ON INDECOMPOSABLE GRAPHS

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1. Introduction. A set of points M of a graph G is a *point cover* if each line of G is incident with at least one point of M. A *minimum cover* (abbreviated m.c.) for G is a point cover with a minimum number of points. The *point covering number* $\alpha(G)$ is the number of points in any minimum cover of G. Let $[V_1, V_2, \ldots, V_r]$, $r > 1$ be a partition of $V(G)$, the set of points of G. Let G_i be the subgraph of G spanned by V_i, for $i = 1, 2, \ldots, r$. Clearly

$$\sum_{i=1}^{r} \alpha(G_i) \leqslant \alpha(G)$$

since any minimum cover for G must cover each G_i. If, in particular,

$$\sum_{i=1}^{r} \alpha(G_i) = \alpha(G),$$

then we say that G is *decomposable* and that $[G_1, G_2, \ldots, G_r]$ is a *decomposition* of G.

The purpose of this paper is to study indecomposable graphs. Erdös and Gallai **(4)** introduced such graphs; their main result is stated in § 3 of this paper. We shall prove that no indecomposable graph is separated by the points of any complete subgraph. Hence, in particular, every indecomposable graph is a block.

The complete graph K_n is easily seen to be indecomposable as is every cycle of odd length. There are, however, many other indecomposable graphs. In the final section, we shall construct several infinite families of such graphs. In the course of this development, we shall study the role of lines which are critical with respect to point covering (cf. **(1, 2, 3, 6)**) in the structure of indecomposable graphs.

All the indecomposable graphs with $p \leqslant 6$ points which are not complete are shown in Figure 1.

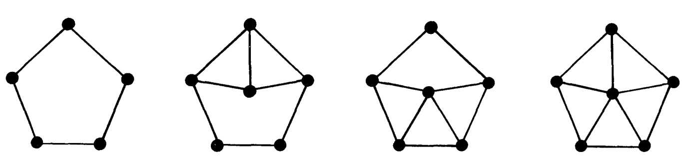

FIGURE 1.

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2. Additional terminology. For the sake of completeness, we shall require additional concepts. A *graph* G consists of a finite non-empty set $V(G)$ of p *points* together with a collection $E(G)$ of *lines* each of which is an unordered pair of points. If x is the line containing points u and v, then we write $x = uv$ and say that u and v are *adjacent*, that x *joins* u and v, and that x is *incident* with the points u and v. Two lines x and y which have a common point are also said to be *adjacent*. The *complete graph* on n points, K_n, is that graph with n points in which every two points are adjacent. The *trivial* graph K_1 has one point and no lines. The *complement* $\bar{G}$ of a graph G is that graph with the same point set as G, in which u and v are adjacent if and only if they are not adjacent in G.

A *path* joining points u and v is an alternating sequence of distinct points and lines beginning with u and ending with v in which each line is incident with the point before it and the point after it. A *cycle* consists of a path containing more than one line together with an additional line joining the first and last points of the path. The *length* of a path or a cycle is the number of lines in it. A cycle is called *even* (*odd*) if it has even (odd) length. We denote the graph consisting of a cycle of length p by C_p. If v is a point and x is a line of G, we denote by $G - v$ and $G - x$ the subgraphs obtained by deleting the point v and the line x respectively. In general, if $W \subset V(G)$, then $G - W$ denotes the graph obtained from G by deleting each point of W. A graph is *connected* if each pair of points are joined by a path. A point v of a connected graph G is a *cutpoint* if $G - v$ is disconnected. In general, a set of points W of G is a *separating set* of G if $G - W$ is disconnected. A graph G is a *block* if it is connected and has no cutpoints.

Let $|A|$ be the number of elements in any set A; by an abuse of notation let $|G| = |V(G)| = p$. If M is a minimum cover for G, we write $\alpha(G) = |M|$. Call v a *critical point* if $\alpha(G) - v < \alpha(G)$. Similarly, x is a *critical line* of G if $\alpha(G - x) < \alpha(G)$. A set of lines X *spans* a graph G if each point of G is incident with a line of X. We denote by $\mathrm{Cr}(G)$ the subgraph of G spanned by the set of critical lines in G. If $\mathrm{Cr}(G) = G$, then G is *line-critical.*

A set of lines X is *independent* if no two of them are adjacent. The *core* of G, $C(G)$, is the union of all sets X of $\alpha(G)$ independent lines in G. If $V(G)$ can be partitioned into two sets V_1, V_2 so that each line of G joins a point of V_1 and a point of V_2, then we say that G is *bipartite*. Finally, the *join* $G + H$ of two disjoint graphs G and H consists of $G \cup H$ together with a line joining each point of G with each point of H.

