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$\mathcal{D}$-FAITHFUL SEMIGROUP-GRADED RINGS

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Abstract A weak form of faithfulness, depending on Green's equivalence $\mathcal{D}$, is introduced for a ring R graded by a semigroup S. Suppose that R satisfies this condition. It is shown that if e and f are $\mathcal{D}$-equivalent idempotents of S and R_e is semiprime (respectively, prime, semiprimitive, right primitive), then R_f is semiprime (respectively, prime, semiprimitive, right primitive). In addition, it is shown that if G and H are maximal subgroups of S lying in the same $\mathcal{D}$-class and R_G is semiprime (respectively, prime, semiprimitive, right primitive), then R_H is semiprime (respectively, prime, semiprimitive, right primitive).

Keywords: graded ring; Green's equivalences; semiprime; prime; semiprimitive; right primitive

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1. Faithfulness and $\mathcal{D}$-faithfulness

Throughout this paper, all rings are associative, but the existence of unity elements is not assumed.

Let S be a semigroup. A ring R is said to be *S-graded* (or *graded by S*) if and only if

(i) the additive group of R is the direct sum $\oplus_{x\in S}R_x$ of a family of subgroups R_x indexed by S, and

(ii) the multiplication in R is such that, for all x and y in S, $R_xR_y \subseteq R_{xy}$.

Suppose that R is such a ring. We call each R_x a *basic summand* of R. For a non-empty subset T of S, we write $R_T := \oplus_{x\in T}R_x$ and note that if T is a subsemigroup of S, then R_T is a subring of R. In particular, if $e = e^2 \in S$, then R_e is a subring of R. For $a \in R$ we denote the R_x-component of a by a_x and we define the *support* of a, $\operatorname{supp}(a)$, to be $\{x \in S : a_x \neq 0\}$.

Following Cohen and Montgomery [**1**] (who introduced the concept for group-graded rings), we say that R is *faithful* (or *faithfully graded by S*) if and only if, for all $x, y \in S$,

$$a \in R_x\backslash 0 \;\Rightarrow\; aR_y \neq 0 \text{ and } R_ya \neq 0.$$

This condition ensures that if R is non-zero, then each basic summand of R is non-zero and there is a non-trivial linkage between any two basic summands.

The semigroup ring $F[S]$ of S over a ring F can be viewed as an S-graded ring R with $R_x = Fx$ for all $x \in S$. It is easily seen that if F has no non-zero left or right annihilator (in particular, if F is an integral domain), then R is faithful.

Since Green's equivalences $\mathcal{H}$ and $\mathcal{D}$ on S are of central importance in this paper, we recall their definitions and some of their key properties. For a detailed account, see [**2**, Chapter II]. First, we define equivalences $\mathcal{L}$ and $\mathcal{R}$ on S by the rule that $x\mathcal{L}y$ (respectively, $x\mathcal{R}y$) if and only if x and y generate the same left (respectively, right) ideal of S. The equivalence $\mathcal{L}\cap\mathcal{R}$ is denoted by $\mathcal{H}$. We adopt the customary notation H_x for the $\mathcal{H}$-class of S that contains the element x. If e is an idempotent of S, then H_e is a subgroup of S with identity e; furthermore, it contains all such subgroups of S. Each subgroup of the form H_e for some $e = e^2 \in S$ is termed a *maximal subgroup* of S. Clearly, if e and f are distinct idempotents of S, then $H_e \cap H_f = \emptyset$.

It can be shown that $\mathcal{L}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{L}$, where $\circ$ denotes the usual composition of relations. We write $\mathcal{D} := \mathcal{L}\circ\mathcal{R}$ and note that this is the smallest equivalence on S that contains both $\mathcal{L}$ and $\mathcal{R}$. Each $\mathcal{D}$-class D is a union of $\mathcal{L}$-classes, a union of $\mathcal{R}$-classes and a union of $\mathcal{H}$-classes. Moreover, all the $\mathcal{H}$-classes in D have the same cardinal; and, for any two distinct idempotents $e, f \in D$, we have that $H_e \cong H_f$.

