety $Y(\mathcal{N}(\Gamma_Q))$ associated to the normal fan of the superpotential polytope; see Figure 11 for a summary. Note that $Y(\mathcal{N}(\Gamma_Q)) = Y(\mathcal{N}(\mathcal{O}(P)))$ is the projective toric variety associated to $\mathcal{O}(P)$, also denoted $Y_{\mathcal{O}(P)}$, and is commonly known as a *Hibi toric variety* as it agrees with the toric variety introduced in [31].

We now show how to describe the Picard group of the toric variety $Y(\mathcal{F}_Q)$, especially in the case where the starred quiver Q comes from a ranked poset.

5.5. The Picard group $\mathrm{Pic}(Y(\mathcal{F}_Q))$ *and a canonical extension of a ranked poset*

Recall that $Y(\mathcal{F}_Q)$ denotes the toric variety associated to $\mathcal{F}_Q$, the face fan of $\mathrm{Root}(Q)$. We now consider the case where Q is the starred quiver associated to an extension of the Hasse diagram of a ranked poset P. In this section, we first give a general algorithm for determining the Cartier divisors of $Y(\mathcal{F}_Q)$, and hence the Picard group, for arbitrary strongly-connected starred quivers Q. We then introduce a new *canonical extension* of the ranked poset P. Finally, we use the canonical extension to describe concretely the Picard group of $Y(\mathcal{F}_Q)$ whenever Q comes from a ranked poset.

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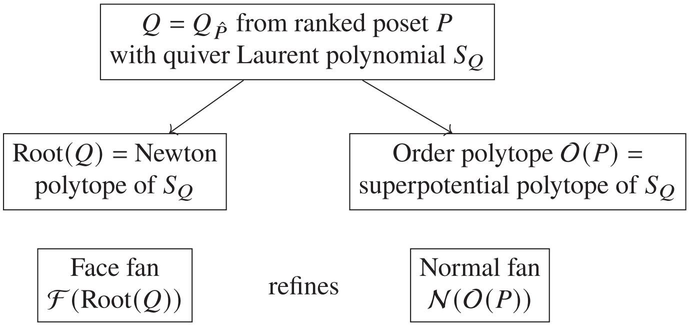

Figure 11. *How the root polytope and order polytope are related to the Laurent polynomial superpotential S_Q of the quiver associated to a ranked poset.*

5.5.1. Cartier divisors in $Y(\mathcal{F}_Q)$ for arbitrary strongly-connected starred quivers Q

By Theorem A and Corollary 2.27, we have the following result.

Corollary 5.8. *Let $\mathcal{F}_Q$ be the face fan of* $\mathrm{Root}(Q)$ *for a starred quiver Q. If Q is strongly-connected, then the toric variety $Y(\mathcal{F}_Q)$ associated to the fan $\mathcal{F}_Q$ is terminal Gorenstein. Moreover, the maximal cones of $\mathcal{F}_Q$ are in bijection with facet arrow-labelings of Q, where the maximal cone associated to a facet arrow-labeling M is the cone spanned by*

$$\{u_a \mid M(a) = -1\}.$$

Note that the facet arrow-labelings are in bijection with the torus-fixed points of $Y(\mathcal{F}_Q)$.

Lemma 5.9. *For any subset S of* $\mathrm{Arr}(Q)$, *let $\Pi(S)$ be the set of paths π with support S in the underlying graph of Q such that either π is a closed path or a path whose endpoints are starred vertices. The linear relations between the vectors $\{u_a \mid a \in S\}$ are generated by relations of the form*

$$\sum_{a\in\pi} \varepsilon(a)u_a = 0,$$

where $\pi \in \Pi(S)$ and the sign $\varepsilon(a) = \pm 1$ is defined to be $+1$ if the orientation of the arrow a agrees with the orientation of the path π, and -1 if the orientations are opposite.

Lemma 5.9 is a straightforward consequence of the definition of the vectors u_a. Cartier divisors on $Y(\mathcal{F}_Q)$ can now be described as follows.

Proposition 5.10. *We use the notation of Lemma 5.9. Let $a \in$* $\mathrm{Arr}(Q)$, *and let D_a denote the Weil divisor of $Y(\mathcal{F}_Q)$ associated to the ray of $\mathcal{F}_Q$ spanned by u_a. The Weil divisor $\sum c_a D_a$ is Cartier if and only if*

$$\sum_{a\in\pi} \varepsilon(a)c_a = 0 \tag{5.3}$$

whenever $\pi \in \Pi(S)$, where $S = F(M)$ for M a facet arrow-labeling of Q.