3. On the structure of an indecomposable graph. The main result of Erdös and Gallai **(4)** on indecomposable graphs is the following.

THEOREM 1. *If a non-trivial graph G with p points is indecomposable, then $\alpha(G) \geqslant p/2$, with equality only if $G = K_2$.*

In particular, since a graph G with a non-empty core, $C(G)$, must have $\alpha(G) \leqslant p/2$, we have the following immediate corollaries.

COROLLARY 1a. *If $C(G) \neq \emptyset$ and $G \neq K_2$, then G is decomposable.*

Those graphs G for which $C(G) = \emptyset$ are characterized in **(6)** and properly contained in this class of graphs are all bipartite graphs. Hence we have the following corollary.

COROLLARY 1b. *Every indecomposable graph contains an odd cycle.*

We next prove our main theorem on indecomposable graphs: a sufficient condition for decomposability of a graph in terms of disconnecting complete subgraphs. Contrapositively, it specifies a structural property of indecomposable graphs.

THEOREM 2. *If G is a connected graph which is separated by the points of a complete subgraph K_n, then G is decomposable.*

Proof. Let the components of $G - V(K_n)$ be denoted by $C_1, \ldots, C_r$, $r \geqslant 2$. Note that some C_i may be trivial. Let M be an m.c. for G. Clearly M must contain at least $n - 1$ of the points in $V(K_n)$.

Case 1. Every minimum cover M of G contains exactly $n - 1$ of the points of $V(K_n)$.

Let $M_i = M \cap V(C_i)$, $i = 1, \ldots, r$. Note that M_i covers C_i for each i. There are two possibilities. If for each i, M_i is an m.c. for C_i, then $[C_i, \ldots, C_r, K_n]$ is a decomposition for G. Suppose, however, that for some i, say $i = 1$, M_1 is not an m.c. for C_1. Then there is an m.c. $N_1$for C_1 such that $|N_1| < |M_1|$. Let w be the point of $V(K_n) - M$. Then

$$M' = (M - M_1) \cup N_1 \cup \{w\}$$

covers G. Also

$$\begin{aligned} |M'| = |(M - M_1) \cup N_1 \cup \{w\}| &= |M| - |M_1| + |N_1| + 1 \\ &\leqslant |M| - |M_1| + |M_1| = |M| = \alpha(G), \end{aligned}$$

and hence M' is an m.c. for G which contains $V(K_n)$. But this contradicts the assumption of Case 1.

Case 2. G has an m.c. M which contains $V(K_n)$. Thus also $M \cap V(C_i) = M_i$ is an m.c. for C_i, for $i = 1, \ldots, r$. Let v be any point in $V(K_n)$. We now show that there is a component C_j of $G - V(K_n)$ such that v is adjacent to at least one point of C_j and, further, that no m.c. for C_j contains all the points of C_j which are adjacent to v. Such a component will be said to be *preferred* by v.

First note that v is adjacent to a point of at least one component of $G - V(K_n)$, since otherwise M would not be a minimum cover for G. Renumbering the components if necessary, let $C_1, \ldots, C_s$ be the components which have at least one point adjacent to v, $1 \leqslant s \leqslant r$. Now suppose that each of $C_1, \ldots, C_s$ has an m.c. N_i which includes all points of C_i adjacent to v in G. For every i, $s+1 \leqslant i \leqslant r$, let N_i be any m.c. for C_i. Then

$$N = V(K_n) \cup N_1 \cup \ldots \cup N_r$$

is an m.c. for G which includes every point adjacent to v in G. But v is in N, and hence $N - \{v\}$ covers G, contradicting the minimality of N. Thus each $v \in V(K_n)$ has at least one preferred component so that at least one of $C_1, \ldots, C_s$, say C_j, has the property that none of its minimum covers contains all the points of C_j adjacent to v.