From these remarks it is clear that there is considerable uniformity within each $\mathcal{D}$-class of S. Now consider a faithful S-graded ring R. The example below illustrates the fact that, for $\mathcal{D}$-equivalent idempotents $e, f \in S$, it may happen that $R_{H_e} \ncong R_{H_f}$.

Example 1.1. Let S denote the bicyclic semigroup, which we take to be the monoid with identity e generated by elements p and q subject to the single relator $pq = e$. It is well known, and easily checked, that S consists of a single $\mathcal{D}$-class and that $H_x = \{x\}$ for all $x \in S$. Let $\mathbb{Z}$ denote the ring of integers and R the subset of the semigroup ring $\mathbb{Z}[S]$ consisting of all those elements $\sum_{x\in S}\alpha_x x$ ($\alpha_x \in \mathbb{Z}$) for which $\alpha_x \in 2\mathbb{Z}$ if $x \neq e$. Then R is a subring of $\mathbb{Z}[S]$ and may also be viewed as an S-graded ring with $R_e = \mathbb{Z}e$ and $R_x = (2\mathbb{Z})x$ for all $x \in S\backslash e$. It is readily seen that R is faithful. Write $f := qp$. Then $f = f^2 \neq e$ and so $R_f \cong 2\mathbb{Z}$. But $R_e \cong \mathbb{Z}$. Hence $R_e \ncong R_f$.

We now introduce a weaker form of the faithfulness condition. Although this is more complicated than the original, it enables us to widen considerably the class of semigroup-graded rings to which our main results apply.

Definition 1.2. Let R be a ring graded by a semigroup S. We say that R is $\mathcal{D}$-*faithful* (or $\mathcal{D}$-*faithfully graded by* S) if and only if, for all $x, y \in S$ with x, y and xy in the same $\mathcal{D}$-class,

$$a \in R_x\backslash 0 \;\Rightarrow\; aR_y \neq 0 \qquad \text{and} \qquad b \in R_y\backslash 0 \;\Rightarrow\; R_x b \neq 0.$$

Let S be a non-trivial semigroup with a zero z and let F be a non-zero ring with no non-zero left or right annihilator. We form the *contracted semigroup ring* $F_0[S]$ of S over F by factoring $F[S]$ by the ideal Fz. Then $F_0[S]$ can be viewed as an S-graded ring R, with $R_x = Fx$ for all $x \in S\backslash z$ and $R_z = 0$. Since there exist $x \in S\backslash z$ and $\alpha \in F\backslash 0$, we have that $\alpha x \in R_x\backslash 0$; but $(\alpha x)R_z = 0$ and so R is not faithful. On the other hand, R

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is $\mathcal{D}$-faithful, as we now show. First, $\{z\}$ is a $\mathcal{D}$-class of S. Let D be any other $\mathcal{D}$-class and let $x, y \in D$ be such that $xy \in D$. Let $a \in R_x \backslash 0$. Then $a = \alpha x$ for some $\alpha \in F \backslash 0$. By hypothesis, there exists $\beta \in F$ such that $\alpha\beta \neq 0$. Hence $a(\beta y) = (\alpha\beta)xy \neq 0$ and so $aR_y \neq 0$. Similarly, we see that if $b \in R_y \backslash 0$ then $R_x b \neq 0$.

The following particular case is of special interest. For a given positive integer n, let S_n denote *the semigroup of* $n \times n$ *matrix units*: that is, $S_n = \{e_{ij} : 1 \leqslant i,\ j \leqslant n\} \cup \{z\}$, with multiplication given by the rule that z is a zero element and

$$e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k, \\ z & \text{if } j \neq k. \end{cases}$$

Then, with F as above, $F_0[S_n] \cong M_n(F)$, the ring of $n \times n$ matrices over F, with the usual matrix operations. Thus $M_n(F)$ can be viewed as a $\mathcal{D}$-faithful S_n-graded ring.