Proof. Let $\mathbf{N}_\mathbb{R}$ denote the vector space containing $\mathrm{Root}(Q)$. Recall that a Weil divisor $\sum c_a D_a$ is Cartier if and only if for each maximal cone σ of the fan $\mathcal{F}_Q$, the function f_σ on primitive generators u_a of rays ρ_a in σ given by $f_\sigma(u_a) = c_a$ extends to a linear map on $\mathbf{N}_\mathbb{R}$. Recall that we have identified the maximal cones in terms of facet arrow-labelings in Corollary 5.8. Let M be a facet arrow-labeling, and let $\{u_a \mid M(a) = -1\}$ be the set of primitive vectors corresponding to rays of the associated maximal cone. Since $\mathrm{Root}(Q)$ is full-dimensional, the maximal cone σ is also full-dimensional and we have that $\{u_a \mid M(a) = -1\}$ spans $\mathbf{N}_\mathbb{R}$. Recalling that $S := \{a \in \mathrm{Arr}(Q) \mid M(a) = -1\}$, let K be the kernel of the linear map $\mathbb{R}^S \to \mathbf{N}_\mathbb{R}$ which sends the standard basis element $e_a \in \mathbb{R}^S$ to u_a. Then $\mathbf{N}_\mathbb{R}$ is identified with

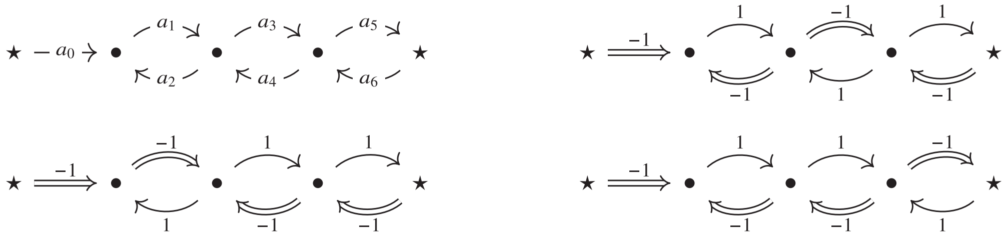

Figure 12. *A starred quiver Q and three of its facet arrow-labelings.*

the quotient $\mathbb{R}^S/K$, and the Weil divisor $\sum c_a D_a$ is Cartier if and only if the linear map $\mathbb{R}^S \to \mathbb{R}$ that takes e_a to c_a is well defined on this quotient. By Lemma 5.9 applied to S, this is the case precisely if the condition (5.3) holds for all $\pi \in \Pi(S)$. □

Remark 5.11. Recall the definition of $\mathbf{N}_Q$ and $\mathbf{N}_{Q,\mathbb{R}}$ and their duals $\mathbf{M}_Q$ and $\mathbf{M}_{Q,\mathbb{R}}$; see Definition 3.6 and Definition 2.17. By Proposition 3.8, we may replace $\mathrm{Root}(Q)$ with the integrally equivalent $\widetilde{\mathrm{Root}}(Q)$ and consider $\mathcal{F}_Q$ as lying in $\mathbf{N}_{Q,\mathbb{R}}$. Thus, $\mathbf{M}_Q$ has an interpretation as the character lattice of the torus acting on $Y(\mathcal{F}_Q)$. Moreover, we may identify $\mathbb{Z}^{\mathrm{Arr}(Q)}$ with the group of torus invariant Weil divisors of $Y(\mathcal{F}_Q)$. The map $\mathbf{M}_Q \to \mathbb{Z}^{\mathrm{Arr}(Q)}$ sending a 0-sum arrow labeling M to the divisor $\sum_{a\in\mathrm{Arr}(Q)} M(a)D_a$ naturally identifies the group of 0-sum arrow labelings $\mathbf{M}_Q$ with the group of torus-invariant principal divisors (the divisors of zeros and poles associated to characters). The class group $\mathrm{Cl}(Y(\mathcal{F}_Q))$ of the toric variety $Y(\mathcal{F}_Q)$ is therefore given by $\mathbb{Z}^{\mathrm{Arr}(Q)}/\mathbf{M}_Q$; see [16, Chapter 4]. The Picard group is the subgroup of $\mathrm{Cl}(Y(\mathcal{F}_Q))$ generated by the Cartier divisor classes. In this way, Proposition 5.10 allows for computation of the Picard group of the toric variety $Y(\mathcal{F}_Q)$.

Example 5.12. Consider the strongly connected starred quiver Q shown in Figure 12. The associated toric variety $Y(\mathcal{F}_Q)$ is a terminal Fano 3-fold with three singular points – namely, corresponding to the maximal cones given by the facet labelings shown in Figure 12. Recall that an element $(c_a) \in \mathbb{Z}^{\mathrm{Arr}(Q)}$ represents a Weil divisor $\sum c_a D_a$ of $Y(\mathcal{F}_Q)$, and the lattice of Weil divisors linearly equivalent to 0 is given by $\mathbf{M}_Q \subset \mathbb{Z}^{\mathrm{Arr}(Q)}$, as explained in Remark 5.11. By Proposition 5.10, the toric Cartier divisors are the divisors $\sum_{i=0}^{6} c_i D_{a_i}$ whose coefficients satisfy

$$c_0 + c_1 = c_4 + c_6, \quad c_0 + c_3 = c_2 + c_6, \quad c_0 + c_5 = c_2 + c_4.$$

