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Some New Results on L^2 Cohomology of Negatively Curved Riemannian Manifolds

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Abstract. The present paper is concerned with the study of the L^2 cohomology spaces of negatively curved manifolds. The first half presents a finiteness and vanishing result obtained under some curvature assumptions, while the second half identifies a class of metrics having non-trivial L^2 cohomology for degree equal to the half dimension of the space. For the second part we rely on the existence and regularity properties of the solution for the heat equation for forms.

1 Introduction

The study of L^2 harmonic forms on a complete Riemannian manifold is a very interesting and important subject; it also has numerous applications in the field of Mathematical Physics (see for example [15, 11]). For topological applications of L^2 harmonic forms on noncompact manifolds see [2, 17].

There is another, probably more compelling reason for considering the study of harmonic L^2 forms on noncompact manifolds. Namely, it can be used to obtain topological information about compact quotients of M. In particular as Dodziuk and Singer show in [4, 16] the study of the L^2-cohomology of negatively curved spaces may be used to prove one of Hopf's famous conjectures.

One of the first results on the vanishing and finite dimensionality of the space of harmonic L^2 forms was obtained by E. Vesentini in [18]. Throughout this paper M is a connected Riemannian manifold, $I_M(\mathcal{R})$ denotes a positive quantity depending on the curvature operator acting on k-forms (see Definition 3.4), λ_1 is the Poincaré constant of the manifold (*i.e.* λ_1 is the infimum of the spectrum of the Laplace operator acting on functions, see [13] for the precise definition), B_r is the geodesic ball of radius $r > 0$ and $\mathcal{H}^k$ is the space of square integrable harmonic k-forms.

The main result of the first part of this paper is:

Theorem 1.1 *Let M be a complete manifold of infinite volume, bounded curvature operator, and $\lambda_1 > 0$. Then we have the following:*

(a) *If $\lambda_1 > I_M(\mathcal{R})$ then $\mathcal{H}^k = 0$,*
(b) *If $\lambda_1 > I_{M\setminus B_r}(\mathcal{R})$ for some $r > 0$, then $\dim \mathcal{H}^k < \infty$.*

As a more practical application of Theorem 1.1, we also prove vanishing and finiteness of the $\mathcal{H}^k$ spaces if the sectional curvature is appropriately pinched outside some compact set (see Corollary 3.7).

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In the last section of the paper we prove a few consequences of the existence and regularity of the solution to the heat equation for forms proven by Gaffney [8]. Among these we find a sufficient condition for the nonvanishing of $\mathcal{H}^{n/2}$ of an n-dimensional (n even), simply connected manifold of negative curvature.

2 Preliminaries

Let M be a smooth, complete and oriented Riemannian manifold. Let $C^\infty\Omega^k$ denote the space of smooth k-forms. The metric on M induces a natural pointwise scalar product on forms and let us denote this by $\langle\alpha(x),\beta(x)\rangle$ where $x \in M$ and $\alpha,\beta \in C^\infty\Omega^k$. Thus we obtain the length at a point $x \in M$ of a form $\alpha \in \Omega^k$ as $\langle\alpha(x),\alpha(x)\rangle \geq 0$. This leads to the definition of a norm of a form (if finite) by

$$|\alpha|^2 = \int_M \langle\alpha(x),\alpha(x)\rangle \, dV(x),$$

where dV is the volume form of the manifold and x represents the variable of integration.

By completing this space with respect to the above mentioned norm we obtain a Hilbert space, henceforth denoted by $L^2\Omega^k$. The inner product in this space will be denoted by

$$(\alpha,\beta) = \int_M \langle\alpha(x),\beta(x)\rangle \, dV(x), \tag{1}$$

where $\alpha,\beta \in L^2\Omega^k$. With the help of the metric one could naturally define the Hodge Laplacian of the manifold $\Delta\colon C^\infty\Omega^k \to C^\infty\Omega^k$, $\Delta = d\delta + \delta d$. In the above formula, d is the exterior derivative and δ the formal adjoint. More details of the definition of Δ can be found in [12, 14].

One of the goals of this paper is to study the space of L^2 harmonic forms (*i.e.,* $\Delta\alpha = 0$ and $\alpha \in L^2\Omega^k$). The heat equation will provide an injection from the space of compactly supported de Rham cohomology classes into the space of L^2 harmonic forms. This will be shown in more detail in what follows.

The definition of the reduced L^2 cohomology groups is a slight modification of de Rham groups. Next we give the precise definition of these groups as well as the definition of the space of L^2 harmonic forms.

Definition 2.1

$$H^k(L^2M) = Z^k(L^2M)/\overline{B^k(L^2M)}$$

where

$$Z^k(L^2M) = \{\alpha \mid \alpha \in C^\infty\Omega^k \cap L^2\Omega^k, d\alpha = 0\},$$

and

$$B^k(L^2M) = \{\beta \mid \beta \in C^\infty\Omega^k \cap L^2\Omega^k, \mu \in C^\infty\Omega^{k-1} \cap L^2\Omega^{k-1}, \beta = d\mu\}.$$

$\overline{B^k(L^2M)}$ is the closure in $Z^k(L^2M)$ with respect to the L^2 norm.