Let $V(K_n) = \{v_1, \ldots, v_n\}$. Without loss of generality, suppose C_1 is a preferred component of v_1. Let

$$V_1 = \{v_i \in V(K_n): C_1 \text{ is a preferred component of } v_i\}.$$

If $V_1 = V(K_n)$, define $V_2 = \emptyset$; if $V_1 \neq V(K_n)$, let $w_1 \in V(K_n) - V_1$. Renumbering components again if necessary, suppose C_2 is preferred by w_1. In this case let

$$V_2 = \{v_i \in V(K_n) - V_1: C_2 \text{ is a preferred component of } v_i\}.$$

If $V_1 \cup V_2 = V(K_n)$, then define $V_3 = \emptyset$. If $V_1 \cup V_2 \neq V(K_n)$, let

$$w_2 \in V(K_n) - (V_1 \cup V_2).$$

Again renumbering if necessary, suppose C_3 is preferred by w_2. Continue in this manner to define $V_1, V_2, V_3, \ldots, V_r$. Note that $V_1 \neq \emptyset$ and that either $V_r \neq \emptyset$ or for some t, $1 \leqslant t \leqslant r$, $V_t = V_{t+1} = \ldots = V_r = \emptyset$. Now

$$V_1 \cup V_2 \cup \ldots \cup V_r = V(K_n).$$

For each $i = 1, \ldots, r$, define G_i to be the subgraph of G spanned by $V(C_i) \cup V_i$. Clearly, these subgraphs G_i are disjoint, and there are at least two of them since $r \geqslant 2$.

To complete the proof for Case 2, we show that $[G_1, \ldots, G_r]$ is a decomposition of G. Recall that M is an m.c. for G with $V(K_n) \subset M$ and that hence $M_i = M \cap V(C_i)$ is an m.c. for C_i. It will suffice to show that $M'_i = M_i \cup V_i$ is an m.c. for G_i, for each $i = 1, \ldots, r$.

For any i, if $V_i = \emptyset$ there is nothing to prove. Thus consider $|V_i| \geqslant 1$. Clearly, M'_i covers G_i, so suppose it is not a minimum cover. Then there is an m.c. N'_i for G_i with $|N'_i| < |M'_i|$. Now $V_i \not\subset N'i$ or else M_i is not an m.c. for C_i. However, at most one point of V_i fails to be in N'_i, say u. Hence $V_i - N'_i = \{u\}$.

We next show that $N_i = N'_i \cap V(C_i)$ is an m.c. for C_i. Clearly N_i covers C_i. Now

$$\begin{aligned}
\alpha(G_i) &= |N'_i| = |N'_i| \cap V(C_i)| + |N'_i| \cap V_i| \\
&= |N_i| + |V_i| - 1 < |M'_i| = |M_i \cup V_i| = |M_i| + |V_i| \\
&= \alpha(C_i) + |V_i| \leqslant |N_i| + |V_i|.
\end{aligned}$$

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Hence

$$|N_i| + |V_i| - 1 < |M_i| + |V_i| \leqslant |N_i| + |V_i|,$$

or

$$|N_i| - 1 < |M_i| \leqslant |N_i|.$$

Thus $|M_i| = |N_i| = \alpha(C_i)$ and N_i is an m.c. for C_i. Now since $u \notin N'_i$, all points of C_i adjacent to u are in N'_i and hence in N_i. Thus N_i is an m.c. for C_i which contains all points of C_i adjacent to u, contradicting the fact that C_i is preferred by u.

Hence M'_i is an m.c. for G_i, for each $i = 1, \ldots, r$. Thus

$$\sum_{i=1}^{r} \alpha(G_i) = \sum_{i=1}^{r} |M'_i| = \sum_{i=1}^{r} |M_i| + n = |M| = \alpha(G)$$

and $[G_1, \ldots, G_r]$ is a decomposition of G. This completes Case 2 and the proof of the theorem.

Putting $n = 1$ in the theorem yields the following result.

COROLLARY 2a. *If G is indecomposable, then G is a block.*

4. Several infinite families of undecomposable graphs. In **(6)**, we constructed an infinite collection of line-critical graphs. In conjunction with the following theorem, this yields our first infinite collection of indecomposable graphs.

THEOREM 3. *If* $\mathrm{Cr}(G)$ *is a connected spanning subgraph of G, then G is indecomposable.*

Proof. Suppose G has a decomposition $[G_1, \ldots, G_r]$, $r \geqslant 2$. Then no line of G joining G_i and G_j, $i \neq j$, is critical in G. Hence it follows that if $\mathrm{Cr}(G)$ spans G, it is disconnected and if $\mathrm{Cr}(G)$ is connected it cannot span G.

The following corollary is immediate.