Modulo linear equivalence we obtain the Picard group of $Y(\mathcal{F}_Q)$, which turns to be rank 1 with generator

$$[D_{a_1} + D_{a_3} + D_{a_5} - D_{a_0}] = [D_{a_2} + D_{a_4} + D_{a_6} + 2D_{a_0}].$$

We see directly that the toric boundary divisor $\sum_{i=0}^{6} D_{a_i}$ is Cartier and, moreover, $-K_{Y(\mathcal{F}_Q)} = [\sum_{i=0}^{6} D_{a_i}] = 2[D_{a_1} + D_{a_3} + D_{a_5} - D_{a_0}]$, showing that $Y(\mathcal{F}_Q)$ has Fano index 2. Additionally, $Y(\mathcal{F}_Q)$ has a small resolution $Y(\widehat{\mathcal{F}}_Q)$ by Theorem 5.4 and a Laurent polynomial superpotential which is read off the quiver Q with coordinates x_1, x_2, x_3 corresponding to the •-vertices v_1, v_2, v_3 in order:

$$S_Q(x_1, x_2, x_3) = x_1 + \frac{x_1}{x_2} + \frac{x_2}{x_3} + x_3 + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{1}{x_3}.$$

Let us change coordinates via $x_1 = xyz, x_2 = yz, x_3 = z$. Then S_Q agrees precisely with the Laurent polynomial

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

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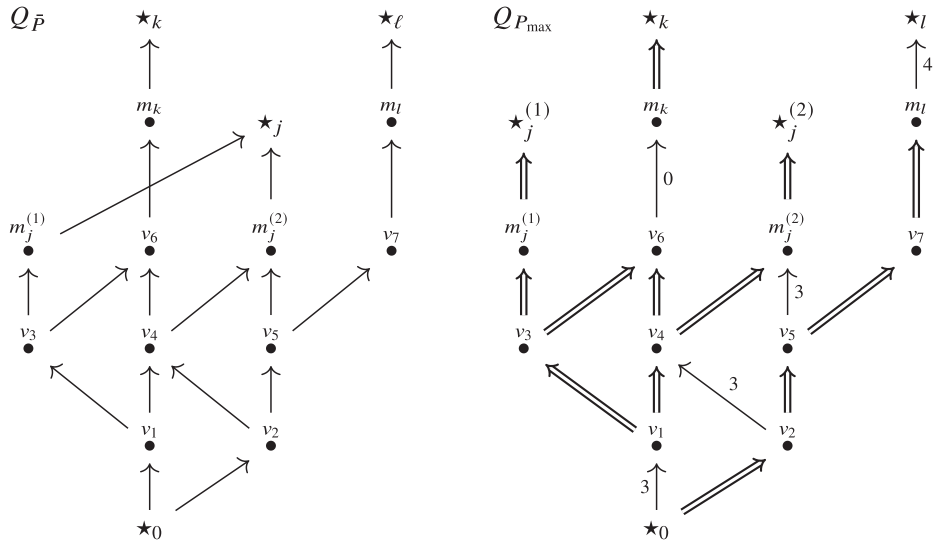

Figure 13. *To the left, we have a quiver $Q_{\bar{P}}$ illustrating the canonical extension of a ranked poset P. To the right, the quiver corresponding to $P_{\max}$ together with a facet arrow-labeling in bold, which connects the two vertices that are equivalent. The notations are as in the proof of Theorem 5.18. Note that $Q = Q_{\bar{P}}$ and $Q_{P_{\max}}$ have the same root polytopes and associated face fan $\mathcal{F}_Q$.*

discussed in [21, Example 5.7]. The constant term series of S_Q (the series in t made up of constant terms of $e^{tS_Q(x)}$) has a geometric interpretation in terms of different types of I-series associated to smooth anticanonical sections in $Y(\widehat{\mathcal{F}}_Q)$ and in a smoothing of $Y(\mathcal{F}_Q)$, see [21].

5.5.2. A canonical extension of a ranked poset

Suppose P is a ranked poset. Recall the bounded extension $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ of P. If P is only ranked but not graded, then $\hat{P}$ is no longer ranked. We may instead consider the naive ranked extension,

$$P_{\max} = P \cup \{\hat{0}\} \cup \{\hat{1}_m \mid m \in P \text{ maximal}\}, \tag{5.4}$$

that adds one element $\hat{1}_m$ above every maximal element m, together with the cover relation $\hat{1}_m \gtrdot m$. From the perspective of toric geometry, however, there is a natural extension of P that is intermediate between the bounded extension and the extension $P_{\max}$, which we now define.