Definition 2.2

$$\mathcal{H}^k = \{\alpha \mid \alpha \in C^\infty\Omega^k \cap L^2\Omega^k, \Delta\alpha = 0\}.$$

The following decomposition theorem due to de Rham is well known.

Theorem (de Rham) *Let M be a complete Riemannian manifold. Then we have the following decomposition:*

$$L^2\Omega^k = \overline{dC_c^\infty\Omega^{k-1}} \oplus \mathcal{H}^k \oplus \overline{\delta C_c^\infty\Omega^{k+1}}.$$

Using the heat flow method, which will be presented shortly, one can show:

Theorem 2.3 *If M is a complete Riemannian manifold then $\mathcal{H}^k \simeq H^k(L^2M)$.*

It is well known (*cf.* [9]) that for the hyperbolic space $\mathbb{H}^{2n}$, $\mathcal{H}^n$ is infinite dimensional (see also [5]). Since $\mathbb{H}^{2n}$ is diffeomorphic to $\mathbb{R}^{2n}$, on which $\mathcal{H}^n = 0$, the $\mathcal{H}^k$'s are not topological invariants. Thus, the $\mathcal{H}^k$'s usually depend on the metric.

What is also known is that in the compact case a form is harmonic ($\Delta\alpha = 0$) if and only if $d\alpha = 0$ and $\delta\alpha = 0$, and this is a consequence of Stokes' Theorem and the very definition of δ operator. The same result remains valid for an L^2 harmonic form (possibly an L^p form) on a complete manifold. The next proposition is due to Andreotti and Vesentini, and its proof can be found in the classical book by de Rham [3].

Proposition 2.4 *Let α be an L^2 harmonic form on a complete Riemannian manifold. Then $d\alpha = 0$ and $\delta\alpha = 0$.*

The following proposition will be used in the following sections and it essentially asserts that d and δ are formally adjoint operators on L^2 forms on a complete manifold. The proof is a typical application of a cut-off function argument. We give the proof in detail.

Proposition 2.5 *Let M be a complete Riemannian manifold and α, β, $d\alpha$, $\delta\beta$ be square integrable forms. Then*

$$(d\alpha, \beta) = (\alpha, \delta\beta).$$

Proof Let α and β as above and let $0 \le \psi_n \le 1$ be a sequence of smooth, compactly supported functions with the following two properties: $|d\psi_n| \le \frac{C}{n}$ for some positive constant $C > 0$ and $\psi_n(x) \to 1$ for every $x \in M$.

We have the pointwise identity

$$\langle d(\psi_n\alpha), \beta\rangle = \langle d\psi_n \wedge \alpha + \psi_n d\alpha, \beta\rangle, \tag{2}$$

and integrating the left-hand side we get

$$\int_M \langle d(\psi_n\alpha), \beta\rangle \, dV = \int_M \langle \psi_n\alpha, \delta\beta\rangle \, dV.$$

Since $\psi_n \to 1$ pointwise we have, according to the Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_M \langle \psi_n \alpha, \delta\beta \rangle \, dV = (\alpha, \delta\beta). \tag{3}$$

Now integrating the right hand side of (2) and using the pointwise estimate

$$\langle d\psi_n \wedge \alpha, \beta \rangle \leq |d\psi_n| \, |\alpha| \, |\beta| \leq \frac{C}{n} |\alpha| \, |\beta|$$

together with Lebesgue's theorem, we get

$$\lim_{n\to\infty} \int_M \langle d\psi_n \wedge \alpha + \psi_n d\alpha, \beta \rangle \, dV = \int_M \langle d\alpha, \beta \rangle \, dV. \tag{4}$$

Using (2), (3), and (4), we get the desired result. ■

In what follows we state the existence and regularity of the solution to the (abstract) heat equation. This result is mainly due to F. Browder [1]. We consider a second order elliptic operator A acting on smooth sections of a vector bundle endowed with a smooth scalar product. Suppose the operator fulfills the following conditions:

(i) $(A\alpha, \beta) = (\alpha, A\beta)$ for any compactly supported α, β,
(ii) $(A\alpha, \alpha) \geq 0$ for any compactly supported α.

We are interested in finding a regular solution to the Cauchy problem:

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} = -A\alpha, \\ \alpha(0) = \alpha_0 \in L^2. \end{cases}$$

More precisely we are interested in finding a path $\alpha\colon [0, \infty) \to L^2\Gamma$ such that the two above conditions are fulfilled. Here Γ denotes the space of sections of the vector bundle. The most important result is contained in the following proposition and it is due to the efforts of Browder and Gaffney [8]:

Proposition 2.6 *There is always a unique solution to the Cauchy problem of the heat equation for forms and the solution has the following properties:*

(a) $\lim_{t\to\infty} \alpha(t) \in \mathcal{H}^k$,
(b) *α, $\delta\alpha$, $d\alpha$, $\delta\alpha$ are all in L^2 at any time $t > 0$,*
(c) *the solution is C^∞ for all $t > 0$,*
(d) *if the initial data α_0 is closed then α is closed for all $t > 0$,*
(e) *the cohomology class is preserved by the flow.*

3 The Finite Dimensionality of $\mathcal{H}^k$'s and Some Vanishing Results

This section is concerned with finding sufficient geometric conditions on the manifold M which will guarantee the finite dimensionality of the L^2 cohomology spaces. The techniques are based on the classical Weitzenböck formula and a few standard PDE techniques. Throughout the whole section, all the operators acting on forms are assumed to act on k forms (*i.e.*, forms of arbitrary degree) if not specified otherwise.