COROLLARY 3a. *If G is line-critical, then G is indecomposable.*

We hasten to point out that there are many graphs G in which $\mathrm{Cr}(G)$ is a connected spanning subgraph, but which fail to be line-critical. The smallest such graph and two others are shown in Figure 2. The non-critical lines of each are dashed.

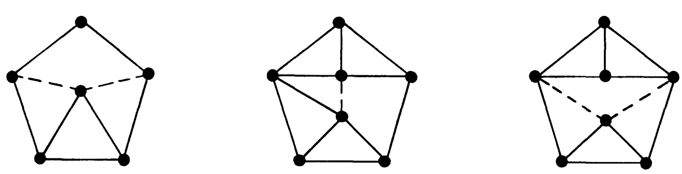

FIGURE 2.

We obtain another class of indecomposable graphs by means of the next theorem.

THEOREM 4. *The join of two indecomposable graphs is also indecomposable.*

Proof. Let G_1 and G_2 be indecomposable and suppose their join $G_1 + G_2$ is decomposable. Then, without loss of generality, we may assume there is a decomposition of $G_1 + G_2$ with precisely two members, $[H_1, H_2]$. Now

$$\alpha(G_1 + G_2) = \min\{|G_1| + \alpha(G_2), \alpha(G_1) + |G_2|\}.$$

Suppose for the sake of this argument that $\alpha(G_1 + G_2) = |G_1| + \alpha(G_2)$. Now let M_2 be an m.c. for G_2. Then $V(G_1) \cup M_2$ is an m.c. for $G_1 + G_2$.

We have several cases to consider.

(1) Suppose $H_1 \cup H_2 \subset G_1$. Then we have

$$\alpha(G_1) < |G_1| \leqslant |G_1| + \alpha(G_2) = \alpha(G_1 + G_2) = \alpha(H_1) + \alpha(H_2) = \alpha(H_1 \cup H_2) \leqslant \alpha(G_1),$$

which is absurd.

(2) Suppose $H_1 \cup H_2 \subset G_2$. Then

$$|G_1| + \alpha(G_2) = \alpha(G_1 + G_2) = \alpha(H_1) + \alpha(H_2) = \alpha(H_1 \cup H_2) \leqslant \alpha(G_2).$$

Thus $|G_1| = 0$, a contradiction.

(3) Now suppose $H_1 \subset G_1$ and $H_2 \subset G_2$. Then

$$|G_1| + \alpha(G_2) = \alpha(G_1 + G_2) = \alpha(H_1) + \alpha(H_2) \leqslant \alpha(G_1) + \alpha(G_2) < |G_1| + \alpha(G_2),$$

which is again absurd.

Hence we must conclude that at least one of the subgraphs H_1 and H_2 must have a non-empty intersection with both G_1 and G_2.

(4) Suppose $H_2 \cap G_1 \neq \emptyset \neq H_2 \cap G_2$. We emphasize that these intersections may consist of points only (no lines).

Now let $F_i = H_i \cap G_1$ and $L_i = H_i \cap G_2$ for $i = 1, 2$. Thus F_2 and L_2 are not empty. First suppose that $L_1 = \emptyset$, so that $F_1 = H_1 \subset G_1$. Now $V(F_2) \cup M_2$ covers H_2. Hence

$$\begin{aligned}\alpha(G_1 + G_2) &= \alpha(H_1) + \alpha(H_2) \leqslant \alpha(H_2) + |V(F_2)| + |M_2| \\ &\leqslant |V(F_1)| - 1 + |V(F_2)| + |M_2| \leqslant |V(G_1)| - 1 + |M_2| \\ &< |V(G_1)| + |M_2| = \alpha(G_1 + G_2),\end{aligned}$$

which is once again absurd, so $L_1 \neq \emptyset$. Now we show that $[L_1, L_2]$ is a decomposition of G_2. We know that $L_1 \cup L_2 \subset G_2$, so suppose that

$$\alpha(L_1) + \alpha(L_2) < \alpha(G_2).$$

Let N_i be an m.c. for L_i, $i = 1, 2$. Then $V(G_1) \cup N_1 \cup N_2$ covers $H_1 \cup H_2$ and thus

$$\begin{aligned}\alpha(H_1) + \alpha(H_2) &= \alpha(H_1 \cup H_2) \leqslant V(G_1)| + |N_1| + |N_2| \\ &= |V(G_1)|. + \alpha(L_1) + \alpha(L_2) < |V(G_1)| + \alpha(G_2) = \alpha(G_1 + G_2),\end{aligned}$$

contradicting the assumption that $[H_1, H_2]$ is a decomposition of $G_1 + G_2$. Thus $[L_1, L_2]$ is a decomposition of G_2, which contradicts the indecomposability of G_2 and completes the proof of the theorem.