Definition 5.13. We define the *canonical extension* $\bar{P}$ of a ranked poset P as follows. Consider the starred quiver $Q_{P_{\max}}$ associated to the ranked poset $P_{\max}$. We call two starred vertices equivalent if there exists a facet arrow-labeling of $Q_{P_{\max}}$ for which they are in the same facet component; see Figure 13. We define $\bar{P}$ to be the quotient poset of $P_{\max}$ obtained by identifying $\hat{1}_m$ with $\hat{1}_{m'}$ whenever the associated starred vertices are equivalent.

Let P be a ranked poset, and $\bar{P}$ its extension from Definition 5.13. Let $Q = Q_{\bar{P}} = (\mathcal{V}, \mathrm{Arr}(Q))$ be the starred quiver associated to $\bar{P}$. So $Q_{\bar{P}}$ has normal vertices $\mathcal{V}_\bullet = P$ and starred vertices $\mathcal{V}_\star = \{\star_0, \star_1, \ldots, \star_r\}$ associated to the elements of $\bar{P} \setminus P$. We assume $\star_0$ is the source vertex so that $\star_1, \ldots, \star_r$ are the sink vertices.

Lemma 5.14. *The poset $\bar{P}$ is ranked. Moreover, every facet-arrow labeling of $Q_{\bar{P}}$ has precisely $r+1$ facet components, one containing each starred vertex.*

Proof. Recall first that the poset $P_{\max}$ is a ranked poset. Fix a facet arrow-labeling M of $Q_{P_{\max}}$. The facet component of $\star_0$ contains no other starred vertex, as follows by the proof of Lemma 4.16. Now consider a facet component which contains two starred vertices $\star_i$ and $\star_j$. Then consider a non-oriented path in the facet component that connects the two starred vertices. We can work out the associated vertex labeling L_M on every vertex of the facet component starting with $L_M(\star_i) = 0$ using that L_M increases by 1 as we reverse along an arrow in the facet component, and decreases by 1 as we follow an arrow upward. In terms of the rank function, this implies that $L_M(v) = \mathrm{rank}(\hat{1}_i) - \mathrm{rank}(v)$ for any $v \in \bar{P}$ lying in the facet component of $\star_i$. Since $\star_j$ is assumed to lie in the same facet component as $\star_i$, and $L_M(\star_j) = 0$ as it is a starred vertex, we deduce that $\mathrm{rank}(\hat{1}_j) = \mathrm{rank}(\hat{1}_i)$.

We now have that $\bar{P}$ is ranked since the equivalence relation identifies only starred vertices that have the same rank.

Next we fix a facet arrow-labeling $\bar{M}$ of $Q_{\bar{P}}$. We have a natural bijection between the arrows of $Q_{\bar{P}}$ and those of $Q_{P_{\max}}$, and the labeling $\bar{M}$ transfers to a facet arrow-labeling M for $Q_{P_{\max}}$. If two starred vertices of $Q_{\bar{P}}$ are in the same facet component for the facet arrow labeling $\bar{M}$, then they have representatives in $P_{\max}$ that are in the same component for M. Therefore, actually the two starred vertices are identified in $\bar{P}$ by construction. It follows that the map from starred vertices of $Q_{\bar{P}}$ to facet components for $\bar{M}$ is injective. But this map is also surjective by Lemma 2.23. □

Remark 5.15. Note that while the quiver $Q = Q_{\bar{P}}$ depends on the chosen extension $\bar{P}$ of the poset P, the root polytope does not,

$$\mathrm{Root}(Q_{\bar{P}}) = \mathrm{Root}(Q_{P_{\max}}) = \mathrm{Root}(Q_{\hat{P}}),$$

so that the toric variety $Y(\mathcal{F}_Q)$ associated to its face fan really just depends on the poset P. The relation between our variety $Y(\mathcal{F}_Q)$ and the other toric variety naturally associated to the poset P – namely, the Hibi toric variety $Y_{\mathcal{O}(P)}$ – is that $Y(\mathcal{F}_Q)$ is a small partial desingularisation of $Y_{\mathcal{O}(P)}$ whenever P is ranked, and if P is graded, the two varieties are isomorphic; see Section 5.4.2.

We will now describe the group of torus-invariant Cartier divisors and the Picard group for $Y(\mathcal{F}_Q)$. Recall that an element $(c_a) \in \mathbb{Z}^{\mathrm{Arr}(Q)}$ represents a Weil divisor $\sum c_a D_a$ of $Y(\mathcal{F}_Q)$, and the lattice of Weil divisors linearly equivalent to 0 is given by $\mathbf{M}_Q \subset \mathbb{Z}^{\mathrm{Arr}(Q)}$, the sublattice of integer 0-sum arrow-labelings of Q from Definition 2.17. We make the following definition.