***Proposition* (Weitzenböck formula)** *Let M be a Riemannian manifold (not necessarily complete). Let e_i be a local orthonormal frame and η^i the associated coframe. Then we have the identity*

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathcal{R}(\alpha), \tag{5}$$

where ∇ represents the covariant derivative acting on forms, ∇^* represents its formal adjoint and $\mathcal{R}(\alpha) = \sum_{i,j} \eta^i \wedge \left(i_{e_j} R(e_i, e_j)\alpha\right)$.

For a proof of this formula see [12].

Definition 3.1 The $\mathcal{R}(\alpha)$ operator defined by the identity in the previous proposition is called the *Weitzenböck curvature operator*. We say $\mathcal{R}$ is positive (negative) if and only if $g\big(\mathcal{R}(\alpha), \alpha\big) > 0$ (< 0) for all $\alpha \neq 0$ where g is the Riemannian metric of M.

We also need the following simple lemma. Here

$$W^{1,2}\Omega^k = \left\{ \alpha \in \Omega^k \;\middle|\; \|\alpha\|_{L^2} + \|d\alpha\|_{L^2} + \|\delta\alpha\|_{L^2} < \infty \right\}.$$

Lemma 3.2 *Let M be as above and $(\alpha_n)_{n\geq 1}$ be a bounded sequence in $W^{1,2}\Omega^k$ and ψ be a smooth compactly supported function. Then the sequence $\psi_n = \psi\alpha_n$ is bounded in $W^{1,2}\Omega^k$ and is compactly supported.*

The proof of the lemma is trivial.

The following proposition is essential in proving the main result of this section.

Proposition 3.3 *On a complete manifold M,* $\dim \mathcal{H}^k < \infty$ *if and only if there exist $p \in M$, $r > 0$ and $C > 0$, such that*

$$\int_{B_r(p)} |\alpha|^2 \, dV \geq C \int_M |\alpha|^2 \, dV$$

for every $\alpha \in \mathcal{H}^k$. Here $B_r(p)$ denotes the geodesic ball centered at p and having radius $r > 0$.

Proof For the "only if" part we observe that both the quantities involved in the above mentioned inequality are norms on a finite dimensional vector space, hence, equivalent. To see that the left hand side of the inequality is a norm, one should note that if a harmonic form is zero on an open ball, it has to be zero everywhere.

For the "if" part, let $r > 0$ and $C > 0$ as in the hypothesis. Let ψ be a cut-off function such that $\psi \equiv 1$ on B_r and $\psi \equiv 0$ on B_{2r}. Assume $\dim \mathcal{H}^k = \infty$ and let α_n be a countable L^2-orthonormal sequence of harmonic forms in $\mathcal{H}^k$. Then, according to Lemma 3.2, the sequence $\psi_n = \psi\alpha_n$ satisfies the conditions of Rellich theorem, hence, we can extract a subsequence convergent in L^2. We will use the same notation for the subsequence ψ_n. Let us now estimate the distance between members of this sequence, namely $d(\psi_n, \psi_m)$ (here $d(\ ,\ )$denotes the L^2 distance). We have

$$\begin{aligned} d(\psi_n, \psi_m) &= \int_M \langle \psi_n - \psi_m, \psi_n - \psi_m \rangle \, dV \\ &= \int_M (|\psi_n|^2 + |\psi_m|^2) \, dV - 2 \int_M \langle \psi_n, \psi_m \rangle \, dV. \end{aligned}$$

Hence,

$$d(\psi_n, \psi_m) \geq \int_{B_r} (|\alpha_n|^2 + |\alpha_m|^2) \, dV - 2 \int_M \langle \psi_n, \psi_m \rangle \, dV.$$

Now applying the inequality from the hypothesis we can estimate the first term of the right-hand side as follows:

$$\int_{B_r} (|\alpha_n|^2 + |\alpha_m|^2) \, dV \geq 2C.$$

So finally we get

$$d(\psi_n, \psi_m) \geq 2C - 2 \int_M \langle \psi_n, \psi_m \rangle \, dV.$$

But the sequence ψ_n is obtained by multiplying an orthonormal sequence in L^2 by a cut-off function, so it is weakly convergent to zero in L^2 and using a diagonal argument we can see that the second term on the right-hand side of the inequality above can be made arbitrarily small (as $n, m \to \infty$). By the Rellich theorem, so is the left-hand side. It then follows that $2C \leq 0$, a contradiction. ■

As this proposition shows, in proving the finite dimensionality of $\mathcal{H}^k$ one could try to get an estimate as above. In fact, Vesentini in [18] obtained the first result of this kind. More precisely he proved that if the curvature operator is positive outside some compact subset of the manifold then the desired inequality holds. In a similar fashion he also proved that if the curvature operator is nonnegative then $\mathcal{H}^k = 0$.