Now if $G_1 = K_m$ and $G_2 = K_n$, then $G_1 + G_2 = K_{m+n}$ and $\mathrm{Cr}(G_1 + G_2) = G_1 + G_2$. If, however, at least one of G_1 and G_2 is not complete, then in general the critical lines of $G_1 + G_2$ form a disconnected subgraph of $G_1 + G_2$. More precisely, we have the next three theorems.

THEOREM 5. *If G_1 and G_2 are not both complete graphs, then*

$$\mathrm{Cr}(G_1 + G_2) \subset \mathrm{Cr}(G_1) \cup \mathrm{Cr}(G_2).$$

Proof. First we show that $\mathrm{Cr}(G_1 + G_2) \subset E(G_1) \cup E(G_2)$. As in the preceding theorem we may let $\alpha(G_1 + G_2) = |G_1| + \alpha(G_2)$. Let $x = v_1 v_2$ be a line with $v_i \in V(G_i)$, $i = 1, 2$. Suppose x is critical in $G_1 + G_2$. Then no m.c. for $(G_1 + G_2) - x$ contains v_1 or v_2. Thus there is a unique m.c. for $(G_1+G_2)-x$, namely $M_x = [V(G_1) \cup V(G_2)] - \{v_1, v_2\}$. Hence

$$\alpha(G_1 + G_2) = |G_1| + \alpha(G_2) = |M_x| + 1 = |G_1| + |G_2| - 1.$$

Hence $\alpha(G_2) = |G_2| - 1$ and it follows that G_2 must be complete. But then $\alpha(G_1) + |G_2| \geqslant |G_1| + \alpha(G_1) = |G_1| + |G_2| - 1$ and hence $\alpha(G_1) \geqslant |G_1| - 1$. Thus G_1 is also complete and we have a contradiction of the hypothesis. Thus $\mathrm{Cr}(G_1 + G_2) \subset E(G_1) \cup E(G_2)$.

Now let y belong to $\mathrm{Cr}(G_1 + G_2)$ and, without loss of generality, let y be an element of $E(G_2)$. We know that

$$\alpha(G_1 + G_2) = \min\{|G_1| + \alpha(G_2), \alpha(G_1) + |G_2|\}$$

and, since $(G_1 + G_2) - y = G_1 + (G_2 - y)$, that

$$\alpha[(G_1 + G_2) - y] = \min\{|G_1| + \alpha(G_2 - y), \alpha(G_1) + |G_2 - y|\}.$$

There are two cases to consider.

Case 1. $\alpha(G_1 + G_2) = |G_2| + \alpha(G_1) < \alpha(G_2) + |G_1|$. Then if

$$\alpha[(G_1 + G_2) - y] = |G_1| + \alpha(G_2 - y),$$

we have

$$\alpha(G_1 + G_2) = |G_2| + \alpha(G_1) \leqslant |G_1| + \alpha(G_2) - 1 \leqslant |G_1| + \alpha(G_2 - y)$$

and $y \notin \mathrm{Cr}(G_1 + G_2)$, contrary to assumption. On the other hand, if

$$\alpha[(G_1 + G_2) - y] = \alpha(G_1) + |G_2 - y| = \alpha(G_1) + |G_2| = \alpha(G_1 + G_2),$$

then again we have the contradiction that $y \notin \mathrm{Cr}(G_1 + G_2)$. Thus Case 1 must be an impossibility.

Case 2. $\alpha(G_1 + G_2) = |G_1| + \alpha(G_2)$. Now if

$$\alpha[(G_1 + G_2) - y] = \alpha(G_1) + |G_2 - y|,$$

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we have

$$\alpha(G_1 + G_2) = |G_1| + \alpha(G_2) \leqslant \alpha(G_1) + |G_2|$$
$$= \alpha(G_1) + |G_2 - y| = \alpha[(G_1 + G_2) - y]$$

so that once again $y \notin \mathrm{Cr}(G_1 + G_2)$. Thus we must have

$$\alpha[(G_1 + G_2) - y] = |G_1| + \alpha(G_2 - y)$$

so that

$$\alpha(G_2 - y) = \alpha[(G_1 + G_2) - y] - |G_1| = \alpha(G_1 + G_2) - |G_1| - 1$$
$$= |G_1| + \alpha(G_2) - |G_1| - 1 = \alpha(G_2) - 1$$

and $y \notin \mathrm{Cr}(G_2)$. This completes the proof of the theorem.

