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ITERATED LIMITS OF LATTICES

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1. Introduction. In this paper the results of [**5**] are extended to classes of lattices. We assume familiarity with [**5**], but we recall for convenience the principal definitions and notations. If $\mathscr{C}$ is a category and if $\mathfrak{s} = \langle I; \{A_i\}; \{\varphi_j^i\}\rangle$ is a direct [resp., inverse] limit system in $\mathscr{C}$, then $\lim_{\rightarrow}(\mathfrak{s}, \mathscr{C})$ [resp., $\lim_{\leftarrow}(\mathfrak{s}, \mathscr{C})$] is the direct [resp., inverse] limit of $\mathfrak{s}$ (determined only up to isomorphism in $\mathscr{C}$). If $\mathfrak{s}$ is an inverse limit system of sets or universal algebras, let $\lim_{\leftarrow} \mathfrak{s}$ denote the canonical construction of inverse limit described for example in [**1**, Chapter 3].

Definition 1. Let H be a class of objects from a category $\mathscr{C}$.

(1) $L_{\rightarrow}(H, \mathscr{C})$ is the class of all objects of the form $\lim_{\rightarrow}(\mathfrak{s}, \mathscr{C})$ where $\mathfrak{s}$ is a direct limit system in $\mathscr{C}$ with objects from H.

(2) For ordinals α we define $L_{\rightarrow}^{\alpha}(H, \mathscr{C})$ inductively as follows: $L_{\rightarrow}^{0}(H, \mathscr{C})$ is the class of all objects in $\mathscr{C}$ isomorphic to objects in H; $L_{\rightarrow}^{\alpha+1}(H, \mathscr{C}) = L_{\rightarrow}(L_{\rightarrow}^{\alpha}(H, \mathscr{C}), \mathscr{C})$; if α is a limit ordinal, then $L_{\rightarrow}^{\alpha}(H, \mathscr{C}) = \bigcup \{L_{\rightarrow}^{\beta}(H, \mathscr{C}) : \beta < \alpha\}$.

(3) Let ∞ be any element which is not an ordinal. Then we define $L_{\rightarrow}$-rank$(H, \mathscr{C})$ to be the smallest ordinal α such that $L_{\rightarrow}^{\alpha}(H, \mathscr{C}) = L_{\rightarrow}^{\alpha+1}(H, \mathscr{C})$ if such an α exists; otherwise, $L_{\rightarrow}$-rank$(H, \mathscr{C}) = \infty$.

(4) Replacing direct limits by inverse limits we similarly define $L_{\leftarrow}(H, \mathscr{C})$, $L_{\leftarrow}^{\alpha}(H, \mathscr{C})$, and $L_{\leftarrow}$-rank$(H, \mathscr{C})$.

Let On denote the class of all ordinals and let On*: = On $\cup \{\infty\}$. (M) denotes the set-theoretic axiom denying the existence of arbitrarily large measurable cardinals. Let $\mathscr{L}$ denote the category of lattices and lattice homomorphisms.

The principal result of the paper is the following.

THEOREM 1. *Let* $\alpha, \beta \in$ On. *Then there exists a class H of lattices such that*

$$L_{\rightarrow}\text{-rank}(H, \mathscr{L}) = \alpha$$

and

$$L_{\leftarrow}\text{-rank}(H, \mathscr{L}) = \beta$$

Furthermore, if (M) *is assumed, the above holds also for* $\alpha, \beta \in$ On*.

Let us outline briefly the proof in [**5**]. We constructed certain categories of sets, $\mathscr{S}_{\alpha\beta}$, which contained subclasses having the desired ranks. Then we described certain full embeddings of categories (the "acceptable" embeddings)

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which preserve the required ranks. Finally, we constructed such embeddings of $\mathscr{S}_{\alpha\beta}$ into various categories of algebras.

The natural approach to Theorem 1, then, would be to attempt to construct acceptable embeddings of $\mathscr{S}_{\alpha\beta}$ into $\mathscr{L}$. This, however, is impossible because of the fact that every constant mapping between two lattices is a homomorphism. In $\mathscr{S}_{\alpha\beta}$, the only morphism from an object to itself is the identity, so there are no full embeddings of $\mathscr{S}_{\alpha\beta}$ into $\mathscr{L}$. Instead, we will construct acceptable embeddings of $\mathscr{S}_{\alpha\beta}$ into the category $\mathscr{L}^*$ of bounded lattices and bound-preserving homomorphisms. These will have the additional property that ranks are preserved by the inclusion functor from $\mathscr{L}^*$ to $\mathscr{L}$. The precise nature of these embeddings is described in § 2 and their relevant properties established. In § 3 the embeddings are constructed under the assumption of existence of certain classes of lattices. In § 4 these classes of lattices are constructed.

2. We begin with some observations concerning the results in [**5**], namely that Theorem 4.5 holds under weakened hypotheses. First we modify the definitions. Recall that the category $\mathscr{S}_{\alpha\beta}$ is the disjoint union of categories $\mathscr{J}_\gamma^{\rightarrow}$ for $\gamma < 1 + \alpha$ and $\mathscr{J}_\gamma^{\leftarrow}$ for $\gamma < 1 + \beta$.

Definition 2. (a) Let $\gamma \in \mathrm{On}$ and let $G:\mathscr{J}_\gamma^{\rightarrow} \rightarrow \mathscr{S}$ be a functor. G is called $\mathscr{J}_\gamma^{\rightarrow}$*-acceptable* if and only if the following hold:

(i) for every morphism $\varphi:X \rightarrow Y$ in $\mathscr{J}_\gamma^{\rightarrow}$, $G(X) \subseteq G(Y)$ and $G(\varphi)$ is the inclusion map;

(ii) if $\mathscr{D}$ is a collection of sets in $\mathscr{J}_\gamma$ directed by inclusion then $G(\bigcup \mathscr{D}) = \bigcup \{G(X):X \in \mathscr{D}\}$.

(b) Let $H:\mathscr{J}_\gamma^{\leftarrow} \rightarrow \mathscr{S}$ be a functor. Then H is called $\mathscr{J}_\gamma^{\leftarrow}$*-acceptable* if and only if the following hold:

(i) if $\varphi:X \rightarrow Y$ is a morphism in $\mathscr{J}_\gamma^{\leftarrow}$, then $G(Y) \subseteq G(X)$ and $G(\varphi)$ is a set retraction (i.e., $x \in G(Y)$ implies $G(\varphi)(x) = x$);

(ii) if $\mathscr{D}$ is a collection of sets in $\mathscr{J}_\gamma$ directed by inclusion, then $H(\bigcup \mathscr{D}) = \bigcup \{H(X):X \in \mathscr{D}\}$;

(iii) if $\mathfrak{s} = \langle I; \{X_i\}; \{\varphi_j^i\}\rangle$ is an inverse limit system in $\mathscr{J}_\gamma^{\leftarrow}$, then the system $H(\mathfrak{s}) = \langle I; \{H(X_i)\}; \{H(\varphi_j^i)\}\rangle$ has the terminal property; i.e. for every $z \in \lim_{\leftarrow} H(\mathfrak{s})$, there exists $i_0 \in I$ such that $z(i) = z(i_0)$ for all $i \geqq i_0$.

(c) An embedding $F:\mathscr{J}_\gamma^{\rightarrow} \rightarrow \mathscr{K}$ [resp., $F:\mathscr{J}_\gamma^{\leftarrow} \rightarrow \mathscr{K}$], where $\mathscr{K}$ is a concrete category, is $\mathscr{J}_\gamma^{\rightarrow}$-acceptable [resp., $\mathscr{J}_\gamma^{\leftarrow}$-acceptable] if and only if $U \circ F$ is, where $U:\mathscr{K} \rightarrow \mathscr{S}$ is the forgetful functor for $\mathscr{K}$.

Now observe that in [**5**], the conclusions of Lemmas 4.1 and 4.3 hold under the weaker hypotheses that $F|\mathscr{J}_\gamma^{\rightarrow}$ be $\mathscr{J}_\gamma^{\rightarrow}$-acceptable and $F|\mathscr{J}_\gamma^{\leftarrow}$ be $\mathscr{J}_\gamma^{\leftarrow}$-acceptable respectively. The proofs in each case are exactly the same. Thus, we have the following strengthening of Theorem 4.5 of [**5**].

THEOREM 2. *Let $\alpha, \beta \in \mathrm{On}^*$, $\mathscr{K}$ a category of algebras, and $F:\mathscr{S}_{\alpha\beta} \rightarrow \mathscr{K}$ a full embedding such that for each $\gamma < 1 + \alpha$, $F|\mathscr{J}_\gamma^{\rightarrow}$ is $\mathscr{J}_\gamma^{\rightarrow}$-acceptable, and for each*

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$\gamma < 1 + \beta$, $F|\mathscr{J}_\gamma^{\leftarrow}$ is $\mathscr{J}_\gamma^{\leftarrow}$-acceptable. Then there is a set K of objects of $\mathscr{K}$ such that

$$L_{\rightarrow}\text{-rank}(K, \mathscr{K}) = \alpha,$$

and

$$L_{\leftarrow}\text{-rank}(K, \mathscr{K}) = \beta.$$

Let $\mathscr{L}_{\mathbf{B}}$ denote the category of bounded lattices with bounds **0** and **1** being values of nullary operations. Let $\mathscr{L}^*$ be the image of $\mathscr{L}_{\mathbf{B}}$ under the forgetful functor into $\mathscr{L}$. Thus $\mathscr{L}^*$ is the category of all lattices having bounds, with all bound-preserving homomorphisms between them.

Since $\mathscr{L}_{\mathbf{B}}$ is an equational category it follows that in $\mathscr{L}^*$ direct and inverse limits are isomorphic to the canonical constructions. That is, if $\mathfrak{s}$ is a direct or inverse limit system in $\mathscr{L}^*$, then $\lim(\mathfrak{s}, \mathscr{L}^*) = \lim(\mathfrak{s}, \mathscr{L})$ (where lim is $\lim_{\rightarrow}$ or $\lim_{\leftarrow}$, respectively).

Definition 3. A direct [resp., inverse] limit system $\mathfrak{s} = \langle I; \{L_i\}; \{\varphi_j^i\}\rangle$ is called *trivial* if and only if for every $i \in I$ there exists some $j \geqq i$ such that φ_j^i [resp., φ_i^j] is constant.