In what follows we will give other geometric conditions that imply the required estimate and also will obtain another useful vanishing result. Before going to the main result we need to make a definition:

Definition 3.4 Let M be a complete manifold and let $\mathcal{R}$ be the curvature operator acting pointwise on k-forms. Let $D \subseteq M$ be a subset. Let

$$\Lambda = -\sup\left\{ c \;\middle|\; \langle \mathcal{R}\alpha(p), \alpha(p) \rangle \geq c|\alpha(p)|^2, \alpha \in \Omega^k_p, p \in D \right\}.$$

Then

$$I_D(\mathcal{R}) = \max(\Lambda, 0).$$

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Remark Obviously, if $D_1 \subset D_2$ then $I_{D_1}(\mathcal{R}) \le I_{D_2}(\mathcal{R})$.

For the proof of the main theorem we will first need to prove one technical lemma.

Lemma 3.5 *Let M be a complete manifold whose Weitzenböck curvature operator on Ω^k is bounded. Then for any $\alpha \in \mathcal{H}^k$ we have $\int_M |\nabla\alpha|^2 < \infty$. Hence, in this case the Weitzenböck formula gives*

$$\int_M |\nabla\alpha|^2\, dV + \int_M \langle \mathcal{R}\alpha, \alpha\rangle\, dV = 0.$$

Proof Let $\psi \in C_0^\infty\Omega^k$ be a compactly supported form. According to the Weitzenböck formula we have

$$\Delta\psi = \nabla^*\nabla\psi + \mathcal{R}\psi.$$

Multiplying both sides by ψ and integrating by parts (we can do this because ψ is compactly supported) we get

$$(\Delta\psi, \psi) = \|\nabla\psi\|^2 + (\mathcal{R}\psi, \psi).$$

We can rewrite the left-hand side of the above identity in terms of d and δ as follows

$$\|d\psi\|^2 + \|\delta\psi\|^2 = \|\nabla\psi\|^2 + (\mathcal{R}\psi, \psi). \tag{6}$$

In all of the above formulas $\|\psi\|$ denotes the L^2 norm of ψ. But the curvature operator is bounded from below, *i.e.*, $\langle\mathcal{R}\psi, \psi\rangle \ge -c|\psi|^2$ and this is pointwise, or equivalently $-\langle\mathcal{R}\psi, \psi\rangle \le c|\psi|^2$. Hence, we get

$$\|\nabla\psi\|^2 = \int_M |\nabla\psi|^2\, dV \le c\int_M |\psi|^2\, dV + \|d\psi\|^2 + \|\delta\psi\|^2 \tag{7}$$

for any ψ compactly supported in M.

Now let $\alpha \in \mathcal{H}^k$ and ϕ be a smooth, compactly supported function such that $\phi \equiv 1$ on B_r and zero outside B_{2r} for arbitrary $r > 0$, and also $|\nabla\phi| \le 1$. Applying (7) (taking into account that $0 \le \phi \le 1$) to the compactly supported form $\phi\alpha$ we get

$$\int_M |\nabla(\phi\alpha)|^2\, dV \le c\int_M |\alpha|^2\, dV + \|d(\phi\alpha)\|^2 + \|\delta(\phi\alpha)\|^2. \tag{8}$$

In order to estimate the last two terms of (8) we proceed as follows:

$$|d(\phi\alpha)| = |d\phi \wedge \alpha| \le |d\phi|\,|\alpha| \le |\alpha|$$

hence,

$$\|d(\phi\alpha)\|^2 \le \|\alpha\|^2$$

and also

$$|\delta(\phi\alpha)| = |*d*(\phi\alpha)| = \left|d\big(\phi(*\alpha)\big)\right| = |d\phi \wedge (*\alpha)| \le |*\alpha| = |\alpha|.$$

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Integrating the above inequality yields

$$\|\delta(\psi\alpha)\| \leq \|\alpha\|.$$

The two previous estimates depend on the facts that $\alpha \in \mathcal{H}^k$ and that $*$ is a pointwise isometry. Also, taking into account that

$$\int_{B_r} |\nabla\alpha|^2 \, dV = \int_{B_r} |\nabla(\phi\alpha)|^2 \, dV \leq \int_M |\nabla(\phi\alpha)|^2 \, dV$$

we get

$$\int_{B_r} |\nabla\alpha|^2 \, dV \leq c\|\alpha\|^2 + 2\|\alpha\|^2. \tag{9}$$

Since $r > 0$ was arbitrarily chosen, letting $r \to \infty$ we get

$$\int_M |\nabla\alpha|^2 \, dV \leq c\|\alpha\|^2 + 2\|\alpha\|^2. \tag{10}$$

Furthermore, since by assumption $\mathcal{R}$ is bounded from above and below it follows from (5) that both $\nabla^*\nabla\alpha$ and $\mathcal{R}\alpha$ are square integrable, and using a standard density argument we can integrate by parts and obtain

$$\int_M |\nabla\alpha|^2 \, dV + \int_M \langle \mathcal{R}\alpha, \alpha \rangle \, dV = 0. \qquad \blacksquare$$

Proof of Theorem 1.1 As a result of Lemma 3.5 we have

$$\int_M |\nabla\alpha|^2 \, dV + \int_M \langle \mathcal{R}\alpha, \alpha \rangle \, dV = 0. \tag{11}$$