The next example continues the matrix theme.

Example 1.3. Let F be a field, let n be a positive integer greater than 2 and let $R = M_n(F)$. Choose distinct positive integers r and s such that $r + s = n$ and partition each matrix into four blocks, with the leading block of type $r \times r$. The rule for block multiplication shows that R is an S_2-graded ring, with basic summands

$$R_{e_{11}} = M_r(F), \qquad R_{e_{12}} = F_{rs}, \qquad R_{e_{21}} = F_{sr}, \qquad R_{e_{22}} = M_s(F), \qquad R_z = 0,$$

where F_{rs} and F_{sr} denote, respectively, the sets of all $r \times s$ and $s \times r$ matrices over F. Again, it can be verified that, as an S_2-graded ring, R is $\mathcal{D}$-faithful. Now e_{11} and e_{22} are $\mathcal{D}$-equivalent idempotents in S_2 and $\mathcal{H}$ is the identity relation on S_2. Note that the rings $R_{e_{11}}$ and $R_{e_{22}}$ are both simple; but, since $r \neq s$, they are not isomorphic.

$\mathcal{D}$-faithfulness also arises naturally in the following situation.

Example 1.4. Let S be a Clifford semigroup [**2**, § IV.2]. Then there exists a semilattice E (a commutative semigroup of idempotents) and a family $G_e (e \in E)$ of pairwise-disjoint subgroups of S such that S is the union of the G_e and, for all e and f in E, $G_eG_f \subseteq G_{ef}$. Let F be a field and let $R := F[S]$. As an S-graded ring, R is faithful. However, it is frequently useful to view R as an E-graded ring, with $R_e = F[G_e]$ for all $e \in E$. Simple examples can be constructed to show that R need not be faithfully E-graded (see, for example, [**6**, § 1]). However, $\mathcal{D}$ is the identity relation on E; also, for all $e \in E$, $a \in R_e \backslash 0$ implies $aR_e \neq 0$ and $R_e a \neq 0$. Hence, as an E-graded ring, R is $\mathcal{D}$-faithful.

The purpose of the present article is to show that, in a ring R that is $\mathcal{D}$-faithfully graded by a semigroup S, some standard properties hold for *all* subrings of a certain type associated with a given $\mathcal{D}$-class of S, provided that they hold for *one* such subring. Let e and f be $\mathcal{D}$-equivalent idempotents of S. It is shown in §§ 3 and 4 that if R_e is semiprime (respectively, prime, semiprimitive, right primitive), then so also is R_f, and if R_{H_e} is semiprime (respectively, prime, semiprimitive, right primitive), then so also is R_{H_f}. Analogous statements are false for simplicity, however: § 5 provides an example in which $\mathcal{H}$ is the identity relation on S and R_e is simple, but R_f is not simple.

2. Two lemmas

This short section comprises two elementary lemmas. The first of these is a well-known result on those $\mathcal{D}$-classes of a semigroup that contain idempotents [**2**, Chapter II: Lemmas 2.1 and 2.2 and Proposition 3.6].

Lemma 2.1.

(i) *Let p and q be elements of a semigroup S such that $pqp = p$, $qpq = q$, and let e, f denote the idempotents pq, qp, respectively. Then p, q, e, f are $\mathcal{D}$-equivalent in S and*

$$pH_f = H_p, \qquad H_pq = H_e, \qquad pH_fq = H_e.$$

(ii) *Let e and f be $\mathcal{D}$-equivalent idempotents of a semigroup S. Then there exist $p, q \in S$ such that*

$$pqp = p, \qquad qpq = q, \qquad pq = e, \qquad qp = f.$$

The applications of $\mathcal{D}$-faithfulness in §§ 3 and 4 make use of the second lemma.