LEMMA 1. *Let $\mathfrak{s}$ be a direct* [resp., inverse] *limit system of lattices. If $\mathfrak{s}$ is trivial, then* $\lim_{\rightarrow}(\mathfrak{s}, \mathscr{L})$ [resp., $\lim_{\leftarrow}(\mathfrak{s}, \mathscr{L})$] *is a one-element lattice.*

Proof. For direct limits, suppose $u \in L_i$ and $v \in L_j$. Choose $k \in I$ with $i \leqq k, j \leqq k$. There exists $l \geqq k$ such that φ_l^k is constant. Then $\varphi_l^i(u) = \varphi_l^k(\varphi_k^i(u)) = \varphi_l^k(\varphi_k^j(v)) = \varphi_l^j(v)$. Hence, u and v represent the same element of the direct limit (considered as a quotient of the disjoint union of the L_i). Thus the direct limit has only one element.

For inverse limits, let $x, y \in \lim_{\leftarrow} \mathfrak{s}$. For any $i \in I$, choose $j \in I, j \geqq i$ such that φ_i^j is constant. Then $x(i) = \varphi_i^j(x(j)) = \varphi_i^j(y(j)) = y(i)$. Thus $x = y$, so $\lim_{\leftarrow} \mathfrak{s}$ has only one element.

Definition 4. A class K of bounded lattices is called *strongly bounded* if and only if whenever $L_1, L_2 \in K$ and $\varphi: L_1 \rightarrow L_2$ is a non-constant homomorphism, then φ is bound-preserving.

LEMMA 2. *Let K be a strongly bounded class of lattices and $H \subseteq K$. Let E be the class of all one-element lattices. Then $L_{\rightarrow}(H, \mathscr{L}) = L_{\rightarrow}(H, \mathscr{L}^*) \cup E$, and $L_{\leftarrow}(H, \mathscr{L}) = L_{\leftarrow}(H, \mathscr{L}^*) \cup E$.*

Proof. By the remark preceding Definition 3 plus the fact that all constant maps are homomorphisms in $\mathscr{L}$, it is clear that $L_{\rightarrow}(H, \mathscr{L}^*) \cup E \subseteq L_{\rightarrow}(H, \mathscr{L})$. Now suppose $\mathfrak{s} = \langle I; \{\mathfrak{U}_i\}; \{\varphi_j^i\}\rangle$ is a direct limit system in H. If $\mathfrak{s}$ is trivial, then $\lim_{\rightarrow} \mathfrak{s} \in E$ by Lemma 1. If $\mathfrak{s}$ is not trivial, then there is some $i_0 \in I$ such that for every $j \geqq i_0$, $\varphi_j^{i_0}$ is non-constant. Then if $k \geqq j \geqq i_0$, φ_k^j is non-constant (for otherwise $\varphi_k^{i_0} = \varphi_j^{i_0} \circ \varphi_k^j$ would be constant). Since K is strongly bounded it follows that φ_k^j is a morphism in $\mathscr{L}^*$ for every $k \geqq j \geqq i_0$. Letting $\mathfrak{s}'$

denote the direct limit system in $\mathscr{L}^*$ obtained from $\mathfrak{s}$ by restriction to $\{j:j \geqq i_0\}$, we have $\lim_{\to}(\mathfrak{s},\mathscr{L}) = \lim_{\to}(\mathfrak{s}',\mathscr{L}^*) \in L_{\to}(H,\mathscr{L}^*)$.

The proof for inverse limits is similar.

By transfinite induction we obtain the following

COROLLARY 1. *Let K be a strongly bounded class of lattices which is closed under formation of direct* [resp., inverse] *limits in $\mathscr{L}^*$. Then for any $H \subseteq K$ and any $\alpha \in$* On*, $L_{\to}^{\alpha}(H,\mathscr{L}) = L_{\to}^{\alpha}(H,\mathscr{L}^*) \cup E$ [resp., $L_{\leftarrow}^{\alpha}(H,\mathscr{L}) = L_{\leftarrow}^{\alpha}(H,\mathscr{L}^*) \cup E$].

COROLLARY 2. *If K is a strongly bounded class of lattices closed under formation of direct* [resp., inverse] *limits in $\mathscr{L}^*$ and if $K \cap E = \emptyset$, then for any $H \subseteq K$,* $L_{\to}$-rank$(H,\mathscr{L}^*) = L_{\to}$-rank$(H \cup E,\mathscr{L})$ [resp., $L_{\leftarrow}$-rank$(H,\mathscr{L}^*) = L_{\leftarrow}$-rank $(H \cup E,\mathscr{L})$].

Combining these results with Theorem 2 we obtain the following.

THEOREM 3. *Let $\alpha, \beta \in$* On* *and let $F:\mathscr{S}_{\alpha\beta} \to \mathscr{L}^*$ be a full embedding such that*

(i) *for each $\gamma < 1 + \alpha$ the restriction $F|\mathscr{J}_{\gamma}^{\to}$ is $\mathscr{J}_{\gamma}^{\to}$-acceptable;*

(ii) *for each $\gamma < 1 + \beta$ the restriction $F|\mathscr{J}_{\gamma}^{\leftarrow}$ is $\mathscr{J}^{\leftarrow}$-acceptable;*

(iii) *the image of F is a strongly bounded class of lattices containing no one-element lattices.*

Then there exists a class H in the image of F such that $L_{\to}$-rank$(H \cup E,\mathscr{L}) = \alpha$ *and* $L_{\leftarrow}$-rank$(H \cup E,\mathscr{L}) = \beta$.

3. We begin this section with a number of notations and definitions. For a bounded Lattice L, let $\mathbf{0}(L)$ and $\mathbf{1}(L)$ denote the least and greatest elements, respectively, of L. The set L-$\{\mathbf{0}(L), \mathbf{1}(L)\}$ is called the *interior* of L, denoted $\mathbf{int}(L)$.

Definition 5. (a) Let L be a bounded lattice and let L_1, L_2 be two sublattices of L which are bounded. L is called the *vertical sum* of L_1 and L_2, written $L = L_1 + L_2$, if and only if

(i) $\mathbf{0}(L) = \mathbf{0}(L_1)$, $\mathbf{1}(L_1) = \mathbf{0}(L_2)$, $\mathbf{1}(L_2) = \mathbf{1}(L)$; and

(ii) $L = L_1 \cup L_2$.

(b) Let L be a bounded lattice and for each $i \in I$ let L_i be a sublattice of L which is bounded. L is called the *horizontal sum* of the lattices $\{L_i:i \in I\}$, written $L = \oplus\{L_i:i \in I\}$, if and only if

(i) for each $i \in I$, $\mathbf{0}(L_i) = \mathbf{0}(L)$ and $\mathbf{1}(L_i) = \mathbf{1}(L)$;

(ii) if $i \neq j$ then $\mathbf{int}(L_i) \cap \mathbf{int}(L_j) = \emptyset$; and

(iii) $L = \bigcup \{L_i:i \in I\}$.

In case $I = \{1, 2\}$ we will also write $L = L_1 \oplus L_2$.

Note that for any pair of bounded lattices L_1, L_2 there is a lattice L uniquely determined up to isomorphism such that $L = L_1' + L_2'$ where $L_1 \cong L_1'$ and $L_2 \cong L_2'$. Thus, given $L_1, L_2 \in \mathscr{L}^*$, $L_1 + L_2$ will refer to any such lattice, unless otherwise specified. Similar remarks apply to the notations $\oplus \{L_i : i \in I\}$ and $L_1 \oplus L_2$. Examples of these constructions are given in Figure 1.

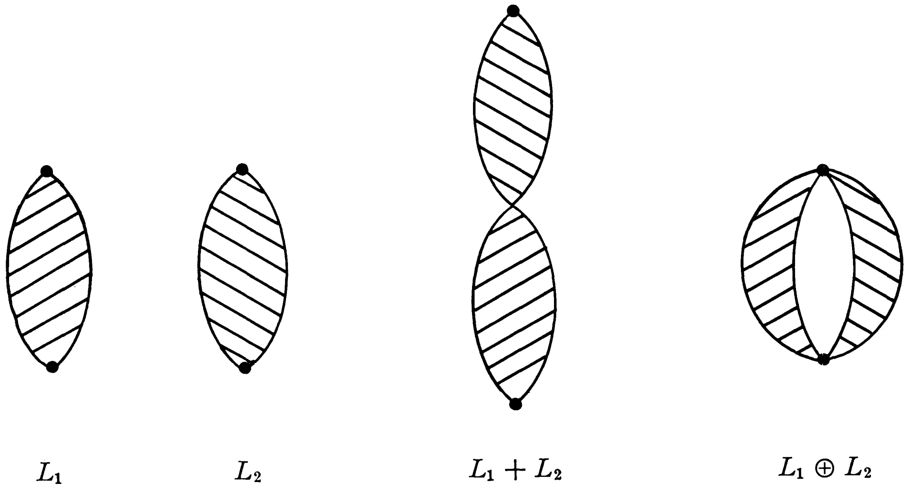

FIGURE 1. Sums of lattices.

Definition 6. A lattice L is called *v-simple* [resp., *h-simple*] if and only if for every pair of lattices $L_1, L_2 \in \mathscr{L}^*$ and every non-constant homomorphism $\varphi: L \rightarrow L_1 + L_2$ [resp., $\varphi: L \rightarrow L_1 \oplus L_2$], either the image of L under φ is contained in L_1 or in L_2.

Let $\gamma \in \mathrm{On}$. We next recall some of the structure of the categories $\mathscr{J}_\gamma^{\rightarrow}$ and $\mathscr{J}_\gamma^{\leftarrow}$. There is a set of sets $\mathscr{J}_\gamma$ whose elements are the objects of $\mathscr{J}_\gamma^{\rightarrow}$ and of $\mathscr{J}_\gamma^{\leftarrow}$. There is a set N_γ such that the sets in $\mathscr{J}_\gamma$ are all subsets of $N_\gamma \times \omega$. (In the notation of [**5**], N_γ is $\bigcup \{J_\delta^\gamma : \delta < \gamma\}$.) If $c \in N_\gamma$, $X \in \mathscr{J}_\gamma$, then $\langle c, 0\rangle \in X$, and if $\langle c, n\rangle \in X$, $n \geqq m$, then $\langle c, m\rangle \in X$. If $X, Y \in \mathscr{J}_\gamma$ then there is a morphism $\varphi: X \rightarrow Y$ in $\mathscr{J}_\gamma^{\rightarrow}$ if and only if $X \subseteq Y$, and in this case φ is the inclusion map. There is a morphism $\varphi: X \rightarrow Y$ in $\mathscr{J}_\gamma^{\leftarrow}$ if and only if $Y \subseteq X$, and in this case φ is the map $\psi(X, Y)$ defined by

$$\psi(X, Y)(\langle c, n\rangle) = \begin{cases} \langle c, n\rangle \text{ if } \langle c, n\rangle \in Y \\ \langle c, 0\rangle \text{ if } \langle c, n\rangle \notin Y \end{cases}$$

for all $\langle c, n\rangle \in X$. Without loss of generality, we assume the sets N_γ, N_δ are disjoint for $\gamma \neq \delta$.