Definition 5.16. Suppose Q is a strongly-connected starred quiver. Let $M : \mathrm{Arr}(Q) \to \mathbb{R}$ be an arrow labeling of Q. We call M an *independent-sum arrow labeling* if for any oriented path from one star vertex to another, the sum $\sum_{a\in\pi} M(a)$ depends only on the endpoints. We write $\mathbf{C}_Q$ for the lattice of $\mathbb{Z}$-valued independent-sum arrow labelings. We may consider $\mathbf{C}_Q$ as a sublattice of $\mathbb{Z}^{\mathrm{Arr}(Q)}$. Note that 0-sum arrow labelings are examples of independent-sum arrow labelings by Lemma 2.18, so that we have $\mathbf{M}_Q \subseteq \mathbf{C}_Q \subseteq \mathbb{Z}^{\mathrm{Arr}(Q)}$.

The following lemma about independent-sum arrow labelings is a generalisation of Lemma 2.18.

Lemma 5.17. *Given a strongly-connected starred quiver Q with vertices $\mathcal{V} = \mathcal{V}_\bullet \sqcup \mathcal{V}_\star$ and arrows* $\mathrm{Arr}(Q)$*, the lattice of independent-sum arrow labelings $\mathbf{C}_Q$ is the image of the map*

$$\begin{array}{ccc} \mathbb{Z}^{\mathcal{V}} & \to & \mathbb{Z}^{\mathrm{Arr}(Q)} \\ (\ell_v)_v & \mapsto & (c_a)_{a\in \mathrm{Arr}(Q)}, \end{array} \tag{5.5}$$

where $c_a = \ell_{h(a)} - \ell_{t(a)}$ for $a \in$ $\mathrm{Arr}(Q)$*. Here, $h(a)$ denotes the head of the arrow a and $t(a)$ the tail.*

Proof. It is immediate that the image of (5.5) consists of independent-sum arrow labeling. Therefore, it remains to show the reverse inclusion.

Suppose M is an independent-sum arrow labeling. Note that for an oriented path π from a starred vertex to itself, the sum $\sum_{a\in\pi} M(a)$ must equal to 0. Therefore, if there is only a single starred vertex, then any independent-sum arrow labeling is actually a 0-sum arrow labeling, and the lemma follows from Lemma 2.18. We can now proceed by induction on the number of starred vertices. Suppose we have $d+1$ starred vertices and the lemma holds whenever there are d starred vertices. Among all of the

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paths in the underlying graph of Q that start at one starred vertex and end at a different starred vertex, and traverse only normal vertices, pick a path π that has a *minimal* number of direction changes. We show indirectly that π must be an oriented path in Q. Namely, consider the subquiver Q' just consisting of the arrows of π. It has two starred vertices $\star_1 \neq \star_2$. If there is a direction change along π at a vertex v, then this vertex is a sink or a source for Q'. Say it is a sink. Since Q is strongly connected, then there is an oriented path in Q from v to a starred vertex $\star_3$. If we replace the part of π running from v to $\star_2$ by the new path to $\star_3$, then we have reduced the number of direction changes. If $\star_3$ happens to equal to $\star_1$, then we replace instead the part of π that runs from $\star_1$ to v by the new path from $\star_1 = \star_3$ to v, again reducing the number of direction changes. We argue analogously if v is a source. Therefore, we have proved that the path between distinct starred vertices with minimal number of direction changes is actually an *oriented* path; it has no direction changes.

By the argument above, there exist two starred vertices $\star_1 \neq \star_2$ that are connected by an oriented path. Let π be such an oriented path from $\star_1$ to $\star_2$. Since M is an independent-sum arrow labeling, the sum $m(\star_1, \star_2) := \sum_{a\in\pi} M(a)$ is well defined (independently of the choice of π).

We now construct a new quiver $\bar{Q}$ as follows. We add to Q an additional arrow – namely from $\star_2$ to $\star_1$ – and then we turn $\star_2$ into a normal vertex. This new quiver is still strongly connected, although it has one fewer starred vertex. We extend the arrow-labeling M to a labeling $\bar{M}$ for $\bar{Q}$ by labeling the new arrow with $-m(\star_1, \star_2)$. This is now an independent-sum arrow labeling for $\bar{Q}$, as is straightforward to check. We can apply the induction hypothesis to obtain a vertex labeling for $\bar{Q}$ that maps to the arrow labeling $\bar{M}$. This vertex labeling recovers all of the arrow labels of our original quiver Q and shows that M is in the image of the map (5.5). □