By the definition of $I_M(\mathcal{R})$ we have

$$\langle \mathcal{R}\alpha, \alpha \rangle \geq -I_M(\mathcal{R})|\alpha|^2$$

or equivalently

$$-\langle \mathcal{R}\alpha, \alpha \rangle \leq I_M(\mathcal{R})|\alpha|^2.$$

This together with (11) gives

$$\int_M |\nabla\alpha|^2 \, dV \leq I_M(\mathcal{R}) \int_M |\alpha|^2 \, dV. \tag{12}$$

Assume there is a nonzero harmonic L^2 form α. Since the volume of M is infinite, it follows that $|\alpha|$ is nonconstant so we may apply the Poincaré inequality (see [13]) to $|\alpha|$. Hence

$$\int_M |\nabla\alpha|^2 \, dV \geq \lambda_1 \int_M |\alpha|^2 \, dV;$$

here we made use of the pointwise inequality: $|\nabla\alpha|^2 \geq \big|\nabla|\alpha|\big|^2$. This together with (12) implies

$$\lambda_1 \int_M |\alpha|^2\, dV \leq I_M(\mathcal{R}) \int_M |\alpha|^2\, dV.$$

Since α is nonzero we have

$$\lambda_1 \leq I_M(\mathcal{R})$$

which contradicts the assumption of part (a) of the theorem.

For part (2) of the theorem let us observe first that there exists constant $C > 0$ such that

$$\langle \mathcal{R}\alpha(p), \alpha(p)\rangle \geq -C|\alpha(p)|^2 \tag{13}$$

for any $\alpha \in \Omega^k$ and $p \in B_r$. To see this, one should consider the continuous function $f(p, v) = \langle \mathcal{R}_p v, v\rangle$ defined on the sphere bundle of $\Omega^k \overline{B_r}$ (since this set is compact, f attains its infimum).

Using Lemma 3.5 we have

$$\int_M |\nabla\alpha|^2\, dV + \int_{M\setminus B_r} \langle \mathcal{R}\alpha, \alpha\rangle dV + \int_{B_r} \langle \mathcal{R}\alpha, \alpha\rangle\, dV = 0.$$

This together with (13) gives

$$\int_M |\nabla\alpha|^2\, dV + \int_{M\setminus B_r} \langle \mathcal{R}\alpha, \alpha\rangle\, dV \leq C \int_{B_r} |\alpha|^2\, dV.$$

Using the definition of $I_{M\setminus B_r}(\mathcal{R})$ we have

$$\int_M |\nabla\alpha|^2\, dV - I_{M\setminus B_r}(\mathcal{R}) \int_{M\setminus B_r} |\alpha|^2\, dV \leq C \int_{B_r} |\alpha|^2\, dV.$$

Now using the Poincaré inequality as in part (1) we get

$$\lambda_1 \int_M |\alpha|^2\, dV - I_{M\setminus B_r}(\mathcal{R}) \int_{M\setminus B_r} |\alpha|^2\, dV \leq C \int_{B_r} |\alpha|^2\, dV.$$

And finally,

$$\lambda_1 \int_M |\alpha|^2\, dV - I_{M\setminus B_r}(\mathcal{R}) \int_M |\alpha|^2\, dV \leq C \int_{B_r} |\alpha|^2\, dV.$$

By the hypothesis of part (b)

$$\int_M |\alpha|^2\, dV \leq \frac{C}{\lambda_1 - I_{M\setminus B_r}(\mathcal{R})} \int_{B_r} |\alpha|^2\, dV \tag{14}$$

But this is exactly what Proposition 3.3 requires. Hence $\dim \mathcal{H}^k < \infty$. ■

Remark Examples of manifolds with bounded curvature operator $\mathcal{R}$ are the manifolds with bounded Riemannian curvature tensor (*i.e.* for which the sectional curvature is bounded).

Next we will show that $I_{\mathbb{H}^n}(\mathcal{R}) = nk - k^2$. This follows easily from the following lemma:

Lemma 3.6 *Let e_i be an orthonormal frame at some point and η_i its associated coframe. Then the following formula holds*

$$\langle \mathcal{R}(\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_k), \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_k \rangle = \sum_{i=1}^{k} \sum_{j=k+1}^{n} K(e_i, e_j). \tag{15}$$

For a proof of this formula, one should consult [14].

By the homogeneity of $\mathbb{H}$ and since $K \equiv -1$ we get the desired formula, namely

$$I_{\mathbb{H}^n}(\mathcal{R}) = nk - k^2.$$

This together with the well-known fact that

$$\lambda_1(\mathbb{H}^n) = \frac{(n-1)^2}{4},$$

imply the vanishing of the $\mathcal{H}^k$ whenever $nk - k^2 < \frac{(n-1)^2}{4}$.