Let $\alpha, \beta \in \mathrm{On}^*$ be fixed throughout the remainder of this section. We assume now the existence of a class $\mathscr{D}$ of bounded lattices with the following properties:

(i) $\mathscr{D}$ is discrete in $\mathscr{L}^*$; that is, if $L_1, L_2 \in \mathscr{D}$ and $\varphi: L_1 \rightarrow L_2$ is an $\mathscr{L}^*$-morphism, then $L_1 = L_2$ and φ is the identity map.

(ii) $\mathscr{D}$ is strongly bounded.

(iii) Each lattice in $\mathscr{D}$ is v-simple and h-simple.

(iv) There is a 1-1 mapping

$$f: \bigcup_{\gamma<1+\alpha} (N_\gamma \times \omega) \cup \bigcup_{\gamma<1+\beta} (N_\gamma \times \omega \times 3) \to \mathscr{D}.$$

Under this assumption we will construct an embedding F satisfying the hypotheses of Theorem 3. In § 4 we will describe the construction of such a class $\mathscr{D}$.

First, let $\gamma < 1 + \alpha$. We will define $F_\gamma:\mathscr{I}_\gamma^{\to} \to \mathscr{L}^*$. If $X \in \mathscr{I}_\gamma$, let

(1) $F_\gamma(X) = \oplus\{f(a):a \in X\}$.

More precisely, we assume (possibly after changing the underlying sets of some lattices in $\mathscr{D}$) that the lattices $f(a)$ for $a \in N_\gamma \times \omega$ all have the same least and greatest elements, and their interiors are pairwise disjoint. Then $\bigcup \{f(a):a \in X\}$ clearly becomes a lattice $F_\gamma(X)$ which satisfies (1). If $\varphi:X \to Y$ is a morphism in $\mathscr{I}_\gamma^{\to}$, then $X \subseteq Y$, so $F_\gamma(X) \subseteq F_\gamma(Y)$. Let $F_\gamma(\varphi)$ be the inclusion map. Clearly F_γ is a functor from $\mathscr{I}_\gamma^{\to}$ to $\mathscr{L}^*$. The sublattices $f(a)$ for all $a \in X$ will be called the *constituents* of $F_\gamma(X)$.

Next let $\gamma < 1 + \beta$. We will construct a functor $G_\gamma:\mathscr{I}_\gamma^{\leftarrow} \to \mathscr{L}^*$. For each $c \in N_\gamma$ we construct the lattice $R(c, \omega)$ pictured in Figure 2. Precisely, $R(c, \omega)$ consists of the disjoint union of the lattices $f(c, n, i)$, $n \in \omega$, $i \in 3$ and a single additional point $k(c)$ with the following identifications of extreme points for each $n \in \omega$:

> the least element of $f(c, n, 1)$ and if $n > 0$ the greatest element of $f(c, n - 1, 1)$ are identified with the least element of $f(c, n, 0)$; the greatest element of $f(c, n, 2)$ and if $n > 0$ the least element of $f(c, n - 1, 2)$ are identified with the greatest element of $f(c, n, 0)$.

The ordering is defined to be the smallest partial order satisfying the above conditions, containing the orderings on each of the lattices $f(c, n, i)$, $n \in \omega$, $i \in 3$, and for which $k(c)$ is less than every element of $f(c, n, 2)$ and greater than every element of $f(c, n, 1)$, for every $n \in \omega$. Again we assume that the underlying sets of the lattices in $\mathscr{D}$ are so modified that for each $n \in \omega$ and $i \in 3$, $f(c, n, i)$ is actually a sublattice of $R(c, \omega)$.

Now if $n \in \omega$, define $R(c, n) \subseteq R(c, \omega)$ as follows:

$$R(c, n) = k(c) \cup \bigcup \{f(c, m, 0):m \in \omega, m \leqq n\}$$
$$\cup \bigcup \{\mathbf{int}(f(c, m, i)):i \in \{1, 2\}, m \in \omega, m \leqq n\}.$$

Then $R(c, n)$, considered as a partially ordered subset of $R(c, \omega)$ is a lattice, but not a sublattice of $R(c, \omega)$. Namely, the least element of $f(c, n, 2)$ and the greatest element of $f(c, n, 1)$ are now replaced by $k(c)$. $R(c, 2)$ is pictured in Figure 3.

Definition 7. Let $X \in \mathscr{I}_\gamma$ and $c \in N_\gamma$. Define

$$Q(c, X) = \begin{cases} R(c, n) \text{ if } \langle c, n\rangle \in X \text{ but } \langle c, n + 1\rangle \notin X, \\ R(c, \omega) \text{ if } \langle c, n\rangle \in X \text{ for all } n \in \omega. \end{cases}$$

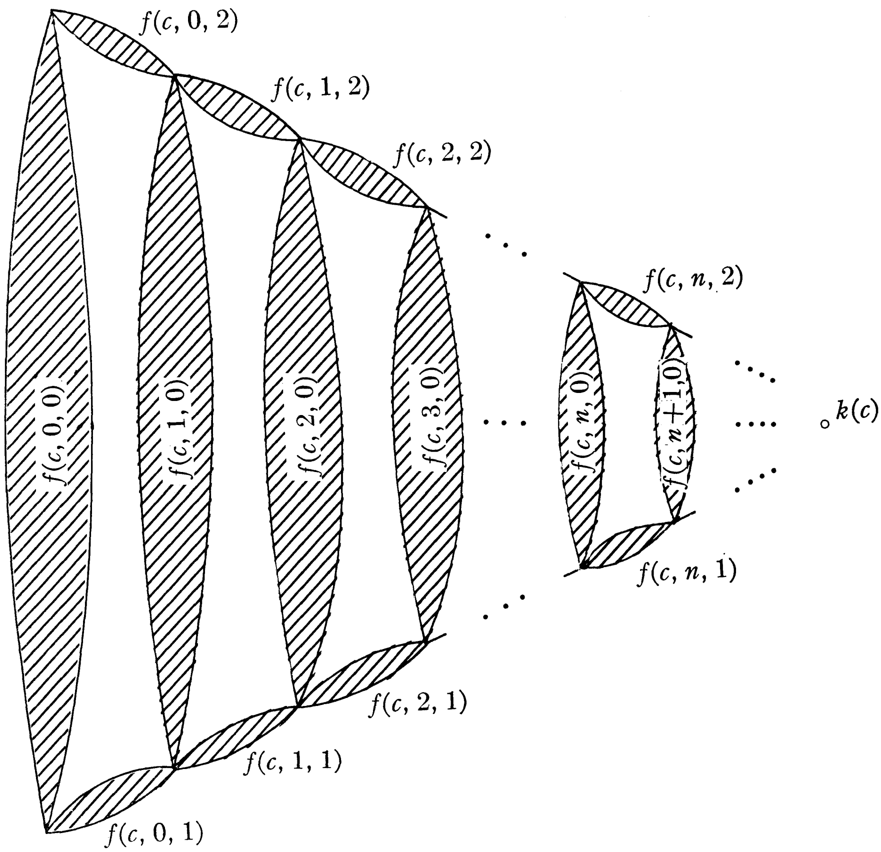

FIGURE 2. $R(c, \omega)$

Finally, we define

$$G_\gamma(X) = \oplus\{Q(c, X) : c \in N_\gamma\}.$$

Again, we assume that $Q(c, X)$ is a sublattice of $G_\gamma(X)$ for each $c \in N_\gamma$, and that for $X, Y \in \mathscr{J}_\gamma$, $G_\gamma(X)$ and $G_\gamma(Y)$ have the same extreme elements.

If $\langle c, n\rangle \in X$ and $i \in 3$, then $f(c, n, i)$ is called a *constituent* of $G_\gamma(X)$. If P is a constituent of $G_\gamma(X)$, there is a unique embedding of P into $G_\gamma(X)$ whose restriction to $\mathbf{int}(P)$ is the inclusion. Its image in $G_\gamma(X)$ will be denoted by P_X and the embedding will be loosely referred to as the inclusion of P into $G_\gamma(X)$.

Observe that if $X, Y \in \mathscr{J}_\gamma$ and $X \subseteq Y$, then $G_\gamma(X) \subseteq G_\gamma(Y)$.

Definition 8. If $X, Y \in \mathscr{J}_\gamma$ and $Y \subseteq X$, define $G_\gamma(\psi(X, Y))$ as follows: if $z \in G_\gamma(X)$

$$G_\gamma(\psi(X, Y))(z) = \begin{cases} z \text{ if } z \in G_\gamma(Y), \\ k(c) \text{ if } z \notin G_\gamma(Y) \text{ and } z \in Q(c, X). \end{cases}$$

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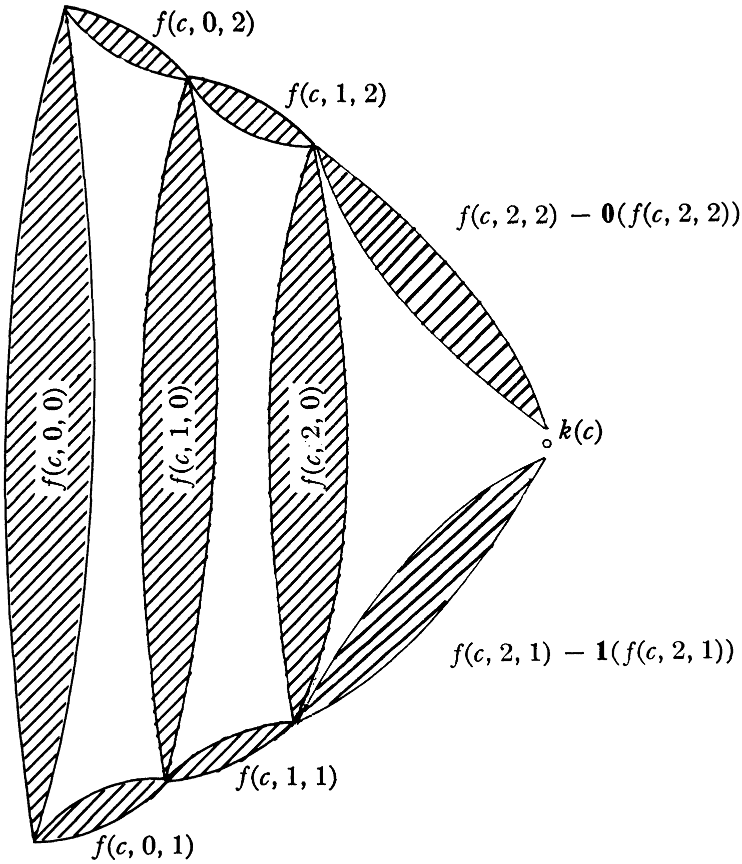

FIGURE 3. $R(c, 2)$

Then $G_\gamma(\psi(X, Y))$ is an $\mathscr{L}^*$-morphism from $G_\gamma(X)$ to $G_\gamma(Y)$ and clearly $G_\gamma:\mathscr{J}_\gamma^{\leftarrow} \to \mathscr{L}^*$ is a functor. Next we examine the structure of $G_\gamma(X)$ more closely.