Theorem 5.18. *Let P be a ranked poset, let $\bar{P}$ be its canonical extension, and let $Q = Q_{\bar{P}}$ be the associated starred quiver, where $Q = (\mathcal{V}, \mathrm{Arr}(Q))$ with $\mathcal{V} = \mathcal{V}_\bullet \cup \mathcal{V}_\star$. Then the lattice $\mathbf{C}_Q$ of independent-sum arrow labelings agrees with the group of torus-invariant Cartier divisors of the toric variety $Y(\mathcal{F}_Q)$ associated to P. Moreover, we have a well-defined map $\mathbf{C}_Q \to \mathbb{Z}^{\mathcal{V}_\star\setminus\{\star_0\}}$ defined by*

$$M \longmapsto \left(\textstyle\sum_{a\in\pi_i} M(a)\right)_{\star_i\in\mathcal{V}_\star\setminus\{\star_0\}}, \tag{5.6}$$

where π_i is an/any oriented path from $\star_0$ to $\star_i$. The sequence

$$0 \longrightarrow \mathbf{M}_Q \longrightarrow \mathbf{C}_Q \longrightarrow \mathbb{Z}^{\mathcal{V}_\star\setminus\{\star_0\}} \longrightarrow 0 \tag{5.7}$$

is exact and identifies the Picard group $\mathrm{Pic}(Y(\mathcal{F}_Q))$ with $\mathbb{Z}^{\mathcal{V}_\star\setminus\{\star_0\}}$.

Remark 5.19. We can also give a direct 'vertex' description of the group of Cartier divisors. Namely, the map in (5.5) induces an isomorphism

$$\mathbb{Z}^{\mathcal{V}\setminus\{\star_0\}} \xrightarrow{\sim} \mathbf{C}_Q$$

sending (ℓ_v) to the arrow-labeling $M \in \mathbf{C}_Q$ with $M(a) := \ell_{h(a)}$ if $t(a) = \star_0$, and $M(a) = \ell_{h(a)} - \ell_{t(a)}$ otherwise. This corresponds to normalising $\ell_{\star_0}$ to 0 in Lemma 5.17 to get an isomorphism. The subgroup $\mathbf{M}_Q$ is then identified with $\mathbb{Z}^{\mathcal{V}_\bullet} \times \prod_{\mathcal{V}_\star\setminus\{\star_0\}}\{0\}$, the sublattice of $\mathbb{Z}^{\mathcal{V}\setminus\{\star_0\}}$ where all starred vertex coordinates $\ell_{\star_j} = 0$. The exact sequence (5.7) becomes simply

$$0 \longrightarrow \mathbb{Z}^{\mathcal{V}_\bullet} \times \prod_{\mathcal{V}_\star\setminus\{\star_0\}} \{0\} \longrightarrow \mathbb{Z}^{\mathcal{V}\setminus\{\star_0\}} \longrightarrow \mathbb{Z}^{\mathcal{V}_\star\setminus\{\star_0\}} \longrightarrow 0.$$

Therefore, from the quiver $Q_{\bar{P}}$ we can read off

- generators for the group of torus-invariant Weil divisors (arrows) of $Y(\mathcal{F}_Q)$,
- generators for the group of torus-invariant Cartier divisors (vertices not equal to $\star_0$),
- generators for the Picard group (starred vertices of $Q_{\bar{P}}$ not equal to $\star_0$).

The generator of the Picard group associated to a starred vertex $\star_j$ is represented by the Cartier divisor D_j given by $D_j = \sum_{a\to\star_j} D_a$, with the sum being over all arrows a with target $\star_j$. The Cartier divisor associated to a normal vertex v is in terms of Weil divisors given by the sum of the incoming arrows minus the sum of the outgoing arrows for v. Directly in terms of the poset $\bar{P}$, we also have

- generators for the group of Weil divisors (covering relations in $\bar{P}$),
- generators for the group of Cartier divisors (elements of $\bar{P} \setminus \{\hat{0}\}$),
- generators for the Picard group (maximal elements of $\bar{P}$).

Remark 5.20. Note that we can define $\mathbf{C}_Q$ for any quiver Q (possibly as the image of the map (5.5) if Q is not strongly connected), and the sequence (5.7) still makes sense. However, $\mathbf{C}_Q$ would not recover the group of Cartier divisors of $Y(\mathcal{F}_Q)$ in general. Even for Q arising from a ranked poset P, two different extensions of P give different $\mathbf{C}_Q$ and different sets of starred vertices, while the variety $Y(\mathcal{F}_Q)$ only depended on the original poset P. Thus, the specific construction of the canonical extension $\bar{P}$ of a ranked poset P is important for this result.

Remark 5.21. If we consider $Q_{\max} := Q_{P_{\max}}$ for the finite ranked poset P, the analogue of the construction from Theorem 5.18 gives us the class group for the same toric variety $Y(\mathcal{F}_Q)$. Namely, the group of torus-invariant Weil divisors is identified with $\mathbb{Z}^{\mathrm{Arr}(Q_{\max})}$, and we have the exact sequence

$$0 \longrightarrow \mathbf{M}_{Q_{\max}} \longrightarrow \mathbb{Z}^{\mathrm{Arr}(Q_{\max})} \longrightarrow \mathrm{Cl}(Y(\mathcal{F}_Q)) \longrightarrow 0.$$

The class group as the cokernel of the map $\mathbf{M}_{Q_{\max}} \longrightarrow \mathbb{Z}^{\mathrm{Arr}(Q_{\max})}$ is seen to be isomorphic to the group $\mathbb{Z}^d$, where d is the number of maximal elements in $P_{\max}$, equivalently in P. Thus, all in all, we have that generators of the class group of $Y(\mathcal{F}_Q)$ are in bijection with maximal elements of $P_{\max}$, and generators of the Picard group of $Y(\mathcal{F}_Q)$ with maximal elements of the canonical extension $\bar{P}$.