As a direct application of Theorem 1.1 we obtain the following result:

Corollary 3.7 *Let M^n be a complete, simply connected, negatively curved manifold with sectional curvature K. If $-1 \leq K$ outside some compact set and $K \leq -1 + \epsilon$ everywhere, for some $\epsilon > 0$. Then the following hold:*

(i) *If $n \geq 6$ and $\epsilon < 1 - \frac{4}{n-1}$, then* $\dim \mathcal{H}^1 = \dim \mathcal{H}^{n-1} < \infty$,

(ii) *If $n \geq 1 + 2k + \sqrt{2k}$, $k \neq 1, n-1$, and*

$$\epsilon < \frac{(n-1)^2 - 4k(n-k)}{(n-1)^2 + 4k^2(n-k)^2[\frac{1}{2} + \frac{1}{3}(k-1)(n-k-1)]},$$

then $\dim \mathcal{H}^k < \infty$.

Proof For the proof of the first part we shall use (15) to estimate $I_{M \setminus B_r}(\mathcal{R})$. Let α be a unit length 1-form at a point outside B_r where the pinching condition is satisfied. There exist $\alpha_2, \ldots, \alpha_n \in T^*M$ such that $\alpha, \alpha_2, \ldots, \alpha_n$ form an orthonormal coframe. Then according to (15) we have

$$\langle \mathcal{R}\alpha, \alpha \rangle = \sum_{i=2}^{n} K(\alpha, \alpha_i).$$

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It follows that $\mathcal{R}$ is bounded from above and below, and

$$I_{M\setminus B_r}(\mathcal{R}) \leq (n-1),$$

and, by McKean's estimate of the Poincaré constant of a negatively curved manifold [13], we have

$$\lambda_1 \geq \frac{(n-1)^2(1-\epsilon)}{4}.$$

Hence, if $\epsilon < 1 - \frac{4}{n-1}$ we have

$$I_{M\setminus B_r}(\mathcal{R}) < \lambda_1,$$

which means, according to Theorem 1.1, $\dim \mathcal{H}^1 < \infty$. This concludes the proof of part one.

For the proof of part two we have to employ more subtle estimates of the curvature operator in terms of the sectional curvature. We shall use the estimates obtained by Elworthy, Li, and Rosenberg in [7]. Let α be a unit length k-form ($k \neq 1, n-1$). According to the proof of Theorem 3.1 in [7] we have

$$\langle \mathcal{R}(\alpha), \alpha \rangle \geq B - (A-B)k(n-k)\Big[\frac{1}{2} + \frac{1}{3}(k-1)(n-k-1)\Big], \tag{16}$$

Where

$$A = \sup\Big\{ \sum_{i=1}^{k} \sum_{k+1}^{n} K(v_i, v_j) \;\Big|\; v_1, \dots, v_n \text{ orthonormal frame} \Big\}$$

and

$$B = \inf\Big\{ \sum_{i=1}^{k} \sum_{k+1}^{n} K(v_i, v_j) \;\Big|\; v_1, \dots, v_n \text{ orthonormal frame} \Big\}.$$

Outside of the compact set the pinching condition is satisfied, so we have

$$A \leq k(n-k)(-1+\epsilon) \text{ and } B \geq -k(n-k),$$

it follows that, outside B_r

$$\langle \mathcal{R}(\alpha), \alpha \rangle \geq -k(n-k)\Big\{ 1 + \epsilon k(n-k)\Big[\frac{1}{2} + \frac{1}{3}(k-1)(n-k-1)\Big]\Big\},$$

which is equivalent to

$$I_{M\setminus B_r}(\mathcal{R}) \leq k(n-k)\Big\{ 1 + \epsilon k(n-k)\Big[\frac{1}{2} + \frac{1}{3}(k-1)(n-k-1)\Big]\Big\}.$$

Using the same estimate of McKean [13] and the hypothesis of the second part of the corollary we obtain again

$$I_{M\setminus B_r}(\mathcal{R}) < \lambda_1,$$

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which by the conclusion of the Theorem 1.1, implies $\dim \mathcal{H}^k < \infty$. This concludes the proof of the corollary. ■

Remarks

(a) If in the hypothesis of Corollary 3.7 one asks for the curvature to be pinched everywhere, then one gets vanishing of the corresponding spaces. However, the ϵ required is much smaller than the one obtained by Donnelly and Xavier in [6].

(b) This result relies heavily on being able to estimate the lower bound of the curvature operator in terms of sectional curvature. A better understanding of this relationship, not easy in general, may lead to new results for the vanishing or finite dimensionality of the L^2 cohomology spaces.

4 On the Heat-Flow Method of Gaffney

As we have seen in Proposition 2.6 the heat flow takes an L^2 form and transforms it into a harmonic L^2 form preserving its cohomology class. A nice differential-topological corollary of the above mentioned fact is the following proposition:

Proposition 4.1 *Let M^n be an n-dimensional noncompact manifold. Then any degree n compactly supported form is exact.*

The proof relies on the following geometric lemma, which is of independent interest:

Lemma 4.2 *Any noncompact manifold admits a complete metric of infinite volume.*

Proof Imbed the manifold into some large Euclidean space $\mathbb{R}^N$ such that the image of the imbedding is closed. This is always possible due to Whitney's Imbedding theorem. We denote this metric by g. If the volume of the manifold with respect to this metric is infinite, we are done. If not, let us fix a point $p \in M$ and let $r(x) = d(x, p)$ be the geodesic distance to the fixed point. Since the image of the imbedding is closed and noncompact, it cannot be bounded, hence $r \to \infty$.