Definition 9. Let $\gamma \in \mathrm{On}$, $X \in \mathscr{J}_\gamma$, $\langle c, n\rangle \in X$. Let $K(c, n, X)$ be the following subset of $Q(c, X)$:

$$K(c, n, X) = \{k(c)\} \cup \bigcup \{f(c, m, i) \cap G_\gamma(X) : i \in 3, m \in \omega, m \geqq n\}.$$

For example, $K(c, 2, X)$ is pictured in Figure 4 for the case where $Q(c, X) = R(c, \omega)$.

We list in a lemma several immediate observations concerning $K(c, n, X)$.

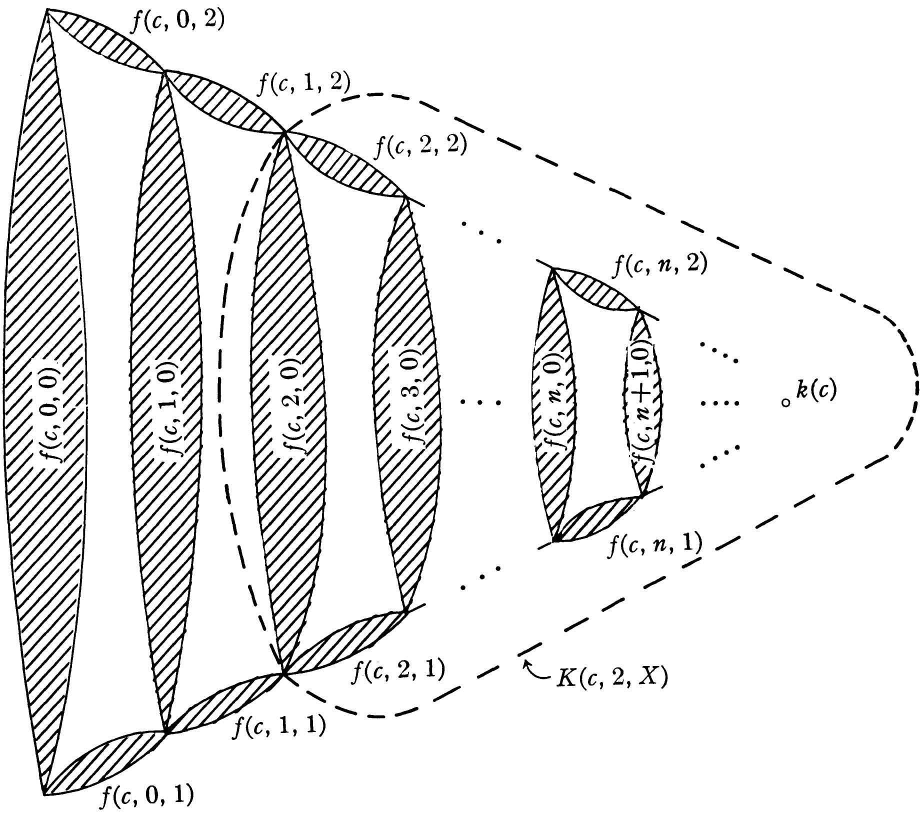

FIGURE 4. $K(c, 2, X)$

LEMMA 3. *Let* $X \in \mathscr{J}_\gamma$, $c \in N_\gamma$.

(1) *For each* $n \in \omega$, $K(c, n, X)$ *is a sublattice of* $Q(c, X)$.

(2) $K(c, n, X) = [f(c, n, 1)_X + K(c, n + 1, X) + f(c, n, 2)_X] \oplus f(c, n, 0)_X$.

(3) $\bigcup \{K(c, n, X): n \in \omega\} = Q(c, X)$.

(4) $\bigcap \{K(c, n, X): n \in \omega\} = \{k(c)\}$.

LEMMA 4. *Let* $\gamma < 1 + \alpha$ [resp., $\gamma < 1 + \beta$], $X \in \mathscr{J}_\gamma$, $P \in \mathscr{D}$, *and let* $\varphi: P \to F_\gamma(X)$ [resp., $\varphi: P \to G_\gamma(X)$] *be a non-constant lattice homomorphism. Then* P *is a constituent of* $F_\gamma(X)$ [resp., $G_\gamma(X)$] *and* φ *is the inclusion map.*

Proof. For $F_\gamma(X)$ the result is immediate since P is h-simple and $\mathscr{D}$ is strongly bounded and $\mathscr{L}^*$-discrete.

For $G_\gamma(X)$, first observe that since P is h-simple, the image of P under φ is contained in $Q(c, X)$ for some $c \in N_\gamma$. Using (3) and (4) of Lemma 3, let n be the largest natural number such that the image of P is contained in $K(c, n, X)$. By (2) of Lemma 3 and since P is v-simple and h-simple, the image is contained in $f(c, n, i)_X$ for some $i \in 3$. In view of the $\mathscr{L}^*$-discreteness and strong boundedness of $\mathscr{D}$, it follows that $P = f(c, n, i)$ and φ is the inclusion.

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THEOREM 4. *Let*

$$F = \bigcup_{\gamma<1+\alpha} F_\gamma \cup \bigcup_{\gamma<1+\beta} G_\gamma.$$

Then F is a full embedding of $\mathscr{S}_{\alpha\beta}$ into $\mathscr{L}^*$ and the image of $\mathscr{S}_{\alpha\beta}$ under F is strongly bounded.

Proof. It is clear that H is an embedding. For P and Q in the image of F, let $\varphi: P \to Q$ be any non-constant lattice homomorphism. We consider two cases.

Case 1. If $P = F_\gamma(X)$, $X \in \mathscr{I}_\gamma$, $\gamma < 1 + \alpha$, then choose any $a \in X$. $f(a)$ is a constituent of $F_\gamma(X)$ and clearly φ restricted to $f(a)_X$ is non-constant. Then, by Lemma 4, $f(a)$ is a constituent of Q. Since f was 1-1, $Q = F_\gamma(Y)$ for some $Y \in \mathscr{I}_\gamma$, and $X \subseteq Y$. Furthermore, by Lemma 4, φ restricted to $f(x)_X$ must be the inclusion for each $x \in X$, hence φ is the inclusion of $F_\gamma(X)$ into $F_\gamma(Y)$, as required.

Case 2. If $P = G_\gamma(X)$, $X \in \mathscr{I}_\gamma$, $\gamma < 1 + \beta$, then let $c \in N_\gamma$. Then $f(c, 0, 0)$ is a constituent of $G_\gamma(X)$, and clearly φ restricted to $f(c, 0, 0)_X$ is non-constant. By Lemma 4, $f(c, 0, 0)$ is a constituent of Q, hence $Q = G_\gamma(Y)$ for some $Y \in \mathscr{I}_\gamma$. We make three claims to complete the proof.

Claim 1. For $n \in \omega$, if φ restricted to $f(c, n, 0)_X$ is non-constant then φ restricted to $f(c, n, i)_X$ is non-constant for $i \in \{1, 2\}$. Indeed, choose any element $x \in \mathbf{int}(f(c, n, 0)_X)$. If, say, φ restricted to $f(c, n, 2)_X$ is constant, then

$$\begin{aligned}
\varphi(\mathbf{0}(f(c, n, 0)_X)) &= \varphi(x \wedge \mathbf{0}(f(c, n, 2)_X)) \\
&= \varphi(x) \wedge \varphi(\mathbf{0}(f(c, n, 2)_X)) \\
&= \varphi(x) \wedge \varphi(\mathbf{1}(f(c, n, 2)_X)) \\
&= \varphi(x \wedge \mathbf{1}(f(c, n, 2)_X)) \\
&= \varphi(x),
\end{aligned}$$

contradicting Lemma 4.

Claim 2. Let $c \in N_\gamma$, $n \in \omega$. If $\langle c, n\rangle \in Y$, then $\langle c, n\rangle \in X$ and φ is non-constant on $f(c, n, 0)_X$. We prove this by induction on n. We have $\langle c, 0\rangle \in X$ by definition and since $\mathbf{0}(f(c, 0, 0)_X) = \mathbf{0}(G_\gamma(X))$ and $\mathbf{1}(f(c, 0, 0)_X) = \mathbf{1}(G_\gamma(X))$, φ is non-constant on $f(c, 0, 0)_X$. Now assume the claim for n and suppose $\langle c, n + 1\rangle \in Y$. By Lemma 4,

$$\begin{aligned}
\varphi(\mathbf{0}(f(c, n, 2)_X)) &= \mathbf{0}(f(c, n, 2)_Y) \\
&\neq \mathbf{1}(f(c, n, 1)_Y) \\
&= \varphi(\mathbf{1}(f(c, n, 1)_X)),
\end{aligned}$$

which implies $\mathbf{0}(f(c, n, 2)_X \neq \mathbf{1}(f(c, n, 1)_X)$, so that $\langle c, n + 1\rangle \in X$. Then also $\varphi(\mathbf{0}(f(c, n + 1, 0)_X) \neq \varphi(\mathbf{1}(f(c, n + 1, 0)_X))$, so φ is non-constant on $f(c, n + 1, 0)_X$.