Proof of Theorem 5.18. Suppose $(c_a) \in \mathbf{C}_Q$. Using Lemma 5.17, choose an $(\ell_v)_v \in \mathbb{Z}^{\mathcal{V}}$ mapping to (c_a) under (5.5). From this presentation of $(c_a)_{a\in\mathrm{Arr}(Q)}$, it follows that $\sum c_a D_a$ is a Cartier divisor in $Y(\mathcal{F}_Q)$, as a direct consequence of the characterisation of Cartier divisors in Proposition 5.10. We now prove that all Cartier divisors lie in $\mathbf{C}_Q$.

Suppose $\sum_{a\in\mathrm{Arr}} c_a D_a$ is a fixed Cartier divisor for $Y(\mathcal{F}_Q)$. Pick an unoriented path π in $Q_{\bar{P}}$ as in Lemma 5.9 with $S = \mathrm{Arr}(Q)$. Suppose first that the path only involves vertices from $\mathcal{V}_\bullet \cup \{\star_0\}$. We construct a facet arrow-labeling for $Q_{\bar{P}}$ as follows. Define a vertex labeling L by $L(v) = -\mathrm{rank}(v)$ if $v \in \mathcal{V}_\bullet$, and $L(v) = 0$ if $v \in \mathcal{V}_\star$. The associated arrow-labeling M labels all arrows not pointing to a $\star$-vertex by -1, while an arrow pointing to a vertex $\star_j$ has label $M(a) = \mathrm{rank}(\star_j) - 1$. This is clearly a facet arrow-labeling. The facet component of $\star_0$ contains all of the arrows that point to normal vertices and, in particular, entirely contains the path π. It follows from the characterisation of Cartier divisors in Proposition 5.10 that for any such path π, the relation

$$\sum_{a\in\pi} \epsilon(a) c_a = 0 \tag{5.8}$$

holds. Given $v \in \mathcal{V}_\bullet$, pick an oriented path π_v from $\star_0$ to v, and define $\ell_v := \sum_{a\in\pi_v} c_a$. Also set $\ell_{\star_0} = 0$. It follows from (5.8) that the element $(\ell_v) \in \mathbb{Z}^{\mathcal{V}_\bullet\cup\{\star_0\}}$ is well defined independently of the paths chosen. Moreover, it determines c_a for any arrow a not pointing to a starred vertex by $c_a = \ell_{h(a)} - \ell_{t(a)}$.

Consider a sink vertex $\star_j$, and pick an arrow pointing to it,

$$\bullet_{m_j^{(1)}} \xrightarrow{a^{(1)}} \star_j.$$

Set $\ell_{\star_j} = \ell_{m_j^{(1)}} + c_{a^{(1)}}$. We now check that if there is another arrow $a^{(2)}$ ending in $\star_j$ the analogous quantity $\ell_{m_j^{(2)}} + c_{a^{(2)}}$ agrees, so that $\ell_{\star_j}$ is well defined.

In the related quiver $Q_{P_{\max}}$, these arrows point to different starred vertices that we shall call $\star_j^{(1)}$ and $\star_j^{(2)}$. By definition of $\bar{P}$, there exists a facet arrow-labeling M of $Q_{P_{\max}}$ and an unoriented path $\pi_j = (a_1, \dots, a_k)$ from $\star_j^{(1)}$ to $\star_j^{(2)}$ for which all arrows are labeled by -1. We may suppose $a_1 = a^{(1)}$ and $a_k = a^{(2)}$, and we have $\varepsilon(a_1) = -1$ and $\varepsilon(a_k) = 1$. Now Proposition 5.10 applies, and we can rewrite (5.3), separating out the first and last summand, to get

$$c_{a^{(1)}} = c_{a^{(2)}} + \sum_{i=2}^{k-1} \varepsilon(a_i) c_{a_i}.$$

The path $(a_2, \dots, a_{k-1})$ runs from $m_j^{(1)}$ to $m_j^{(2)}$ and can be assumed to only traverse normal vertices. Therefore, we can replace $\sum_{i=2}^{k-1} \varepsilon(a_i) c_{a_i}$ by $\ell_{m_j^{(2)}} - \ell_{m_j^{(1)}}$. It follows that

$$c_{a^{(1)}} + \ell_{m_j^{(1)}} = c_{a^{(2)}} + \ell_{m_j^{(2)}}.$$

Therefore, we have defined an element $(\ell_v)_v \in \mathbb{Z}^{\mathcal{V}}$ that maps to $(c_a)_a$. This implies the independent-sum condition and proves that $(c_a)_a$ lies in $\mathbf{C}_Q$. Thus, $\mathbf{C}_Q$ agrees with the group of Cartier divisors.