THEOREM 6. *A critical line of G_2 is critical in $G_1 + G_2$ if and only if*

$$\alpha(G_1 + G_2) = |G_1| + \alpha(G_2).$$

Proof. We first show that if some critical line x of G_2 is critical in $G_1 + G_2$, then $\alpha(G_1 + G_2) = |G_1| + \alpha(G_2)$. Assume that

$$\alpha(G_1 + G_2) = |G_2| + \alpha(G_1) < |G_1| + \alpha(G_2).$$

Again there are two possibilities.

If $\alpha[(G_1 + G_2) - x] = |G_2 - x| + \alpha(G_1)$, then

$$\alpha[(G_1 + G_2) - x] = |G_2| + \alpha(G_1) = \alpha(G_1 + G_2)$$

and $x \notin \mathrm{Cr}(G_1 + G_2)$. On the other hand, if

$$\alpha[(G_1 + G_2) - x] = |G_1| + \alpha(G_2 - x) = |G_1| + \alpha(G_2) - 1$$
$$\geqslant \alpha(G_1) + |G_2| = \alpha(G_1 + G_2),$$

then again $x \notin \mathrm{Cr}(G_1 + G_2)$.

To prove the converse, let $\alpha(G_1 + G_2) = |G_1| + \alpha(G_2)$ and let $x \in \mathrm{Cr}(G_2)$. Then

$$\alpha[(G_1 + G_2) - x] \leqslant |G_1| + \alpha(G_2 - x) = |G_1| + \alpha(G_2) - 1 = \alpha(G_1 + G_2) - 1$$

and hence $x \in \mathrm{Cr}(G_1 + G_2)$, completing the proof.

We may now combine Theorems 5 and 6 to decide when

$$\mathrm{Cr}(G_1 + G_2) = \mathrm{Cr}(G_1) \cup \mathrm{Cr}(G_2).$$

THEOREM 7. *The condition* $\mathrm{Cr}(G_1 + G_2) = \mathrm{Cr}(G_1) \cup \mathrm{Cr}(G_2)$ *holds if and only if:*

(1) $\mathrm{Cr}(G_1) = \mathrm{Cr}(G_2) = \emptyset$, *or*
(2) $\mathrm{Cr}(G_1) = \emptyset \neq \mathrm{Cr}(G_2)$ *and* $\alpha(G_1 + G_2) = |G_1| + \alpha(G_2)$, *or*
(3) $\mathrm{Cr}(G_1), \mathrm{Cr}(G_2) \neq \emptyset$ *and* $|G_1| - \alpha(G_1) = |G_2| - \alpha(G_2)$.

This theorem illustrates the fact that there are indecomposable graphs obtainable from the method of Theorem 4 which do not satisfy the hypotheses of Theorem 3.

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We now present a sufficient condition for the indecomposability of the join of a decomposable graph with an indecomposable one.

THEOREM 8. *Let* G_1 *be decomposable and let* G_2 *be indecomposable. Then, if* $|V(G_1)| - \alpha(G_1) \leqslant |V(G_2)| - \alpha(G_2)$, $G_1 + G_2$ *is indecomposable.*

Proof. The hypothesis implies that $\alpha(G_1 + G_2) = |V(G_1)| + \alpha(G_2)$. Assume that $G_1 + G_2$ is decomposable. Then if one proceeds as in the proof of Theorem 4, a contradiction of the fact that G_2 is indecomposable is obtained.

It is not known if the converse of Theorem 8 holds in general. We do, however, present two special cases in which this converse does hold.