Claim 3. Let $c \in N_\gamma$ and $n \in \omega$. If $\langle c, n\rangle \notin Y$ and $\langle c, n\rangle \in X$, then φ maps all of $K(c, n, X)$ onto $k(c)$. The proof is again by induction on n. It holds vacuously for $n = 0$. Assume it for n, and suppose $\langle c, n+1\rangle \in X$, $\langle c, n+1\rangle \notin Y$. There are two cases: (i) if $\langle c, n\rangle \notin Y$, then, since $\langle c, n\rangle \in X$, φ maps $K(c, n, X)$ onto $k(c)$ by inductive hypothesis. Therefore φ maps $K(c, n+1, X)$ to $k(c)$. (ii) If $\langle c, n\rangle \in Y$, then by Lemma 4 and Claim 2,

$$\begin{aligned}\varphi(\mathbf{1}(f(c, n+1, 0)_X)) &= \varphi(\mathbf{0}(f(c, n, 2)_X))\\ &= \mathbf{0}(f(c, n, 2)_Y)\\ &= k(c)\\ &= \mathbf{1}(f(c, n, 1)_Y)\\ &= \varphi(\mathbf{1}(f(c, n, 1)_X))\\ &= \varphi(\mathbf{0}(f(c, n+1, 0_X)).\end{aligned}$$

Thus, φ maps $K(c, n+1, X)$ to $k(c)$.

Claims 1 to 3 imply that $Y \subseteq X$ and $\varphi = G_\gamma(\psi(X, Y))$.

It remains finally to show that F_γ is $\mathscr{J}_\gamma^{\rightarrow}$-acceptable for all $\gamma < 1 + \alpha$ and G_γ is $\mathscr{J}_\gamma^{\leftarrow}$-acceptable for all $\gamma < 1 + \beta$. To this end we state a lemma, the proof of which is trivial.

LEMMA 5. *Let $\gamma \in$ On. Let B be a set and for each $x \in N_\gamma \times \omega$ let A_x be a set such that $A_x \cap B = A_x \cap A_y = \emptyset$ for all $x \neq y$ in $N_\gamma \times \omega$. For each $X \in \mathscr{J}_\gamma$, define*

$$M(X) = B \cup \bigcup \{A_x : x \in X\}.$$

Then

(i) *for $X, Y \in \mathscr{J}_\gamma$, $X \subseteq Y$ if and only if $M(X) \subseteq M(Y)$, and*

(ii) *M preserves directed unions; i.e., if $\mathfrak{X} \subseteq \mathscr{J}_\gamma$ is directed by inclusion, then $M(\bigcup\mathfrak{X}) = \bigcup \{M(X) : X \in \mathfrak{X}\}$.*

THEOREM 5. *F_γ is $\mathscr{J}_\gamma^{\rightarrow}$-acceptable for all $\gamma < 1 + \alpha$ and G_γ is $\mathscr{J}_\gamma^{\leftarrow}$-acceptable for all $\gamma < 1 + \beta$.*

Proof. For F_γ, apply Lemma 5 where $B = \{\mathbf{0}(F_\gamma(X)), \mathbf{1}(F_\gamma(X))\}$ and for each $x \in X$, $A_x = \mathbf{int}(f(x))$. Since F_γ obviously preserves inclusions, F_γ is $\mathscr{J}_\gamma^{\rightarrow}$-acceptable.

For G_γ, apply Lemma 5 with $B = \{k(c) : c \in N_\gamma\} \cup \{\mathbf{0}(G_\gamma(X)), \mathbf{1}(G_\gamma(X))\}$ and for each $\langle c, n\rangle \in N_\gamma \times \omega$, $A_{\langle c,n\rangle} = \bigcup \{f(c, n, i) : i \in 2\} - \{\mathbf{0}(f(c, n, 2)), \mathbf{1}(f(c, n, 1)), \mathbf{0}(G_\gamma(X)), \mathbf{1}(G_\gamma(X))\}$. Then $M(X)$ is the underlying set of $G_\gamma(X)$, so G_γ preserves directed unions. Since G_γ preserves set retractions by its definition, it remains only to establish (iii) of Definition 2(b). First note that for any morphism $\varphi : X \to H$ in $\mathscr{J}_\gamma^{\leftarrow}$ and any $x \in G_\gamma(X)$, either $G_\gamma(\varphi)(x) = x$ or else $G_\gamma(\varphi)(x) \in B$. Furthermore, if $x \in B$, then $G_\gamma(\varphi)(x) = x$. Now let $\mathfrak{S} = \langle I; \{X_i\}; \{\varphi_j^i\}\rangle$ be an inverse limit system in $\mathscr{J}_\gamma^{\leftarrow}$. If $g \in \lim_{\leftarrow} G_\gamma(\mathfrak{S})$ we have two possibilities.

(a) If $g(i) \in B$ for all $i \in I$, then for all $i \geqq j$, $g(j) = G_\gamma(\varphi_j^i)g(i) = g(i)$ by the preceding comment.

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(b) If $g(i_0) \notin B$, $i_0 \in I$, then for all $j > i_0$, $G_\gamma(\varphi_{i_0}{}^j)(g(j)) = g(i_0) \notin B$. Hence by the remark above, $g(j) \notin B$, so $G_\gamma(\varphi_{i_0}{}^j)(g(j)) = g(j)$. Thus $j > i_0$ implies $g(j) = g(i_0)$ as required.

4. In this section we show how to construct the strongly bounded class $\mathscr{D}$ of lattices required for § 3. We will prove the following

THEOREM 6. *For each graph $\mathfrak{G}$ there is a lattice $L(\mathfrak{G})$ such that if D is a discrete class of graphs then $\{L(\mathfrak{G}):\mathfrak{G} \in D\}$ is a strongly bounded class of v-simple and h-simple lattices which is discrete in $\mathscr{L}^*$.*

To complete the proof of Theorem 1, it suffices to show that there exists a discrete category of graphs whose objects are in one-to-one correspondence with the elements of the class

$$A = \bigcup_{\gamma<1+\alpha} (N_\gamma \times \omega) \cup \bigcup_{\gamma<1+\beta} (N_\gamma \times \omega \times 3).$$

This follows from known results in category theory. More precisely, in [**2**] it is shown that any small category can be fully embedded into the category of graphs. Thus, if $\alpha, \beta < \infty$, then we can find the category $\mathscr{D}$. If $\alpha = \infty$ or $\beta = \infty$, then A is a proper class. But Lemma 1 of [**3**] shows that, assuming (M), the discrete category whose objects are the ordinal numbers is fully embeddable into the category of all universal algebras of some fixed type. In [**2**] it is also proved that every such category of algebras can be fully embedded into the category of graphs. It only remains to mention the bijection f called for in § 3, but it is easy to see that the class A can be well-ordered and can be put in one-to-one correspondence with the class of all ordinals.

In Theorem 6 and elsewhere "graph" means a directed graph, i.e., a pair $\langle X; T\rangle$ where X is a set and $T \subseteq X \times X$. Let $\mathfrak{G} = \langle X; T\rangle$ be a fixed graph. Define

$$X^* = X \times 2 \,\dot\cup\, X \times X \,\dot\cup\, 2 \times 2,$$

where without loss of generality we have assumed the sets 2 and X are disjoint. To simplify notation, we denote $X \times \{0\}$ by X_- and $X \times \{1\}$ by X^-, and if $x \in X$, denote $\langle x, 0\rangle$ by x_- and $\langle x, 1\rangle$ by x^-. Also, let $a = \langle 0, 0\rangle$, $b = \langle 0, 1\rangle$, $c = \langle 1, 0\rangle$, $d = \langle 1, 1\rangle$. Thus, in this notation we have

$$X^* = X^2 \cup X_- \cup X^- \cup \{a, b, c, d\},$$

and these are disjoint unions. Define $T^* \subseteq (X^*)^2$ as follows:

(i) $\{\langle a, b\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle c, d\rangle\} \subseteq T^*$;

(ii) for $x, y \in X$, $\{\langle a, x_-\rangle, \langle x_-, x^-\rangle, \langle x_-, \langle x, y\rangle\rangle, \langle\langle x, y\rangle, y^-\rangle\} \subseteq T^*$;

(iii) for $\langle x, y\rangle \in T$, $\langle b, \langle x, y\rangle\rangle \in T^*$;

(iv) T^* contains only those pairs already specified in parts (i)–(iii).

If we denote the fact that $\langle u, v\rangle \in T^*$ by drawing an arrow from u to v, then the diagram of the graph $\langle X^*; T^*\rangle$ is illustrated in Figure 5. Let $\langle X; T\rangle^*$ denote $\langle X^*; T^*\rangle$.

In the following proof and elsewhere, if φ is a function with domain A and B is any set, let $\varphi''(B)$ denote $\{\varphi(x) : x \in B \cap A\}$.

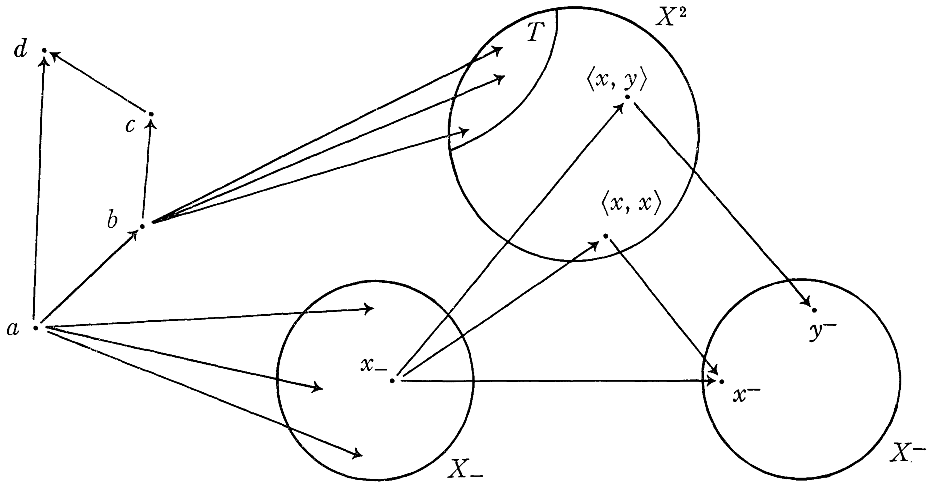

FIGURE 5. The graph $\langle X^*; T^*\rangle$

LEMMA 6. *If $\mathscr{D}$ is a discrete class of graphs, then $\{\mathfrak{G}^* : \mathfrak{G} \in \mathscr{D}\}$ is a discrete class of graphs.*

Proof. Let $\mathfrak{G}_i = \langle X_i; T_i\rangle \in \mathscr{D}$ for each $i = 1, 2$, and suppose $\varphi : \mathfrak{G}_1^* \rightarrow \mathfrak{G}_2^*$ is a homomorphism. We introduce some notation. If $i \in \{1, 2\}$, $u \in X_i^*$, and $n \in \{1, 2, 3\}$, define $C_n^i(u)$ to be the set of all $v \in X_i^*$ such that there exists a sequence $z_0, z_1, \ldots, z_n$ with $z_0 = u$, $z_n = v$, and such that $\langle z_{j-1}, z_j\rangle \in T_i^*$ for each $j = 1, 2, \ldots, n$. In other words, $C_n^i(u)$ is the set of elements of X_i^* which can be reached from u through a "T_i^*-path" of length n. We now make the following observations, which are immediate from the definitions. For $x, y \in X_i$

(i) $C_1^i(a) = \{b, d\} \cup X_{i-}$;
(ii) $C_2^i(a) = \{c\} \cup X_i^2 \cup X_i^-$;
(iii) $C_3^i(a) = \{d\} \cup X_i^-$;
(iv) $C_1^i(b) = \{c\} \cup T_i$;
(v) $C_2^i(b) = \{d\} \cup \{z^- : \langle w, z\rangle \in T_i$ for some $w \in X_i\}$;
(vi) $C_1^i(c) = \{d\}$;
(vii) $C_1^i(x_-) = \{x\} \times X_i \cup \{x^-\}$;
(viii) $C_2^i(x_-) = X_i^-$;
(ix) $C_1^i(\langle x, y\rangle) = \{y^-\}$;
(x) $C_n^i(u) = \emptyset$ in all other cases not covered by (i)–(ix).