Lemma 2.2. *Let p, q, e, f be elements of a semigroup S such that $pqp = p$, $qpq = q$, $pq = e$, $qp = f$, and let R be a ring $\mathcal{D}$-faithfully graded by S. Then, for all $a \in R_{H_f}\backslash 0$, there exist $x \in R_p$ and $y \in R_q$ such that $xay \in R_{H_e}\backslash 0$.*

Proof. Let $a \in R_{H_f}\backslash 0$ and let $k \in \operatorname{supp}(a)$. By Lemma 2.1 (i), p, q, e, f lie in the same $\mathcal{D}$-class, D say, of S; further, $pk \in pH_f = H_p \subseteq D$ and $pkq \in H_pq = H_e \subseteq D$. Hence, by the $\mathcal{D}$-faithfulness of R, since $p, k, pk \in D$ and $a_k \neq 0$, there exists $x \in R_p$ such that $xa_k \neq 0$; and, since $pk, q, pkq \in D$, there exists $y \in R_q$ such that

$$xa_ky \in R_{pkq}\backslash 0. \tag{2.1}$$

Now

$$xay = xa_ky + \sum_{h \in \operatorname{supp}(a)\backslash k} xa_hy \tag{2.2}$$

(with the convention that if $\operatorname{supp}(a)\backslash k = \emptyset$, then the sum on the right-hand side is 0). But, for all $h \in \operatorname{supp}(a)$, $xa_hy \in R_{phq}$; and if $phq = pkq$, then $h = qphqp = qpkqp = k$, since $qp = f$ and $h, k \in H_f$. Hence, from (2.1) and (2.2), $xay \neq 0$. Also, $xay \in R_{pH_fq} = R_{H_e}$, by Lemma 2.1 (i). Thus $xay \in R_{H_e}\backslash 0$. □

3. Semiprimeness, primeness and semiprimitivity

In this section we establish the first of our main results (Theorem 3.3).

Recall that an element a of a ring R is *right quasiregular* if and only if there exists $b \in R$ such that $a + b = ab$. The Jacobson radical $J(R)$ of R can be described as the set

$$\{a \in R : \forall r \in R,\ ar \text{ is right quasiregular}\}$$

and is the largest ideal of R consisting of right quasiregular elements. We say that R is *semiprimitive* if and only if $J(R) = 0$.

A characterization of the right quasiregular elements of R is given by the following lemma [**4**, Lemma 6.3].

Lemma 3.1. *An element r in a ring R is right quasiregular if and only if $\{rx - x : x \in R\} = R$.*

Lemma 3.2. *Let p and q be elements of a semigroup S such that $pqp = p$, $qpq = q$ and let e, f denote the idempotents pq, qp, respectively. Let T and U be subsemigroups of H_e and H_f, respectively, such that $T = pUq$ and let R be a ring $\mathcal{D}$-faithfully graded by S. If R_T is semiprime (respectively, prime, semiprimitive), then R_U is semiprime (respectively, prime, semiprimitive).*

Proof. First observe that $qpUqp = fUf = U$, since $U \subseteq H_f$. Hence

$$qTp = U. \tag{3.1}$$

We divide the proof into three parts.

(i) Assume that R_T is semiprime. Since the zero ring is trivially semiprime, we may suppose that $R_U \neq 0$. Let $a \in R_U \backslash 0$. By Lemma 2.2, since $a \in R_{H_f} \backslash 0$ there exist $x \in R_p$ and $y \in R_q$ such that $xay \neq 0$. Also $xay \in R_p R_U R_q \subseteq R_{pUq} = R_T$. Hence, since R_T is semiprime, $xayR_Txay \neq 0$. Consequently, $ayR_Txa \neq 0$. But $yR_Tx \subseteq R_q R_T R_p \subseteq R_{qTp} = R_U$, by (3.1). Hence $aR_Ua \neq 0$. Thus R_U is semiprime.