It follows immediately from the definition of $\mathbf{C}_Q$ that the map (5.6) is well defined. The divisors D_j from Remark 5.19 map to the standard generators of $\mathbb{Z}^{\mathcal{V}_\star \setminus \{0\}}$, which implies surjectivity. It follows from the definitions that the kernel of (5.6) is precisely $\mathbf{M}_Q$. Thus, (5.7) is an exact sequence, and the rest of the theorem follows. □

Corollary 5.22. *Let P be a ranked poset with order polytope $\mathcal{O}(P)$. The Hibi projective toric variety $Y_{\mathcal{O}(P)}$ associated to the order polytope $\mathcal{O}(P)$ has a small toric partial desingularization by a terminal Gorenstein Fano variety whose Picard rank equals the number of maximal elements in the canonical extension $\bar{P}$ of P.*

Proof. The partial desingularization is given by $Y(\mathcal{F}_Q)$ for the quiver $Q = Q_{\hat{P}}$. Namely, the face fan $\mathcal{F}_Q$ of $\mathrm{Root}(Q)$ refines the normal fan of $\mathcal{O}(P)$ by Theorem 4.15. Since no rays are added, this desingularization is small. And $Y(\mathcal{F}_Q)$ is terminal Gorenstein by Corollary 5.8. The canonical extension $\bar{P}$ defines a different quiver $Q_{\bar{P}}$ but with the same root polytope; that is, $\mathrm{Root}(Q_{\hat{P}}) = \mathrm{Root}(Q_{\bar{P}})$. Therefore, the Picard rank of $Y(\mathcal{F}_Q)$ is given in Theorem 5.18, and we see that it agrees with the number of maximal elements of $\bar{P}$. □

Let P_D denote the polytope associated to an ample divisor D of a projective toric variety, as in [16].

Corollary 5.23. *Let P be a finite ranked poset with canonical extension $\bar{P}$ and maximal elements of $\bar{P}$ denoted $\{\star_1, \dots, \star_s\}$. For each maximal element $\star_i$ of $\bar{P}$, we have a generator $[D_i]$ of the Picard group of $Y(\mathcal{F}_Q)$ as in Remark 5.19. Consider the quiver Laurent polynomial $S_{Q_{\bar{P}}}$ with one quantum parameter q_i associated to every sink $\star$-vertex $\star_i$. Suppose $D = \sum_i r_i D_i$ is ample. Then the associated polytope P_D is equal to the superpotential polytope $\Gamma^{\mathbf{r}}_{Q_{\bar{P}}}$, where $\mathbf{r} = (r_1, \dots, r_s)$.*

Remark 5.24. [3, Proposition 19] gives a description of the Cartier divisors and Picard group of the projective toric variety associated to a flow polytope; it also gives a concrete description of the Picard group in the case of *flag quivers* [3, Definition 21]. We note that in the special case when our quiver Q comes from a *graded, planar* poset, its dual quiver $Q^\vee$ is a flag quiver.

We now make use of the resolution of singularities of $Y(\mathcal{F}_Q)$ from Theorem 5.4.

Proposition 5.25. *Let P be a finite, ranked poset with order polytope $\mathcal{O}(P)$. The Hibi toric variety $Y_{\mathcal{O}(P)}$ with fan $\mathcal{N}(\mathcal{O}(P))$ has a small resolution of singularities $Y(\widehat{\mathcal{F}}_Q) \to Y(\mathcal{F}_Q)$. The Picard rank ρ of $Y(\widehat{\mathcal{F}}_Q)$ is the number of maximal elements in P.*

Proof. Let $Q = Q_{\bar{P}}$, and consider the fan $\widehat{\mathcal{F}}_Q$ from Theorem 5.4. The small desingularisation of $Y_{\mathcal{O}(P)}$ is the composition of the small desingularisation $Y(\widehat{\mathcal{F}}_Q) \to Y(\mathcal{F}_Q)$ in Theorem 5.4 and the small partial desingularisation $Y(\mathcal{F}_Q) \to Y(\mathcal{N}(\mathcal{O}(P)))$ from Theorem 5.18. Now $Y(\widehat{\mathcal{F}}_Q)$ has an isomorphic group of torus-invariant Weil divisors to $Y(\mathcal{F}_Q)$, but since it is smooth, this identifies it with the group of torus-invariant Cartier divisors of $Y(\widehat{\mathcal{F}}_Q)$. Therefore, the calculation of the class group of $Y(\mathcal{F}_Q)$ from Remark 5.21 computes the Picard group of $Y(\widehat{\mathcal{F}}_Q)$. □

Remark 5.26. In the *graded* case, the construction of a small resolution for a Hibi toric variety is discussed in [52, Section 2.4].

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