Observe also that by repeated application of the definition of homomorphism for graphs, we have $\varphi''(C_n^1(u)) \subseteq C_n^2(\varphi(u))$ for any $u \in X_1$ and $n \in \{1, 2, 3\}$. Hence, since $d \in C_3^1(a)$, we must have $\varphi(d) \in C_3^2(\varphi(a))$, so $C_3^2(\varphi(a)) \neq \emptyset$. Thus, $\varphi(a) = a$ by inspection of (i)–(ix). Then $C_3^2(\varphi(a)) \cap C_1^2(\varphi(a)) = \{d\}$, so

$\varphi(d) = d$. Now $d \in C_2^1(b)$, so $d = \varphi(d) \in C_2^2(\varphi(b))$. But $d \in C_2^2(u)$ is possible only if $u = b$, so $\varphi(b) = b$. Then since $c \in C_1^1(b)$, we have $\varphi(c) \in C_1^2(b) = \{c\} \cup T_2$. But $d \in C_1^1(c)$, so $d = \varphi(d) \in C_1^2(\varphi(c))$, hence $\varphi(c) \notin T_2$ by (ix). Thus, $\varphi(c) = c$. Next we observe that if $u \in X_i^* - X_{i-}$, then $C_1^i(u) \cap C_2^i(u) = \emptyset$. Let $x \in X_1$. Then $x^- \in C_1^1(x_-) \cap C_2^1(x_-)$, hence

$$\varphi(x^-) \in C_1^2(\varphi(x_-)) \cap C_2^2(\varphi(x_-)) \neq \emptyset.$$

Consequently, $\varphi(x_-) \in X_{2-}$, say $\varphi(x_-) = y_-$, where $y \in X_2$. Put $f(x) = y$. This defines a function $f: X_1 \to X_2$ such that $\varphi(x_-) = f(x)_-$ for all $x \in X_1$. Now as we saw, if $x \in X_1$, then $\varphi(x^-) \in C_1^2(f(x))_- \cap C_2^2(f(x))_- = \{f(x)\}^-$, hence $\varphi(x^-) = f(x)^-$. If $x, y \in X_1$, then $\langle x, y\rangle \in C_1^1(x_-)$, hence $\varphi(\langle x, y\rangle) \in C_1^2(f(x)_-)$. Clearly, $C_1^1(\langle x, y\rangle) \neq \emptyset$, so $C_1^2(\varphi\langle x, y\rangle) \neq \emptyset$, hence $\varphi(\langle x, y\rangle) \notin X_2^-$. Then by (vii), $\varphi(\langle x, y\rangle) \in \{f(x)\} \times X_2$, say $\varphi(\langle x, y\rangle) = \langle f(x), z\rangle$, where $z \in X_2$. Since $y^- \in C_1^1(\langle x, y\rangle)$, it follows that

$$f(y)^- = \varphi(y^-) \in C_1^2(\varphi(\langle x, y\rangle)) = C_1^2(\langle f(x), z\rangle) = \{z^-\}.$$

Thus, $f(y) = z$, and therefore $\varphi(\langle x, y\rangle) = \langle f(x), f(y)\rangle$. Finally, if $\langle x, y\rangle \in T_1$, then $\langle x, y\rangle \in C_1^1(b)$, so $\langle f(x), f(y)\rangle = \varphi(\langle x, y\rangle) \in C_1^2(b) = T_2 \cup \{c\}$, hence $\langle f(x), f(y)\rangle \in T_2$. Thus, f is a homomorphism from $\mathfrak{X}_1$ to $\mathfrak{X}_2$. Consequently, $\mathfrak{X}_1 = \mathfrak{X}_2$, and $f(x) = x$ for all $x \in X_1$. Then clearly $\varphi(u) = u$ for all $u \in X_1^*$, so φ is the identity map. This completes the proof.

Let $\mathfrak{X} = \langle X; T\rangle$ and $\mathfrak{X}^* = \langle X^*; T^*\rangle$ be as defined above. If we partially order X^* according to the diagram in Figure 6, then it becomes a lattice. We will denote join and meet in this lattice by $\vee^*$ and $\wedge^*$, respectively, and the partial order by $\leqq^*$.

In the next lemma we note two obvious facts about T^* and $\leqq^*$.

LEMMA 7. (1) *T^* is asymmetric. That is, if $\langle x, y\rangle$ is in T^*, then $\langle y, x\rangle$ is not.* (2) *T^* is compatible with $\leqq^*$. That is, if $\langle x, y\rangle \in T^*$, then $x \leqq^* y$.*

For $x \in X^*$, let $x_0 = \langle x, 0\rangle$, $x_1 = \langle x, 1\rangle$, and if $\langle x, y\rangle$ is in T^* (respectively, $\langle z, x\rangle$ is in T^*), let $x^y = \langle x, \langle x, y\rangle\rangle$ (respectively, $x_z = \langle x, \langle z, x\rangle\rangle$). Then define

$$S(x) = \{x, x_0, x_1\} \cup \{x^y : \langle x, y\rangle \in T^*\} \cup \langle x_z : \langle z, x\rangle \in T^*\}.$$

Finally, put $L(\mathfrak{X}) = \bigcup \{S(x) : x \in X^*\}$. We will describe a partial ordering on $L(\mathfrak{X})$ which makes it a lattice.

The ordering, which we denote by $\leqq$, can be roughly described as follows: for $x \in X^*$, order $S(x)$ as in Figure 7. The elements of X^* are ordered as in Figure 6. For every $u \in L(\mathfrak{X})$ we put $a_0 \leqq u$. Finally, if $\langle x, y\rangle \in T^*$, we require that $x^y \leqq y_x$. Thus, for each element $x \in X^*$ we "hang" a copy of $S(x) - \{x\}$ below the occurrence of x in Figure 6. For $\langle x, y\rangle \in T^*$, the elements $x, x_0, x_1, y, y_0, y_1, x^y, y_x$ form a configuration like that depicted in Figure 8. Note that x_0 and y_0 both cover a_0, and are incomparable (unless, of course, $x = a$). Figure 9 gives a partial diagram of $L(\mathfrak{X})$. In it, elements of X^* are represented by

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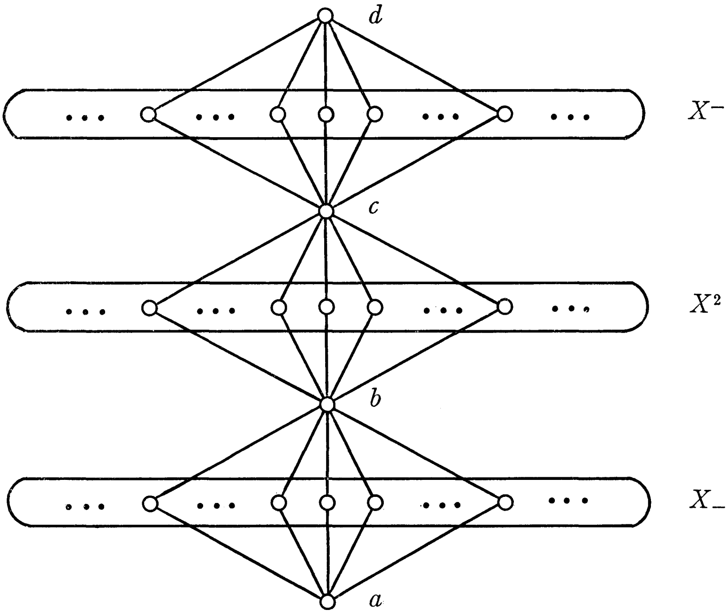

FIGURE 6. The lattice $\langle X^*; \vee^*, \wedge^*\rangle$

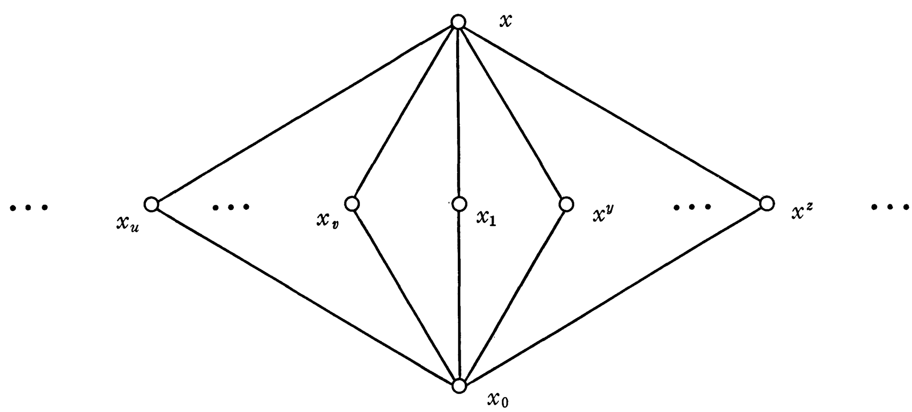

FIGURE 7. $S(x)$

squares, and for $u \in X^*$, $\{u, u_1, u_0\}$ are depicted as in Figure 9a. The downward arrows denote coverings of a_0.

The precise definition of the partial ordering is given in terms of the principal dual ideals.