(ii) Assume that R_T is prime. Again we may suppose that $R_U \neq 0$. Let $a_1, a_2 \in R_U \backslash 0$. By Lemma 2.2, since $a_i \in R_{H_f} \backslash 0$ there exist $x_i \in R_p$ and $y_i \in R_q$ such that $x_i a_i y_i \neq 0$ $(i = 1, 2)$. Also, $x_i a_i y_i \in R_T$ $(i = 1, 2)$. Thus, since R_T is prime, $x_1 a_1 y_1 R_T x_2 a_2 y_2 \neq 0$ and so $a_1 y_1 R_T x_2 a_2 \neq 0$. But $y_1 R_T x_2 \subseteq R_U$, by (3.1). Hence $a_1 R_U a_2 \neq 0$. This shows that R_U is prime.

(iii) Assume that $J(R_U) \neq 0$. We shall prove that $J(R_T) \neq 0$. Choose $a \in J(R_U)\backslash 0$. By Lemma 2.2, there exist $x \in R_p$ and $y \in R_q$ such that

$$xay \neq 0. \tag{3.2}$$

Let $w \in R_T$. Then $ywx \in R_{qTp} = R_U$, by (3.1). Thus, since $J(R_U)$ is an ideal of R_U, $aywx \in J(R_U)$. Hence $aywx$ is right quasiregular in R_U and so, by Lemma 3.1, there exists $b \in R_U$ such that $a + b = (aywx)b$. Consequently,

$$xayw + xbyw = (xayw)(xbyw). \tag{3.3}$$

However, $xayw, xbyw \in R_{pUq}R_T \subseteq R_T^2 \subseteq R_T$. Hence, by (3.3), $xayw$ is a right quasiregular element of R_T. Since $xay \in R_T$ and w is arbitrary in R_T, we have that $xay \in J(R_T)$. Thus, by (3.2), $J(R_T) \neq 0$. From this argument, it follows that if R_T is semiprimitive, then so also is R_U.

□

Theorem 3.3. *Let e and f be $\mathcal{D}$-equivalent idempotents of a semigroup S and let R be a ring $\mathcal{D}$-faithfully graded by S.*

(i) *If R_e is semiprime (respectively, prime, semiprimitive), then R_f is semiprime (respectively, prime, semiprimitive).*

(ii) *If R_{H_e} is semiprime (respectively, prime, semiprimitive), then R_{H_f} is semiprime (respectively, prime, semiprimitive).*

Proof. Note first that, by Lemma 2.1 (ii), there exist $p, q \in S$ such that $pqp = p$, $qpq = q$, $pq = e$, $qp = f$. We may therefore apply Lemma 3.2, with suitable choices for T and U. For (i), take $T = \{e\}$ and $U = \{f\}$. For (ii), having observed that, by Lemma 2.1 (i), $pH_fq = H_e$, take $T = H_e$ and $U = H_f$. □

We end this section with an application. Kelarev [**3**] has shown that if R is a ring that is faithfully graded by an *inverse* semigroup S and if R_G is semiprime (respectively, semiprimitive) for all maximal subgroups G of S, then R is semiprime (respectively, semiprimitive). In view of Theorem 3.3 (ii), the same conclusion holds under the weaker hypothesis that R_G is semiprime (respectively, semiprimitive) for one maximal subgroup G in each $\mathcal{D}$-class of S. In a sequel to [**3**], the author [**5**] has proved that if R is a ring that is faithfully graded by a *bisimple* inverse semigroup S (an inverse semigroup consisting of a single $\mathcal{D}$-class) and if R_G is prime for some maximal subgroup G of S, then R is prime.

4. Right primitivity

A ring R is said to be *right primitive* if and only if $R \neq 0$ and there exists a faithful irreducible right R-module. Our second main result, Theorem 4.3, concerns this property and is the exact analogue of Theorem 3.3.

We begin by noting the following standard result.