Now choose $f \in C^\infty$ such that

$$f(x)^n \geq \frac{1}{V_{m+1} - V_m} + 1 \quad \text{for } m \leq r(x) \leq m+1. \tag{17}$$

(In (17), V_r denotes the volume of the geodesic ball of radius r). This is always possible.

Let us consider now the metric $\tilde{g} = f^2 g$ and denote the corresponding volume elements by $d\tilde{V}$ and by dV. Then we have the following identity $d\tilde{V} = f^n dV$ where n represents the dimension of the manifold.

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Now, obviously,

$$\int_M 1\, d\tilde{V} \geq \sum_{m=0}^{\infty} \int_{\overline{B(p,m+1)} \setminus B(p,m)} f^n \, dV \tag{18}$$
$$\geq \sum_{m=0}^{\infty} \Big(\frac{1}{V_{m+1} - V_m} + 1 \Big) (V_{m+1} - V_m)$$
$$\geq \sum_{m=0}^{\infty} 1 \geq \infty.$$

In the inequalities above $B(p,r)$ denotes the geodesic ball with respect to the metric g. As a conclusion we see that $(M, \tilde{g})$ has infinite volume. On the other hand, $\tilde{g} \geq g$ which implies that any Cauchy sequence with respect to $\tilde{g}$ is Cauchy with respect to g, hence a convergent sequence. This concludes the proof. ■

Proof of Proposition 4.1 Let us assume the contrary. Endow the manifold M^n with a complete metric of infinite volume. Let $\alpha \in C_0{}^{\infty}\Omega^n$ and $[\alpha] \neq 0$, then let $\alpha(t)$ be the solution to the heat equation with initial data α and let $[\alpha_\infty] = \lim_{t \to \infty} \alpha(t)$. Then we obtain α_∞ a harmonic L^2 n-form which is nontrivial, a contradiction. ■

Observations

(a) It is well known that on a noncompact manifold every top-degree form must be exact. For example see [10]. The proof we offered above makes no use of algebraic topology techniques.

(b) The fact that the existence of a compactly supported nontrivial de Rham class induces a nontrivial L^2 harmonic form was used by Segal in [15] and by Hitchin in [11]. The method used by Segal to prove this is not based on the heat flow method initiated by Gaffney.

Proposition 4.3 *Let M be a complete Riemannian manifold. The following two conditions are equivalent:*

(i) $\mathcal{H}^k = \{0\}$,

(ii) *closed L^2 forms are orthogonal to coclosed L^2 forms.*

Proof Suppose $\mathcal{H}^k \neq 0$. Then there exists $\alpha \in \mathcal{H}^k$ and $\alpha \neq 0$. But $\alpha \in L^2, d\alpha = 0, \delta\alpha = 0$ and by assumption $(\alpha, \alpha) \neq 0$. For the converse, let us assume there exist $\alpha, \beta \in L^2, d\alpha = 0, \delta\beta = 0$, and $(\alpha, \beta) \neq 0$. Let μ denote the solution to the heat flow having as initial data $\mu(0) = \alpha$. Now consider $Q(t) = (\mu(t), \beta)$. Due to the properties of the solution to the heat equation this is a smooth function in t, for $t > 0$ and continuous for, $t \geq 0$.

Differentiating Q we get:

$$\dot{Q}(t) = (\dot{\mu}, \beta) = -(\Delta\mu, \beta).$$

Since μ is closed for all $t > 0$ it follows

$$\dot{Q}(t) = -(d\delta\mu, \beta) = -(\delta\mu, \delta\beta) = 0.$$

This means $Q(t) = Q(0)$ and $(\alpha_\infty, \beta) \neq 0$. Therefore $\mathcal{H}^k \neq 0$. ■

Next we will introduce the concept of the heat-flow map.

Proposition 4.4 *Let M be a manifold as before and α in L^2Z a closed form on M. Let $\dot{\mu} = -\Delta\mu$ be the solution to the heat equation having initial data α. Then the following map $H\colon L^2Z^k \to \mathcal{H}^k$, $H(\alpha) = \lim_{n\to\infty} \mu(t) = \mu_\infty$ is well defined and linear.*

Proof Obvious from the uniqueness of the heat flow. ■

Remarks

(a) It is obvious that the heat-flow map is nothing other than the Hilbert projection onto the space of harmonic forms, although it is preferred to interpret this map as a continuous transformation when proving the results of this section.

(b) When considering different metrics on the same manifold we will indicate that the spaces or operators are taken with respect to the metric g by an appropriate index. For example the space of harmonic L^2 forms with respect to the metric g will be indicated as $\mathcal{H}^k_g$, the Laplacian with respect to the metric g will be denoted by Δ_g, the Hodge-star operator by $*_g$, *etc.*

The next theorem is another immediate application of the principle we used to prove Proposition 4.3.

Theorem 4.5 *Let M be a noncompact manifold and let g and $\tilde{g}$ be two complete metrics on M such that*

$$L^2\Omega^k_{\tilde{g}} \subset L^2\Omega^k_g \text{ and } L^2\Omega^{n-k}_{\tilde{g}} \subset L^2\Omega^{n-k}_g. \tag{19}$$

Then the map $H_{\tilde{g}}\colon \mathcal{H}^k_{\tilde{g}} \to \mathcal{H}^k_g$ is linear and injective.