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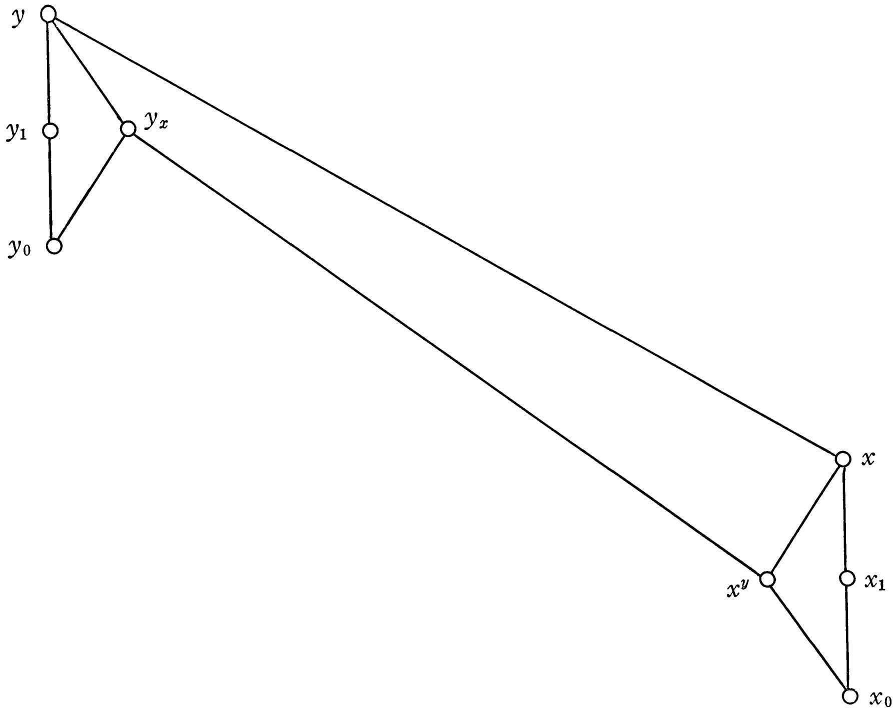

FIGURE 8. Configuration for $\langle x, y\rangle \in T^*$, $x \neq a$.

Definition 10. (1) For $x \in X^*$ define $(x]^*$ to be $\{y \in X^* : y \leqq^* x\}$.

(2) Let $x \in X^*$. We define $(u]$ for all $u \in S(x)$ as follows:

(i) $(x] = \bigcup \{S(z) : z \in (x]^*\}$;

(ii) $(x_1] = \{x_1, x_0, a_0\}$;

(iii) if $\langle z, x\rangle \in T^*$, then $(x_z] = \{x_z, x_0, z^x, z_0, a_0\}$;

(iv) if $\langle x, z\rangle \in T^*$, then $(x^z] = \{x^z, x_0, a_0\}$;

(v) $(x_0] = \{x_0, a_0\}$.

(3) For u and v in $L(\mathfrak{S})$ define $u \leqq v$ to hold if and only if $(u] \subseteq (v]$.

The proof of the next result is a tedious but routine examination of cases, and can be found in [**4**].

LEMMA 8. (1) $\leqq$ *is a partial ordering of* $L(\mathfrak{S})$ *under which* $L(\mathfrak{S})$ *becomes a lattice.*

(2) *Joins of incomparable pairs are described as follows.*

(a) *Let* $\langle x, y\rangle \in T^*$. *Then* $x_0 \vee y_0 = x^y \vee y_0 = y_0 \vee x^y = y_x$.

(b) *If* $u \in S(w)$, $v \in S(z)$, *u is incomparable with v, and* $u \vee v$ *is not determined by* (a), *then* $u \vee v = w \vee^* z$.

(3) *Meets of incomparable pairs are described as follows.*

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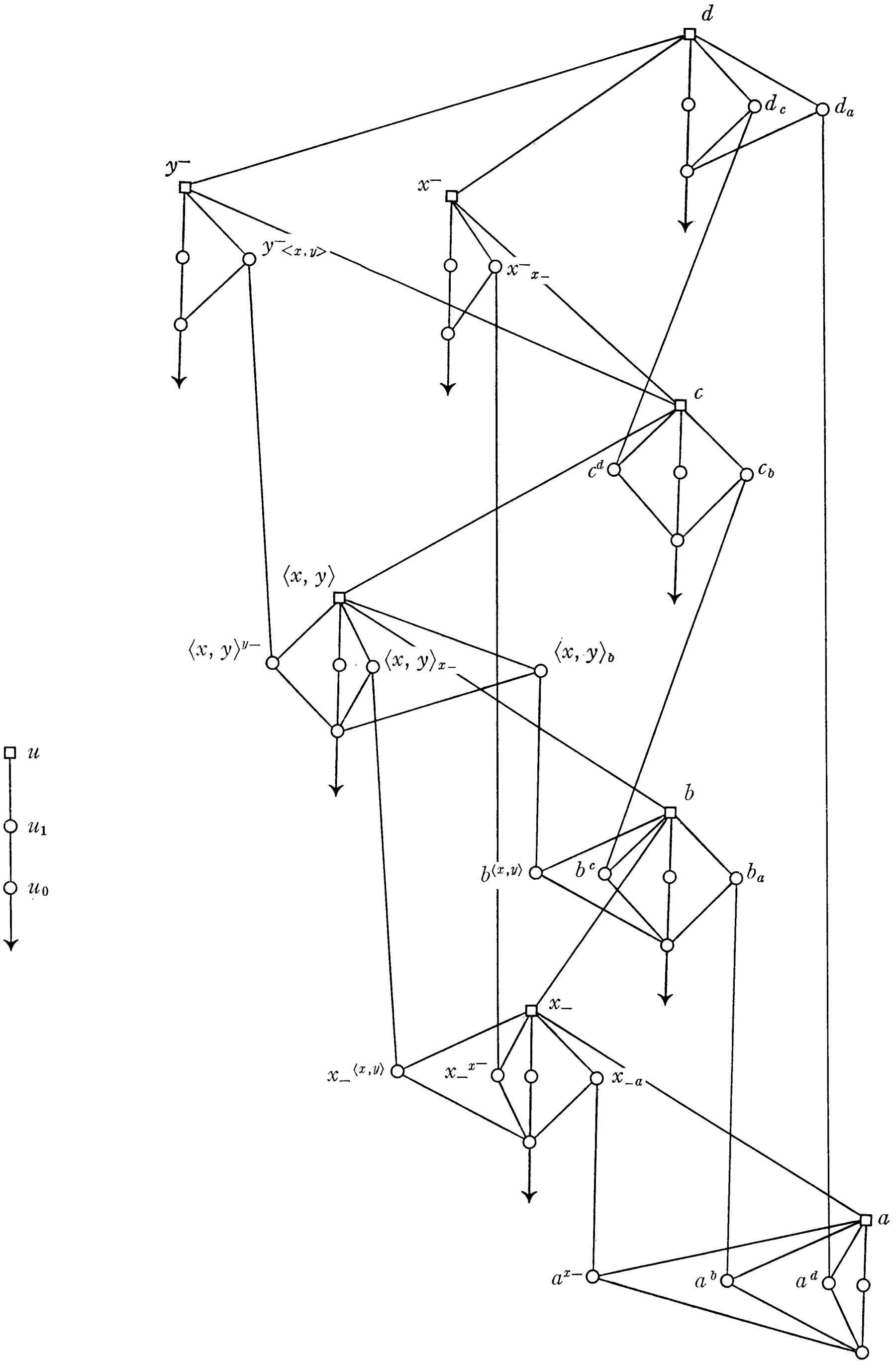

FIGURE 9. Partial diagram of $L(\mathfrak{S})$.

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(a) *Let* $x, y, z \in X^*$ *and* $x \neq y$. *Then*

(i) $x \wedge y = x \wedge^* y$;

(ii) *for* $\langle x, y\rangle \in T^*$, $x_1 \wedge y_x = y_x \wedge x_1 = x_0$;

(iii) *for* $\langle x, y\rangle \in T^*$, $\langle x, z\rangle \in T^*$, *and* $y \neq z$ $x^z \wedge y_x = y_x \wedge x^z = x_0$;

(iv) *for* $\langle x, y\rangle \in T^*$ *and* $\langle z, x\rangle \in T^*$, $x_z \wedge y_x = y_x \wedge x_z = x_0$;

(v) *if* $\langle z, y\rangle \in T^*$ *and* $z \leqq^* x$ *but not* $y \leqq^* x$, *then* $x \wedge y_z = y_z \wedge x = z^y$;

(vi) *if* $\langle z, x\rangle \in T^*$, $\langle z, y\rangle \in T^*$, *and* $x \neq y$, *then* $x_z \wedge y_z = y_z \wedge x_z = z_0$.

(b) *If* $u, v \in S(x)$ *and* u *is incomparable with* v, *then* $u \wedge v = x_0$.

(c) *If* $u, v \in L(\mathfrak{G})$, u *is incomparable with* v, *and if* $u \wedge v$ *is not determined by* (*a*) *or* (*b*), *then* $u \wedge v = a_0$.

The proof of Theorem 6 is established by the next three lemmas.

LEMMA 9. $L(\mathfrak{G})$ *is simple. That is, any lattice homomorphism with domain* $L(\mathfrak{G})$ *is either constant or* 1-1.

Proof. Let $\sim$ denote the congruence relation on $L(\mathfrak{G})$ induced by some homomorphism. It is enough to show that if there exist $u, v \in L(\mathfrak{G})$ with $u \neq v$ and $u \sim v$, then $u' \sim v'$ holds for all $u', v' \in L(\mathfrak{G})$. Since in this case $u \wedge v \sim u$ and $u \sim u \vee v$, and at least two of these three are distinct, we may without loss of generality assume $u < v$. Then there are two cases to consider.

Case 1. If $u, v \in S(x)$ for some $x \in X^*$, then since $S(x)$ is a simple sublattice of $L(\mathfrak{G})$ (see Figure 7), it follows that $x_0 \sim x$. Then $a_0 = a \wedge x_0 \sim a \wedge x = a$, so $a_0 \sim a$. Since $a < d$, we have $d = d_0 \vee a \sim d_0 \vee a_0 = d_0$, so $d_0 \sim d$. Now since $|X| \geqq 2$, we may choose $y, x \in X^-$ with $y \neq z$. Then $y \vee z = y \vee^* z = d$, and since neither $\langle y, z\rangle$ nor $\langle z, y\rangle$ is in T^*, we have $y_0 \vee z_0 = d$. Thus, $z_0 = z_0 \wedge d \backsim z_0 \wedge d_0 = a_0$. Then $d = z_0 \vee y_0 \backsim a_0 \vee y_0 = y_0$. Hence, $d_0 = d \wedge d_0 \backsim y_0 \wedge d_0 = a_0$, so $d \backsim a_0$. Finally, if $w \in L(\mathfrak{G})$, then we have $w = w \wedge d \backsim w \wedge a_0 = a_0$, so there is only one congruence class.