Lemma 4.1. *Let R be a right primitive ring. Then R contains a proper right ideal M such that the right R-module R/M is faithful and irreducible: thus, for all $a \in R\backslash 0$, $Ra \nsubseteq M$, and, for all $a \in R\backslash M$, $(a + M)R = R/M$.*

Lemma 4.2. *Let p and q be elements of a semigroup S such that $pqp = p$, $qpq = q$ and let e, f denote the idempotents pq, qp, respectively. Let T and U be submonoids of H_e and H_f, respectively, such that $T = pUq$ and let R be a ring $\mathcal{D}$-faithfully graded by S. If R_T is right primitive, then R_U is right primitive.*

Proof. Observe that

$$Tp = pU, \qquad qT = Uq, \qquad qTp = U. \tag{4.1}$$

Assume that R_T is right primitive. Thus $R_T \neq 0$ and, by Lemma 4.1, R_T contains a proper right ideal M such that

$$(\forall a \in R_T\backslash 0), \quad R_Ta \nsubseteq M \tag{4.2}$$

and

$$(\forall a \in R_T\backslash M), \quad (a + M)R_T = R_T/M. \tag{4.3}$$

Note also that, since $R_T \neq 0$, it follows from Lemma 2.2, with p and q interchanged and e and f interchanged, that there exist $y \in R_q$ and $x \in R_p$ such that $yR_Tx \neq 0$. But $yR_Tx \subseteq R_{qTp} = R_U$, by (4.1). Hence $R_U \neq 0$. We construct a faithful irreducible right R_U-module.

Consider the subgroup R_TR_{Tp} of $(R,+)$. Since, by (4.1), $TpU = T^2p \subseteq Tp$, we have that $(R_TR_{Tp})R_U \subseteq R_TR_{Tp}$. Hence R_TR_{Tp} is a right R_U-module under the multiplication induced by that in R.

Now write

$$N := \{v \in R_TR_{Tp} : vR_{qT} \subseteq M\}.$$

This is readily seen to be a subgroup of $(R_TR_{Tp},+)$. Also, from (4.1), $UqT = qT^2 \subseteq qT$ and so, for all $v \in N$ and all $r \in R_U$,

$$(vr)R_{qT} \subseteq vR_UR_{qT} \subseteq vR_{qT} \subseteq M.$$

Hence $vr \in N$. Consequently, N is an R_U-submodule of R_TR_{Tp}.

We may therefore form the right R_U-module R_TR_{Tp}/N. To complete the proof, we show that this is faithful and irreducible.

Let $a \in R_U \backslash 0$. By Lemma 2.2, there exist $x \in R_p$ and $y \in R_q$ such that $xay \neq 0$. Now $xay \in R_{pUq} = R_T$. Hence, by (4.2), there exists $r \in R_T$ such that $rxay \notin M$. But $R_p \subseteq R_{Tp}$, since $p = ep$ and $e \in T$. Thus $rx \in R_TR_{Tp}$ and so $rxa \in R_TR_{Tp}$. Also, $R_q \subseteq R_{qT}$, since $q = qe$ and $e \in T$. Hence $y \in R_{qT}$. Since $rxay \notin M$, it follows that $rxa \notin N$; that is, $(rx + N)a \neq N$. This shows that R_TR_{Tp}/N is faithful.

Next, let $v \in R_TR_{Tp}$, $v \notin N$. Then there exists $y \in R_{qT}$ such that $vy \notin M$. Consider an arbitrary element $w \in R_TR_{Tp}$. We have that $w = \sum_{i=1}^{n} r_ix_i$ for some positive integer n and some $r_i \in R_T$, $x_i \in R_{Tp}$ $(i = 1,2,\dots,n)$. Now $vy \in R_TR_{Tp}R_{qT} \subseteq R_{T^2pqT} \subseteq R_T$. Hence, since $vy \notin M$, we see from (4.3) that there exist $s_i \in R_T$ such that $(vy + M)s_i = r_i + M$; that is, $vys_i - r_i \in M$ $(i = 1,2,\dots,n)$. Write $z := \sum_{i=1}^{n} s_ix_i$. Since $vys_ix_i - r_ix_i \in MR_{Tp}$ $(i = 1,2,\dots,n)$, we have that $vyz - w \in MR_{Tp} \subseteq R_TR_{Tp}$. Also,