THEOREM 9. *If* G *is decomposable, then* $G + K_n$ *is decomposable.*

Proof. First we note that

$$\alpha(G + K_n) = \min\{|V(G)| + n - 1, \alpha(G) + n\} = \alpha(G) + n.$$

Now let $[D_1, D_2]$ be a decomposition of G. We shall show that $[D_1 + K_n, D_2]$ is a decomposition of $G + K_n$. We begin by noting that

$$\alpha(D_1 + K_n) = \min\{|V(D_1)| + n - 1, \alpha(D_1) + n\} = \alpha(D_1) + n.$$

Next, let M be an m.c. for G and hence $M \cup V(K_n)$ is an m.c. for $G + K_n$. Let $M_i = M \cap V(D_i)$ for $i = 1, 2$. Then M_i is an m.c. for D_i. In particular, $M_1 \cup V(K_n)$ is an m.c. for $D_1 + K_n$. Thus

$$\begin{aligned}\alpha(D_1 + K_n) + \alpha(D_2) &= |M_1 \cup V(K_n)| + |M_2| = |M_1| + |M_2| + |V(K_n)| \\ &= |M| + |V(K_n)| = \alpha(G) + n = \alpha(G + K_n)\end{aligned}$$

and the theorem is proved.

THEOREM 10. *Let* G *be an indecomposable graph. Then* $G + \bar{K}_n$ *is indecomposable if and only if* $n \leqslant |V(G)| - \alpha(G)$.

Proof. Suppose first that $n \leqslant |V(G)| - \alpha(G)$. If $n = 1$, since $\bar{K}_1 = K_1$, it follows that $G + \bar{K}_1$ is indecomposable by Theorem 4. If $n > 1$, then $\bar{K}_n$ is decomposable since it is disconnected. Hence, since $n = |V(\bar{K}_n)| - \alpha(\bar{K}_n)$, $G + \bar{K}_n$ is indecomposable by Theorem 8.

To prove the converse, assume that $n > |V(G)| - \alpha(G)$. Hence

$$\alpha(G + \bar{K}_n) = \min\{|V(G)| + 0, \alpha(G) + n\} = |V(G)| < \alpha(G) + n,$$

and $G + \bar{K}_n$ has $V(G)$ as its only minimum cover. Now suppose that $v \in V(\bar{K}_n)$. We observe that $[v, G + (\overline{K_n - v})]$ is a decomposition for $G + \bar{K}_n$. To see this, we need only point out that

$$\alpha(G + (\overline{K_n - v})) = \min\{|V(G)| + 0, \alpha(G) + n - 1\} = |V(G)|,$$

and since $\alpha(v) = 0$, we are finished.

Next suppose G_1 and G_2 are both decomposable. What can be said about $G_1 + G_2$?

THEOREM 11. *Let G_1 and G_2 be decomposable and $|G_1| - \alpha(G_1) \leqslant |G_2| - \alpha(G_2)$. Then a sufficient condition that $G_1 + G_2$ be decomposable is that there exist decompositions $[A_1, B_1]$ for G_1 and $[A_2, B_2]$ for G_2 such that*

$$|A_1| - \alpha(A_1) \leqslant |A_2| - \alpha(A_2)$$

and

$$|B_1| - \alpha(B_1) \leqslant |B_2| - \alpha(B_2).$$

Proof. We need only observe that

$$\begin{aligned}\alpha(A_1 + A_2) + \alpha(B_1 + B_2) &= \min\{|A_1| + \alpha(A_2), |A_2| + \alpha(A_1)\} \\ &\qquad + \min\{|B_1| + \alpha(B_2), |B_2| + \alpha(B_1)\} \\ &= |A_1| + \alpha(A_2) + |B_1| + \alpha(B_2) = |G_1| + \alpha(G_2) \\ &= \min\{|G_1| + \alpha(G_2), |G_2| + \alpha(G_1)\} \\ &= \alpha(G_1 + G_2)\end{aligned}$$

and the theorem is proved.

We conjecture that the condition of Theorem 11 is also necessary, but the proof appears to be difficult. We conclude by mentioning that it is not true that the join of two decomposable graphs is decomposable. The graphs G_1 and G_2 of Figure 3 are both decomposable but it can be verified that their join is indecomposable.

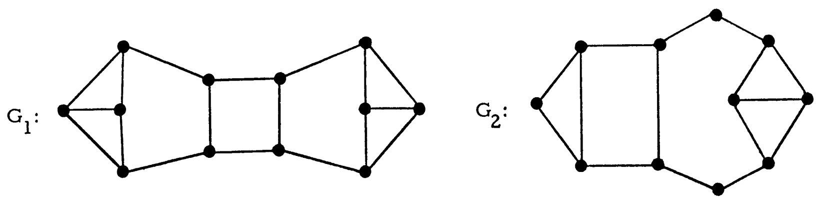

FIGURE 3.

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