Case 2. If $u \in S(x)$, $v \in S(y)$, and $x \neq y$, then it follows that $x < y$, and if $u' \in S(x)$ and $v' \in S(y)$, then not $v' \leqq u'$. Thus, $x = x \vee u \backsim x \vee v = y$. Now $x \wedge y_0 = a_0$, so $y_0 = y \wedge y_0 \backsim x \wedge y_0 = a_0$. Then since $x_1 \vee y_0 = y$, we have $x_1 = x_1 \vee a_0 \backsim x_1 \vee y_0 = y$. Hence, $x_1 \backsim x$, and we refer to Case 1.

LEMMA 10. *Let* $\mathfrak{G}$ *be a graph. Then* $L(\mathfrak{G})$ *is v-simple and h-simple.*

Proof. Suppose $\varphi: L(\mathfrak{G}) \to L$ is a non-constant lattice homomorphism and $L = L_1 + L_2$. Setting $\mathfrak{G} = \langle X, T\rangle$ and using the above notation, choose any element $x \in X^-$. Then $\langle x, d\rangle \notin T^*$, so $x_0 \vee d_0 = d = \mathbf{1}$ $(L(\mathfrak{G}))$, $x_0 \wedge d_0 = a_0 = \mathbf{0}$ $(L(\mathfrak{G}))$, and x_0 and d_0 are incomparable in $L(\mathfrak{G})$. Since $L(\mathfrak{G})$ is simple, $\varphi(x_0)$ and $\varphi(d_0)$ are incomparable in L, hence for some $i \in \{1, 2\}$, $\{\varphi(x_0), \varphi(d_0)\} \subseteq L_i$. Since L_i is a sublattice, $\varphi(d) = \varphi(x_0) \vee \varphi(d_0)$ and $\varphi(a_0) = \varphi(x_0) \wedge \varphi(d_0)$ are in L_i, hence the image of φ lies in L_i.

Next, let $\varphi': L(\mathfrak{G}) \to L' = L_1 \oplus L_2$ be a non-constant lattice homomorphism. Observe that if $x, y \in L'$ and x and y are comparable, then $\{x, y\} \subseteq L_i$ for some $i \in \{1, 2\}$. Define C to be the transitive closure of the comparability relation on the interior of $L(\mathfrak{G})$, that is

$$C = \{\langle x, y\rangle : (\exists\, n \in \omega)(\exists\, z_1, \ldots, z_n)(z_i \in \mathbf{int}\,(L(\mathfrak{G}))\forall i,$$
$$x = z_1, y = z_n, \text{ and } z_i \text{ is comparable with } z_{i+1} \text{ for } 1 \leqq i < n)\}.$$

Then it follows from the preceding observation that if $\varphi(x) \in L_i$ and $\langle x, y\rangle \in C$, then $\varphi(y) \in L_i$. Now every element of X^* is comparable with b and every element of $L(\mathfrak{G}) - S(d)$ is comparable with some element of $X^* \cap \mathbf{int}\, L(\mathfrak{G})$. The element d_a is comparable with a^d, and d_0 is comparable with d_a. Every element of $S(d)$ is comparable with d_0. Thus $\langle x, a\rangle \in C$ for every $x \in \mathbf{int}\,(L(\mathfrak{G}))$. It follows that $L(\mathfrak{G})$ is h-simple.

LEMMA 11. *Let $\mathfrak{G} = \langle X; T\rangle$ and $\mathfrak{G}' = \langle Y; U\rangle$ be members of a discrete class $\mathscr{D}$ of graphs. If $\varphi: L(\mathfrak{G}) \to L(\mathfrak{G}')$ is a non-constant lattice homomorphism, then $\mathfrak{G} = \mathfrak{G}'$, and φ is the identity map.*

Proof. By Lemma 9 φ is 1-1. Consequently, for $u, v \in L(\mathfrak{G})$, $u \leqq v$ if and only if $\varphi(u) \leqq \varphi(v)$. The proof of the lemma consists of five steps.

(1) $\varphi''(X^*) \subseteq Y^*$: If $x \in X^*$, then by inspection of Definition 10 we conclude that $(x]$ contains at least six elements (indeed, $(a] \subseteq (x]$, and if $y \in X_-$, then $(a]$ contains $\{a_0, a_1, a\, a^b, a^d, a^y\}$). If $\varphi(x) \notin Y^*$, then $(\varphi(x)]$ has at most five elements. Since φ is 1-1 and since clearly $\varphi''((x]) \subseteq (\varphi(x)]$, this is impossible. (Here $\varphi''(A)$ denotes the image of a set A under a mapping φ.) Hence $\varphi(x) \in Y^*$.

Thus, the restriction $\varphi|X^*$ is a lattice isomorphism of X^* into Y^*. By inspection of Figure 6 it is evident that $\varphi(a) = a$, $\varphi(b) = b$, $\varphi(c) = c$, $\varphi(d) = d$, $\varphi''(X_-) \subseteq Y_-$, $\varphi''(X^2) \subseteq Y^2$, and $\varphi''(X^-) \subseteq Y^-$. (For a rigorous proof of these facts, note that in X^* all maximal chains have the same finite length, and the level of an element in such a chain must be preserved by φ.)

(2) If $x \in X^*$, then $\varphi''(S(x)) \subseteq S(\varphi(x))$: First we note that if $z \in X_-$ (respectively, $z \in X^2$, $z \in X^-$), then $S(z) = (z] - (a]$ (respectively, $(z] - (b]$, $(a] - (c]$), and similar statements hold in $L(\mathfrak{G}')$. Let $x \in X_-$. Since $u \leqq v$ for $u, v \in L(\mathfrak{G})$ if and only if $\varphi(u) \leqq \varphi(v)$, we have $\varphi''(S(x)) = \varphi''((x] - (a]) \subseteq (\varphi(x)] - (\varphi(a)] = (\varphi(x)] - (a]$. Since we have $\varphi(x) \in Y_-$, this is equal to $S(\varphi(x))$. Similar arguments apply if $x \in X^2$ or $x \in X^-$.

Since $S(a) = (a]$, we have $\varphi''(S(a)) = \varphi''((a]) \subseteq (\varphi(a)] = (a] = S(a)$.

Next, suppose $x = b$. For $u \in S(b)$ we have $u \leqq b$ but not $u \leqq a$. Hence, in $L(\mathfrak{G}')$ we have $\varphi(u) \leqq b$ but not $\varphi(u) \leqq a$. Thus, $\varphi(u) \in S(z)$ for some $z \in \{b\} \cup Y_-$. If $u, v \in S(b) - \{b, b_0\}$ and $u \neq v$, then let $\varphi(u) \in S(z)$ and $\varphi(v) \in S(w)$. Then at most one of z and w can be in Y_-. Indeed, suppose $z, w \in Y_-$. If $z = w$, then $\varphi(u) \vee \varphi(v)$ is less than or equal to z, which is less than b. But we have $\varphi(u \vee v) = \varphi(b) = b$, contradicting the homomorphism property. If $z \neq w$, then $z \wedge w = a$, so $\varphi(b_0) = \varphi(u \wedge v) = \varphi(u) \wedge \varphi(v) \leqq z \wedge w = a$. But $b_0 \leqq a$ is false, so this is a contradiction. Now it is clear that $S(b) - \{b, b_0\}$ contains at least three elements. If u, v, and w are distinct elements of $S(b) - \{b, b_0\}$, then by the above discussion, at least two of $\varphi(u)$, $\varphi(v)$, and $\varphi(w)$ are in $S(b)$, say $\varphi(u), \varphi(v) \in S(b)$. Then $\varphi(u) \wedge \varphi(v) =$

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$\varphi(u \wedge v) = \varphi(b_0) \in S(b)$. But $u \wedge w = b_0$, so $\varphi(u) \wedge \varphi(w) = \varphi(b_0) \in S(b)$. Since φ is 1-1, we have $\varphi(b_0) < \varphi(w) < b$, so $\varphi(w) \in S(b)$. Since u, v, and w were arbitrarily chosen, we have proved $\varphi''(S(b))$ is included in $S(b)$.

The proofs for $x = c$ and $x = d$ are the same as for $x = b$, but with X_- replaced by X^2, X^- and a by b, c, respectively.

(3) If $x \in X^*$, then $\varphi(x_0) = (\varphi(x))_0$: Indeed, we have $x_0 < x_1 < x$, so $\varphi(x_0) < \varphi(x_1) < \varphi(x)$. Since these are all members of $S(\varphi(x))$, $\varphi(x_0)$ must be $(\varphi(x))_0$.

(4) The restriction $\varphi|X^*$ is a graph homomorphism from $\mathfrak{S}^*$ to $(\mathfrak{S}')^*$: Indeed, if $\langle x, y\rangle \in T^*$, then let $x' = \varphi(x)$ and $y' = \varphi(y)$. Then $x_0 \vee y_0 = y_x \neq y$, so $(x')_0 \vee (y')_0 = \varphi(x_0) \vee \varphi(y_0) = \varphi(x_0 \vee y_0) = \varphi(y_x) \neq y'$. Then by inspection of Lemma 8 $\langle x', y'\rangle \in U^*$, in view of $x \leqq y$ by Lemma 7.

In view of Lemma 6, we have $\mathfrak{S} = \mathfrak{S}'$, and $\varphi(x) = x$ for all $x \in X^*$.

(5) $\varphi(u) = u$ for all $u \in L(\mathfrak{S})$: Let $x \in X^*$. Then we already have $\varphi(x) = x$, so by (3), $\varphi(x_0) = x_0$. If $\langle x, y\rangle \in T^*$, then $x_0 \vee y_0 = y_x$, so $\varphi(y_x) = \varphi(x_0 \vee y_0) = \varphi(x_0) \vee \varphi(y_0) = x_0 \vee y_0 = y_x$. Since $x \wedge y_x = x^y$, we have $\varphi(x^y) = \varphi(x \wedge y_x) = \varphi(x) \wedge \varphi(y_x) = x \wedge y_x = x^y$. Finally, if $u \in S(x) - \{x_1\}$, we have shown $\varphi(u) = u$. Since $\varphi''(S(x)) \subseteq S(x)$ and φ is 1-1, it follows that $\varphi(x_1) = x_1$.

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