$$(vyz - w)R_{qT} \subseteq MR_{Tp}R_{qT} \subseteq MR_T \subseteq M,$$

and so $vyz - w \in N$. Further, $yz \in R_{qT}R_TR_{Tp} \subseteq R_{qTp} = R_U$, by (4.1). Hence $(v+N)yz = w + N$. Thus R_TR_{Tp}/N is irreducible. $\square$

Theorem 4.3. *Let e and f be $\mathcal{D}$-equivalent idempotents of a semigroup S and let R be a ring $\mathcal{D}$-faithfully graded by S.*

(i) *If R_e is right primitive, then R_f is right primitive.*

(ii) *If R_{H_e} is right primitive, then R_{H_f} is right primitive.*

Proof. By Lemma 2.1 (ii), there exist $p, q \in S$ such that $pqp = p$, $qpq = q$, $pq = e$, $qp = f$. We may therefore apply Lemma 4.2, with suitable choices for T and U. For (i), take $T = \{e\}$ and $U = \{f\}$. For (ii), first note that $pH_fq = H_e$, by Lemma 2.1 (i), then take $T = H_e$ and $U = H_f$. $\square$

In passing, we remark that it is shown in [**5**] that if R is a ring faithfully graded by a bisimple inverse semigroup S, and if, for some maximal subgroup G of S, R_G is right primitive and such that $a \in aR_G$ for all $a \in R_G$, then the whole ring R is right primitive.

5. Simplicity

To conclude, we give an example to show that there is no analogue of Theorem 4.3 for the property of simplicity. This is related to our earlier Example 1.3.

Example 5.1. Let F be a field and let $\mathbb{N}$ be the set of all positive integers. Denote by $M_{\mathbb{N}}(F)$ the set of all $\mathbb{N} \times \mathbb{N}$ matrices over F with at most finitely many non-zero entries in each row and column. Under the usual matrix operations, $M_{\mathbb{N}}(F)$ is a ring. Let $F_{1,\mathbb{N}}$ and $F_{\mathbb{N},1}$ denote, respectively, the sets of all $1 \times \mathbb{N}$ and $\mathbb{N} \times 1$ matrices over F with at most finitely many non-zero entries. Now define an S_2-graded ring R with the property that

$$R_{e_{11}} = F, \qquad R_{e_{12}} = F_{1,\mathbb{N}}, \qquad R_{e_{21}} = F_{\mathbb{N},1}, \qquad R_{e_{22}} = M_{\mathbb{N}}(F), \qquad R_z = 0,$$

where, for all $i, j, k \in \{1, 2\}$, all $a \in R_{e_{ij}}$ and all $b \in R_{e_{jk}}$, the product ab is obtained by matrix multiplication. It is a routine matter to prove that R is $\mathcal{D}$-faithful. Also, e_{11} and e_{22} are $\mathcal{D}$-equivalent idempotents in S_2 and $R_{e_{11}}$ is a simple ring. However, $R_{e_{22}}$ is not a simple ring; for

$$\{a \in M_{\mathbb{N}}(F) : \text{rank of } a \text{ is finite}\}$$

is a non-trivial proper ideal of $M_{\mathbb{N}}(F)$.

We note that, by Theorem 4.3, since $R_{e_{11}}$ is right primitive, so also is $R_{e_{22}}$; that is, $M_{\mathbb{N}}(F)$ is right primitive—a result that can readily be proved directly. Further, it is easy to see that $R \cong M_{\mathbb{N}}(F)$; hence R itself is right primitive, as predicted by [**5**, Theorem 4.2].

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