Proof All we have to prove is that $\ker H_g = 0$. Let $\alpha \in \mathcal{H}^k_{\tilde{g}}$ such that $H_g(\alpha) = 0$. Since $\alpha \in \mathcal{H}^k_{\tilde{g}}$, it implies α is closed and coclosed with respect to the $\tilde{g}$ metric. This means $*_{\tilde{g}}\alpha$ is also closed, hence $*_g *_{\tilde{g}} \alpha$ is coclosed with respect to the g metric. All the forms here are also L^2 since the $*$-operator preserves length. Let $\beta = *_g *_{\tilde{g}} \alpha$. We have

$$0 = (H_g(\alpha), \beta) = (\alpha, \beta)_g = \int_M \alpha \wedge *_g\beta.$$

But $*_g\beta = \pm *_{\tilde{g}} \alpha$. In conclusion we have

$$\int_M \alpha \wedge *_{\tilde{g}}\alpha = (\alpha, \alpha)_{\tilde{g}} = 0.$$

Hence $\alpha = 0$. Therefore H_g is injective. ■

In order to give an interesting application of Theorem 4.5 we need to make the following definition:

Definition 4.6 Let V be a finite dimensional real vector space and let g and h be two positive definite inner products on V. Let $G\colon V \to V^*$ and $H\colon V \to V^*$ denote the metric isomorphisms induced by g and h respectively. Let $A\colon V \to V$ denote the composition $A = H^{-1}G$. It is well known that A is orthogonally diagonalizable with respect to the metric h and hence let $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n$ be its eigenvalues. We will call these positive numbers the eigenvalues of g with respect to h.

Corollary 4.7 *Let (M^n, g) be an even dimensional, complete, simply connected Riemannian manifold of negative sectional curvature. Let h denote the complete metric of constant -1 sectional curvature. Let $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n$ be the eigenvalues of g with respect to h as functions on the manifold (i.e., $\mu_i = \mu_i(x)$, $x \in M$).*

If $\sup_{x\in M} \frac{\mu_n}{\mu_1} < \infty$ then

$$\dim \mathcal{H}^{n/2}(M, g) = \infty.$$

Proof Let us fix a point $x \in M$ and let $e_1, e_2, \ldots, e_n$ be a set of eigenvectors for g which are orthonormal with respect to h. Let $e^1, e^2, \ldots, e^n$ be the associated dual coframe. It follows that

$$\eta_i = \frac{1}{\sqrt{\mu_i}} e_i$$

is an orthonormal frame with respect to g having associated coframe

$$\eta^i = \sqrt{\mu_i} e^i.$$

Hence, if we denote by ω_h and ω_g the volume forms for the two metrics respectively, we have

$$\omega_g = \eta^1 \wedge \eta^2 \wedge \cdots \wedge \eta^n = \sqrt{\mu_1 \cdots \mu_n}\, \omega_h.$$

Let α be an $n/2$-form expressed at $x \in M$ as

$$\alpha = \alpha_{i_1 \cdots i_{n/2}} e^{i_1} \wedge \cdots \wedge e^{i_{n/2}} = \alpha_{i_1 \cdots i_{n/2}} \frac{1}{\sqrt{\mu_{i_1} \cdots \mu_{i_{n/2}}}} \eta^{i_1} \wedge \cdots \wedge \eta^{i_{n/2}},$$

it follows that

$$|\alpha|_h^2 = \sum_{i_1 < i_2 < \cdots < i_{n/2}} \alpha^2_{i_1 \cdots i_{n/2}} \text{ and } |\alpha|_g^2 = \sum_{i_1 < i_2 < \cdots < i_{n/2}} \alpha^2_{i_1 \cdots i_{n/2}} \frac{1}{\mu_{i_1} \cdots \mu_{i_{n/2}}}.$$

Next we compare the L^2 norms, we have

$$|\alpha|_g^2 \omega_g = \sum_{i_1 < i_2 < \cdots < i_{n/2}} \alpha^2_{i_1 \cdots i_{n/2}} \frac{1}{\mu_{i_1} \cdots \mu_{i_{n/2}}} \sqrt{\mu_1 \cdots \mu_n}\, \omega_h,$$

hence

$$|\alpha|_g^2 \omega_g \le \frac{\mu_n^{n/2}}{\mu_1^{n/2}} |\alpha|_h^2 \omega_h. \tag{20}$$

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Using the hypothesis that $\sup_{x\in M} \frac{\mu_n}{\mu_1} < \infty$ and integrating we obtain for $\alpha \in C_0^\infty \Omega^{n/2}$

$$\sup_{x\in M}\left(\frac{\mu_n}{\mu_1}\right)^{n/2} \|\alpha\|_h^2 \geq \|\alpha\|_g^2,$$

hence

$$L^2\Omega_h^{n/2} \subset L^2\Omega_g^{n/2}$$

and applying the conclusion of Theorem 4.5 we get

$$\dim \mathcal{H}_g^{n/2} = \dim \mathcal{H}_h^{n/2} = \infty.$$

This concludes the proof of the corollary. ■

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